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A GEOMETRIC INTERPRETATION OF ISOPARAMETRIC HYPERSURFACES

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1. Isoparametric hypersurfaces in a real space form.

This part is a joint work with Sadahiro Maeda [KM]. In differential geometry it is interesting to know the shape of a Riemannian submanifold by observing the extrinsic shape of geodesics of the submanifold. For example: A hypersurface M^n isometrically immersed into a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c (that is, $\widetilde{M}^{n+1}(c) = \mathbb{R}^{n+1}$, $S^{n+1}(c)$ or $H^{n+1}(c)$ according as the curvature c is zero, positive, or negative) is totally umbilic in $\widetilde{M}^{n+1}(c)$ if and only if every geodesic of M, considered as a curve in the ambient space $\widetilde{M}^{n+1}(c)$, is a circle. Here we treat a geodesic as a circle of null curvature.

In this talk we are interested in a hypersurface M^n of a real space form $\widetilde{M}^{n+1}(c)$ satisfying that there exists such an *orthonormal* basis $\{v_1, \cdots, v_n\}$ at each point p of the hypersurface M^n that all geodesics of M^n through p in the direction v_i $(1 \leq i \leq n)$ are circles in the ambient space $\widetilde{M}^{n+1}(c)$. The classification problem of such hypersurfaces is concerned with studies about isoparametric hypersurfaces M^n's in a real space form $\widetilde{M}^{n+1}(c)$ (that is, all principal curvatures of M^n in $\widetilde{M}^{n+1}(c)$ are constant).

Theory of isoparametric submanifolds is one of the most interesting objects in differential geometry. In particular, É. Cartan studied extensively isoparametric hypersurfaces in a standard sphere. The classification problem of isoparametric hypersurfaces in a sphere is still open (for details, see [CR]).

The initial purpose of this talk is to provide a characterization of all isoparametric hypersurfaces by observing the extrinsic shape of geodesics of hypersurfaces in a real space form.

Theorem 1. *Let M^n be a connected hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c. Then M^n is isoparametric in $\widetilde{M}^{n+1}(c)$ if and only if there exists such an orthonormal basis $\{v_1, \cdots, v_m\}$ of the orthogonal complement of* $\ker A$ *in* $T_p(M)$ $(m = \operatorname{rank} A)$ *that all geodesics of M through p in the direction v_i $(1 \leq i \leq m)$ are circles of nonzero curvature in the ambient space $\widetilde{M}^{n+1}(c)$.*

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Theorem 2. *Let M^n be a connected hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c. Then M^n is isoparametric with nonzero constant principal curvatures in $\widetilde{M}^{n+1}(c)$ if and only if for each point p of M, there exists such an orthonormal basis $\{v_1, \cdots, v_n\}$ of $T_p(M)$ that all geodesics of M through p in the direction v_i $(1 \leq i \leq n)$ are circles of nonzero curvature in the ambient space $\widetilde{M}^{n+1}(c)$.*

Theorem 3. *Let M^n be a connected hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c. Then M^n is isoparametric in $\widetilde{M}^{n+1}(c)$ if and only if for each point p of M, there exists such an orthonormal basis $\{v_1, \cdots, v_n\}$ of $T_p(M)$ of principal curvature vectors that all geodesics of M through p in the direction v_i $(1 \leq i \leq n)$ are circles in the ambient space $\widetilde{M}^{n+1}(c)$.*

2. Homogeneous real hypersurfaces in a complex projective space.

This part is a joint work with Toshiaki Adachi and Sadahiro Maeda [AKM]. Let $P_n(\mathbb{C})$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4, and let M be a real hypersurface of $P_n(\mathbb{C})$. Then M has an almost contact metric structure (ϕ, ξ, η, g) inherited from the Kaehler structure of $P_n(\mathbb{C})$. Many differential geometers have studied M by using this structure (cf. [O]). Typical examples of real hypersurfaces in $P_n(\mathbb{C})$ are homogeneous real hypersurfaces, that is, real hypersurfaces given as orbits under subgroups of the projective unitary group $PU(n+1)$.

Takagi ([T]) classified homogeneous real hypersurfaces in $P_n(\mathbb{C})$. Due to his work, we find that a homogeneous real hypersurface in $P_n(\mathbb{C})$ is locally congruent to one of the six model spaces of type A_1, A_2, B, C, D and E. They are realized as tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or 2. A homogeneous real hypersurface of type A_1 is usually called a *geodesic hypersphere.*

In the study of real hypersurfaces in $P_n(\mathbb{C})$, there can be the following two problems:

(A) Give a characterization of homogeneous real hypersurfaces in $P_n(\mathbb{C})$.

(B) Construct non-homogeneous nice real hypersurfaces in $P_n(\mathbb{C})$ and characterize such examples.

In this talk we are interested in Problem (A). In differential geometry it is interesting to know the shape of a Riemannian submanifold by observing the extrinsic shape of geodesics of the submanifold. ¿From this point of view we here recall the fact that a hypersurface M^n in $\mathbb{R}^{n+1}$ is locally a standard sphere if and only if all geodesics of M are circles of positive curvature in $\mathbb{R}^{n+1}$. We shall provide a characterization of all homogeneous real hypersurfaces in $P_n(\mathbb{C})$ by observing the shape of geodesics on the real hypersurfaces as curves in $P_n(\mathbb{C})$.

The purpose of this part is to prove the following result which is an improvement of the previous paper [MO].

Theorem 4. *Let M be a connected real hypersurface of $P_n(\mathbb{C})$. Then M is congruent to a homogeneous real hypersurface if and only if there exist such orthonormal vectors $v_1, v_2, \cdots, v_{2n-2}$ orthogonal to ξ at each point p of M that all geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i (1 \leqq i \leqq 2n-2)$ are circles in $P_n(\mathbb{C})$ with positive curvature.*

In the hypothesis of our Theorem we do not need to suppose that we take the vectors $\{v_1, \cdots, v_{2n-2}\}$ as *a local field of orthonormal frames* in M. However, for all homogeneous real hypersurfaces $M's$ in $P_n(\mathbb{C})$, we can take a local field of orthonormal frames in M satisfying the hypothesis of our Theorem.

It is well-known that there does not exist a real hypersurface all of whose geodesics are circles in $P_n(\mathbb{C})$. Every circle in Theorem is a simple closed curve which lies on some totally real totally geodesic $P^2(\mathbb{R})$ in $P_n(\mathbb{C})$. We note that for any homogeneous real hypersurface M, at each point p of M the geodesic $\gamma = \gamma(s)$ with $\gamma(0) = p$ and $\dot\gamma(0) = \xi$ is also a circle in $P_n(\mathbb{C})$ which is a simple closed curve lying on some holomorphic totally geodesic $P_1(\mathbb{C})$ in $P_n(\mathbb{C})$. All circles in $P_n(\mathbb{C})$ are simple curves. However, a circle in $P_n(\mathbb{C})$ is not necessarily closed (see, [AMU]).

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