SINGULARITIES OF SOLUTIONS TO ELASTIC WAVE PROPAGATION PROBLEMS IN STRATIFIED MEDIA II

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ABSTRACT. In this paper we shall study elastic mixed or initial-interface value problems and give an inner estimate of the location of singularities of reflected and refracted Riemann functions by making use of the localization method.

In this paper, we shall continue our study in the previous paper “Singularities of Solutions to Elastic Wave Propagation Problems in Stratified media” in RIMS Kokyuroku 994 (1997), Spectral and Scattering Theory and Its Related Topics pp.104-120.

1. Introduction

We consider elastic wave propagation problems in the following plane-stratified media $\mathbb{R}^3$ with the planar interface $x_3 = 0$:

\[
(\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} 
(\lambda_1, \mu_1, \rho_1) & \text{for } x_3 < 0, \\
(\lambda_2, \mu_2, \rho_2) & \text{for } x_3 > 0.
\end{cases}
\]

Here the constants $\lambda_1, \lambda_2, \mu_1, \mu_2$ are called the Lamé constants and the constants $\rho_1, \rho_2$ are densities. We shall denote the lower half-space $\mathbb{R}_-^3$ by Medium I and the upper half-space $\mathbb{R}_+^3$ by Medium II, respectively, as in Figure 1.

\[
\begin{array}{cccc}
& & \mu_2 & \rho_2 \\
\lambda_2 & II & & \\
& & 0 & \\
I & & \mu_1 & \rho_1
\end{array}
\]

Figure 1 Stratified media I and II

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We assume that

\[ \lambda_i + \mu_i > 0, \quad \mu_i > 0, \quad \rho_i > 0, \quad i = 1, 2. \]

(1.1) is the natural assumption in practical situation. From the roots of the characteristic equations of \( P^I(D) \) and \( P^{II}(D) \) which are defined below \( 3 \times 3 \) matrix valued hyperbolic partial differential operators in Medium I and Medium II, respectively, we obtain two speeds correspond to Pressuer or Primary wave (P wave) and Share or Secondary wave (S wave) on each medium. \( c_{p1} \) denotes the speed of P wave in Medium I and \( c_{s1} \) denotes the speed of S wave in Medium I. \( c_{p2} \) and \( c_{s2} \) denote the speed of P and S wave in Medium II, respectively. They are given by

\[
c_{p_{i}}^{2} = \frac{\lambda_{i} + 2 \mu_{i}}{\rho_{i}}, \quad c_{s_{i}}^{2} = \frac{\mu_{i}}{\rho_{i}}, \quad i = 1, 2.
\]

By assumption (1.1), the speed of P wave is greater than that of S wave in each medium. On account of this, these are six cases of the order relation of the speeds of \( \{c_{p1}, c_{s1}, c_{p2}, c_{s2}\} \). Here we assume that

\[ c_{s1} < c_{p1} \leq c_{s2} < c_{p2}. \]

(1.2)

It is the standard case (cf. [Sh, Section 3]). The other cases can be treated in a similar manner.

Let \( x = (x_0, x_1, x_2, x_3) = (x', x_3) = (x_0, x_3) = (x_0, x_3) \in \mathbb{R}^4 \). The variable \( x_0 \) will play a role of time, and \( x'' = (x_1, x_2, x_3) \) will play that of space. \( \xi \) is a real dual variable of \( x \) and is equal to \( (\xi_0, \xi_1, \xi_2, \xi_3) = (\xi', \xi_3) = (\xi_0, \xi') = (\xi_0, \xi_1, \xi_2, \xi_3) \) in \( \mathbb{R}^4_\xi \). We use the differential symble \( D_j = i^{-1} \partial / \partial x_j \) \((j = 0, 1, 2, 3)\), where \( i = \sqrt{-1} \).

We shall denote by \( \mathbb{R}^n_+ \) the half-space \( \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n | x_n < 0 \} \) and by \( \mathbb{R}^n_+ \) the half-space \( \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n | x_n > 0 \} \), and also use the notation \( |x| = \sqrt{x_1^2 + \cdots + x_n^2} \).

Let \( u(x) = (u_1(x), u_2(x), u_3(x)) \in \mathbb{R}^3 \) be the displacement vector at time \( x_0 \) and position \( x'' \). The propagation problems of elastic waves in the stratified media is formulated as mixed (initial-interface value) problem:

\[
\begin{aligned}
P^I(D)u(x) &= f(x), \quad x_0 > 0, \quad x'' = (x_1, x_2, x_3) \in \mathbb{R}^3, \\
P^{II}(D)u(x) &= f(x), \quad x_0 > 0, \quad x'' = (x_1, x_2, x_3) \in \mathbb{R}^3, \\
u(x)|x_3=0 &= u(x)|x_3=+0, \quad x_0 > 0, \quad x''' \in \mathbb{R}^2, \\
B^I(D)u(x)|x_3=0 &= B^{II}(D)u(x)|x_3=+0, \quad x_0 > 0, \quad x''' \in \mathbb{R}^2, \\
D^k_0 u(x)|x_3=0 &= g_k(x''), \quad k = 0, 1, \quad x'' \in \mathbb{R}^3.
\end{aligned}
\]

(1.3)

Here

\[
P^I(D)u = -D_0^2 Eu + \frac{\lambda_1 + \mu_1}{\rho_1} \nabla_{x''} (\nabla_{x''} \cdot u) + \frac{\mu_1}{\rho_1} \Delta_{x''} u,
\]

is a \( 3 \times 3 \) matrix valued second order hyperbolic differential operator with constant coefficients where \( E \) is a \( 3 \times 3 \) identity matrix,

\[
(B^I(D)u)_k = i \lambda_1 (\nabla_{x''} \cdot u) \delta_{k3} + 2 \mu_1 \varepsilon_{k3}(u), \quad k = 1, 2, 3,
\]

is a \( 3 \times 3 \) matrix valued second order hyperbolic differential operator with constant coefficients where \( E \) is a \( 3 \times 3 \) identity matrix,
is the k-th component of symmetric stress tensors $B^I(D)u$ where

$$\varepsilon_{k3}(u) = \frac{i}{2}(D_3u_k + D_ku_3), \quad k = 1, 2, 3,$$

are strain tensors. The $P^{II}(D)u$ and $B^{II}(D)u$ are defined by replacing $\lambda_1$, $\mu_1$, $\rho_1$ by $\lambda_2$, $\mu_2$, $\rho_2$, respectively.

If we put unit impulse Dirac’s delta $\delta(x - y)$ with $x_0 \geq y_0$ and $y_3 < 0$, that is, put it in Medium I, then the Riemann function of this elastic mixed problem is given by the following:

$$G(x, y) = \begin{cases} E^I(x - y) - F^I(x, y) & \text{for } x_3 < 0, \\ F^{II}(x, y) & \text{for } x_3 > 0, \end{cases}$$

where $E^I(x)$ is the fundamental solution in Medium I describing an incident wave, is defined by

$$E^I(x) = (2\pi)^{-4} \int_{\mathbb{R}_\xi^4} e^{i(x - y)} P^I(\xi + i\eta)^{-1} d\xi, \quad \eta \in -s\vartheta - \Gamma,$$

with a positive real $s$ large enough. Here $\vartheta$ and $\Gamma$ are defined below. Taking partial Fourier-Laplace transform with respect to $x'$ for the mixed problem, we obtain a interface value problem for ordinary differential equation with parameters. Then taking partial inverse Fourier-Laplace transform for the solution, we obtain explicit expressions of reflected and refracted Riemann functions $F^I(x, y)$ and $F^{II}(x, y)$ which describe reflected and refracted waves, respectively.

In this paper, we give an inner estimate of the location of singularities of reflected and refracted Riemann functions $F^I(x, y)$ and $F^{II}(x, y)$ by making use of the localization method. This method is first studied by M. F. Atiyah, R. Bott, L. Gårding [A-B-G] for initial value problem, then studied by M. Matsumura [Ma 1], M. Tsuji [Ts], and S. Wakabayashi [Wa 1], [Wa 2] for half-space mixed problem. Matsumura studied the singularities of the ordinary wave propagation problems in the stratified media by applying above methods [Ma 2], [Ma 3]. They are useful references to our study.

We define a localization of polynomials according to Atiyah-Bott-Gårding (cf. [A-B-G]):

**Definition 1.** Let $P(\xi)$ be a polynomial of degree $m \geq 0$ and develop $\nu^m P(\nu^{-1}\xi + \eta)$ in ascending power of $\nu$:

$$\nu^m P(\nu^{-1}\xi + \eta) = \nu^p P_\xi(\eta) + O(\nu^{p+1}) \quad \text{as } \nu \to 0,$$

where $P_\xi(\eta)$ is the first coefficient that does not vanish identically in $\eta$. The polynomial $P_\xi(\eta)$ is the localization of $P$ at $\xi$, the number $p$ is the multiplicity of $\xi$ relative to $P$.

Moreover we introduce the following:
Definition 2. \( \Gamma = \Gamma(P, \vartheta) \) is the component of \( \mathbb{R}^n \setminus \{ \eta \in \mathbb{R}^n, P(\eta) = 0 \} \) which contains \( \vartheta = (1, 0, \cdots, 0) \in \mathbb{R}^n \). Moreover \( \Gamma' = \Gamma'(P, \vartheta) = \{ x \in \mathbb{R}^n \mid x \cdot \eta \geq 0, \eta \in \Gamma \} \) is the dual cone of \( \Gamma \) and is called the propagation cone.

We obtain the following Main theorem I. This Main Theorem I is corresponding to Main Theorem in the previous paper. It means singular supports of reflected and refracted Riemann functions \( F^I(x, y) \) and \( F^{II}(x, y) \) are estimated innerly by localizations \( F_{\xi_0}^I(x, y) \) and \( F_{\xi_0}^H(x, y) \) of \( F^I(x, y) \) and \( F^{II}(x, y) \) at \( \xi^0 \), respectively.

Main Theorem I. For \( \xi^0 \in \mathbb{R}_\xi^4 \) satisfying \( (\text{det} P_j^I)(\xi^0) = 0 \) \((j \in \{p_1, s_1\})\), that is
\[
(\text{det} P_{p_1}^I)(\xi^0) = \xi_{0}^{0^2} - c_{p_1}^2 |\xi^{0''}|^2 = 0,
\]
or
\[
(\text{det} P_{s_1}^I)(\xi^0) = \xi_{0}^{0^2} - c_{s_1}^2 |\xi^{0''}|^2 = 0,
\]
we have the following:
(1) For the reflected Riemann function \( F^I(x, y) \), we have
\[
(1.5) \quad \lim_{\nu \to \infty} \nu^{\frac{3}{2}} e^{-i\nu((x'-y') \cdot \xi^0') + c_{p_1}^2 |\xi^{0''}|^2} F^I_j(x, y) = F^{I}_{j\xi_0^0 k}(x, y),
\]
and if \( \xi^{0'} \) are zeros of \( \tau_{m}^{+}((\xi') \cdot \eta') \), that is, \( \xi^{0'} \) satisfy \( |\xi^{0''}| = \frac{\xi_{0}^{0^2}}{c_m} \) \((m \in \{p_1, p_2, s_2\})\), then we have
\[
(1.6) \quad \lim_{\nu \to \infty} \nu^{\frac{3}{2}} e^{-i\nu((x'-y') \cdot \xi^0') + c_{p_1}^2 |\xi^{0''}|^2} F^I_j(x, y) = F^{I}_{j\xi_0^0 km}(x, y),
\]
in the distribution sense with respect to \( (x, y) \in \mathbb{R}_-^4 \times \mathbb{R}_-^4 \).

Moreover we have
\[
(1.7) \quad \bigcup_{\xi^0 \neq 0} (\text{supp} F^I_{j\xi_0^0 k}(x, y) \cup \text{supp} F^{I}_{j\xi_0^0 km}(x, y)) \subset \text{sing supp} F^I(x, y),
\]
and
\[
(1.8) \quad \text{supp} F^I_{j\xi_0^0 k}(x, y) = (\Gamma_{j\xi_0^0})^I_k \left\{ (x, y) \in \mathbb{R}_-^4 \times \mathbb{R}_-^4 : \right\}
\]
\[
((x'-y') + c_{p_1} \text{grad}_{\xi}^{-}(\xi^0')) \cdot \eta' - y_3 \eta_3 \geq 0, \eta \in \Gamma_{j\xi_0^0},
\]
\( j \in \{p_1, s_1\}, \quad k \in \{p_1, s_1\} \)
for \( \xi^0 \) satisfying \( F^I_{j\xi_0^0 k}(x, y) \neq 0 \) \((j \in \{p_1, s_1\}, k \in \{p_1, s_1\})\),
(1.9) \quad \text{supp} F^{I}_{j\xi_0^0 km}(x, y) = (\Gamma_{j\xi_0^0 m})^I_k \left\{ (x, y) \in \mathbb{R}_-^4 \times \mathbb{R}_-^4 : \right\}.
\[ ((x' - y') + x_3 \text{grad}_x \tau_k^+(\xi^0')) \cdot \eta' - y_3 \eta_3 \geq 0, \eta \in \Gamma_{j\xi^0 m}, \]
\[ j \in \{p_1, s_1\}, \quad k \in \{p_1, s_1\}, \quad m \in \{p_1, p_2, s_2\} \]

for \( \xi^0 \) satisfying \( F_{j\xi^0 km}(x, y) \neq 0 (j \in \{p_1, s_1\}, k \in \{p_1, s_1\}, m \in \{p_1, p_2, s_2\}) \).

(2) For the refracted Riemann function \( F^{II}(x, y) \), we have

\[ \lim_{\nu \to \infty} \nu e^{-i\nu((x' - y') \cdot \xi^0' + x_3 \tau_k^+(\xi^0') - y_3 \xi_3^0)} F^{II}(x, y) = F_{j\xi^0 k}(x, y), \]
\[ j \in \{p_1, s_1\}, \quad k \in \{p_1, s_1\}, \quad m \in \{p_2, s_2\}, \]

and if \( \xi^0' \) are zeros of \( \tau_m^+(\zeta') (m \in \{p_2\}) \), then we have

\[ \lim_{\nu \to \infty} \left\{ \nu^\frac{3}{2} e^{-i\nu((x' - y') \cdot \xi^0 + x_3 \tau_k^+(\xi^0') - y_3 \xi_3^0)} - \nu^\frac{1}{2} F^{II}_{j\xi^0 k}(x, y) \right\} \]
\[ = F_{j\xi^0 km}(x, y), \quad j \in \{p_1, s_1\}, \quad k \in \{p_1, s_1\}, \quad m \in \{p_2\}, \]
in the distribution sense with respect to \( (x, y) \in \mathbb{R}_+^4 \times \mathbb{R}_-^4 \).

Moreover we have

\[ \bigcup_{\xi^0 \neq 0} (\text{supp } F^{II}_{j\xi^0 k}(x, y) \cup \text{supp } F^{II}_{j\xi^0 km}(x, y)) \subset \text{sing supp } F^{II}(x, y), \]

and

\[ \text{supp } F^{II}_{j\xi^0 k}(x, y) = (\Gamma_{j\xi^0 k}^{II}) = \{(x, y) \in \mathbb{R}_+^4 \times \mathbb{R}_-^4 : \}
\[ ((x' - y') + x_3 \text{grad}_x \tau_k^+(\xi^0')) \cdot \eta' - y_3 \eta_3 \geq 0, \eta \in \Gamma_{j\xi^0} \}, \]
\[ j \in \{p_1, s_1\}, \quad k \in \{p_2, s_2\} \]

for \( \xi^0 \) satisfying \( F^{II}_{j\xi^0 k}(x, y) \neq 0 (j \in \{p_1, s_1\}, k \in \{p_2, s_2\}, m \in \{p_2\}) \), and

\[ \text{supp } F^{II}_{j\xi^0 km}(x, y) = (\Gamma_{j\xi^0 m}^{II}) = \{(x, y) \in \mathbb{R}_+^4 \times \mathbb{R}_-^4 : \}
\[ ((x' - y') + x_3 \text{grad}_x \tau_k^+(\xi^0')) \cdot \eta' - y_3 \eta_3 \geq 0, \eta \in \Gamma_{j\xi^0 m} \}, \]
\[ j \in \{p_1, s_1\}, \quad k \in \{p_2, s_2\}, \quad m \in \{p_2\} \]

for \( \xi^0 \) satisfying \( F^{II}_{j\xi^0 km}(x, y) \neq 0 (j \in \{p_1, s_1\}, k \in \{p_2, s_2\}, m \in \{p_2\}) \).

Here

\[ \Gamma_{j\xi^0} = \Gamma(\text{det } P^I_j, \xi^0(\eta), \vartheta), \quad \vartheta = (1, 0, 0, 0), \quad j \in \{p_1, s_1\}, \]
\begin{align}
\Gamma_{j\xi^{0}m} &= \Gamma((\det P_{j}^{I})_{\xi^{0}(\eta)}, \vartheta) \cap \left\{ \Gamma \left( \frac{\xi_{0}^{0}}{c_{m}^{2}} \eta_{0} - \xi_{1}^{0} \eta_{1} - \xi_{2}^{0} \eta_{2}, \vartheta' \right) \times \mathbb{R}_{\eta} \right\}, \\
\vartheta' &= (1, 0, 0), \ j \in \{p_{1}, s_{1}\}, \ m \in \{p_{2}\}, \\
\tau_{p_{1}}^{\pm}(\xi') &= \text{sgn}(\pm \xi_{0}) \sqrt{\frac{\xi_{0}^{2}}{c_{p_{1}}^{2}} - |\xi^{0''}|^{2}}, \ \text{if} \ \frac{\xi_{0}^{2}}{c_{p_{1}}^{2}} - |\xi^{0''}|^{2} \geq 0,
\end{align}

and \( \tau_{p_{1}}^{\pm}(\xi') \) is taken a branch of \( \sqrt{\frac{\xi_{0}^{2}}{c_{p_{1}}^{2}} - |\xi^{0''}|^{2}} \) such that \( \pm \text{Im} \tau_{p_{1}}^{\pm}(\xi') > 0 \) if \( \frac{\xi_{0}^{2}}{c_{p_{1}}^{2}} - |\xi^{0''}|^{2} < 0 \). \( \tau_{p_{1}}^{\pm}(\xi') \), \( \tau_{s_{1}}^{\pm}(\xi') \), and \( \tau_{s_{2}}^{\pm}(\xi') \) are defined as the same as \( \tau_{p_{1}}^{\pm}(\xi') \) substituting \( c_{p_{1}} \) for \( c_{s_{1}} \), \( c_{p_{2}} \), and \( c_{s_{2}} \), respectively.

**Remark.1.** The \( (\Gamma_{j\xi^{0}})_{k}^{I} \ (j \in \{p_{1}, s_{1}\}, \ k \in \{p_{1}, s_{1}\}) \) represent k reflected wave for j incident wave. The \( (\Gamma_{j\xi^{0}m})_{k}^{I} \ (j \in \{p_{1}, s_{1}\}, \ m \in \{p_{1}, p_{2}, s_{2}\}) \) represent m lateral wave of k reflected wave for j incident wave. The \( (\Gamma_{j\xi^{0}})_{k}^{II} \ (j \in \{p_{1}, s_{1}\}, \ k \in \{p_{2}, s_{2}\}) \) represent k refracted wave for j incident wave. The \( (\Gamma_{j\xi^{0}m})_{k}^{II} \ (j \in \{p_{1}, s_{1}\}, \ k \in \{p_{2}, s_{2}\} \ m \in \{p_{2}\}) \) represent m lateral wave of k refracted wave for j incident wave.

**Remark.2.** The \( \tau_{p_{1}}^{\pm}(\xi') \), \( \tau_{s_{1}}^{\pm}(\xi') \), \( \tau_{p_{2}}^{\pm}(\xi') \), and \( \tau_{s_{2}}^{\pm}(\xi') \) arise from

\begin{align}
\text{det} P^{I}(\xi) &= \text{det} P_{1}^{I}(\xi) \times \text{det} P_{2}^{I}(\xi) \\
&= \{(-\xi_{0}^{2} + c_{p_{1}}^{2}|\xi''|^{2})(-\xi_{0}^{2} + c_{s_{1}}^{2}|\xi''|^{2})\} \times (-\xi_{0}^{2} + c_{s_{2}}^{2}|\xi''|^{2}) \\
&= \text{det} P_{p_{1}}^{I}(\xi) \times \text{det} P_{s_{1}}^{I}(\xi) \\
&= \{c_{p_{1}}^{2}c_{s_{1}}^{2}(\xi_{3} - \tau_{p_{1}}^{+}(\xi'))(\xi_{3} - \tau_{p_{1}}^{-}(\xi'))(\xi_{3} - \tau_{s_{1}}^{+}(\xi'))(\xi_{3} - \tau_{s_{1}}^{-}(\xi'))\} \\
&\times \{c_{s_{2}}^{2}(\xi_{3} - \tau_{s_{2}}^{+}(\xi'))(\xi_{3} - \tau_{s_{2}}^{-}(\xi'))\},
\end{align}

and the factor of \( \text{det} P^{II}(\xi) \) given with replaced \( p_{1}, s_{1} \) by \( p_{2}, s_{2} \), respectively.

**Remark.3.** If \( (\text{det} P^{I})_{j}(\xi^{0}) \neq 0, \ (j \in \{p_{1}, s_{1}\}) \) then \( (\text{det} P^{I})_{j\xi^{0}(\eta)} = (\text{det} P^{I})_{j}(\xi^{0}) \) and is constant. So \( \Gamma_{j\xi^{0}} = \Gamma_{j\xi^{0}m} = \mathbb{R}^{4} \) and thus \( (\Gamma_{j\xi^{0}})_{k}^{I} = (\Gamma_{j\xi^{0}m})_{k}^{I} = \{0\} \subset \mathbb{R}^{4} \times \mathbb{R}^{4} \ (j \in \{p_{1}, s_{1}\}, \ k \in \{p_{1}, s_{1}\} m \in \{p_{1}, p_{2}, s_{2}\}) \) and \( (\Gamma_{j\xi^{0}})_{k}^{II} = (\Gamma_{j\xi^{0}m})_{k}^{II} = \{0\} \subset \mathbb{R}^{4} \times \mathbb{R}^{4} \ (j \in \{p_{1}, s_{1}\}, \ k \in \{p_{2}, s_{2}\} m \in \{p_{2}\}).

**Remark.4.** By the assumption (1.2), there are not any real \( \xi \) that are roots of \( \xi_{0}^{2} - c_{p_{1}}^{2}|\xi''|^{2} = 0 \) and zeros of \( \tau_{s_{1}}^{\pm}(\xi') \). The sets of \( \xi^{0} \) cause singularities are given in (3.1)-(3.19).

**Remark.5.** In (1.8), \( \xi^{0} \) satisfying \( F_{j\xi^{0}k}(x, y) \neq 0 \) is equivalent to \( (Q_{1}(\xi^{0}), Q_{2}(\xi^{0})) \neq 0 \) in (3.18) below, or is equivalent to \( Q_{1}(\xi^{0}) \neq 0 \) in (3.20) below. In (1.9), \( \xi^{0} \) satisfying \( F_{j\xi^{0}km}(x, y) \neq 0 \) is equivalent to \( T_{1}(\xi^{0})R_{1}(\xi^{0}) - Q_{1}(\xi^{0})S(\xi^{0}) \neq 0 \) or \( T_{2}(\xi^{0})R_{1}(\xi^{0}) - Q_{2}(\xi^{0})S(\xi^{0}) \neq 0 \) in (3.21) below.

Lateral waves, in other words, glancing wave, arise from the presence of branch points of \( \tau_{p_{1}}^{\pm}(\xi'), \tau_{s_{1}}^{\pm}(\xi'), \tau_{s_{2}}^{\pm}(\xi') \), and \( \tau_{s_{2}}^{\pm}(\xi') \). In our problem, many lateral waves are appeared.
Concerning the Lopatinski determinant, there are two cases: One is the case that Lopatinski determinant has one real zero, and the other is the case that it has no zero. It depends on the Lamé constants and densities. We remark that the speed of the Stoneley wave $c_{St}$ is less than or equal to $c_{s_{1}}$ which is the minimum speed of $\{c_{s_{1}}, c_{p_{1}}, c_{s_{2}}, c_{p_{2}}\}$ (cf. [Sh Section 3]). If we put unit impulse $\delta$ on the interface $x_{3} = 0$, then we have the Riemann functions

$$H_{l}(x) = \begin{cases} H_{l}^{I}(x) & \text{for } x_{3} < 0, \\ H_{l}^{I}(x) & \text{for } x_{3} > 0, \end{cases}$$

given (4.1)-(4.5) below. The singularity appears which corresponds to the interface wave of which name is the Stoneley wave, in the case that the Lopationski determinant has one real zero. For this case we have the following.

**Main Theorem II.** For $\xi^{0'} \in \mathbb{R}_{\xi}^{3}$ satisfying $\xi_{0}^{2} - c_{s_{1}}^{2}(\xi_{1}^{2} + \xi_{2}^{2}) = 0$, we have the following:

(1) For the Riemann function $H_{l}^{I}(x)$ ($l = 1, 2$), we have

$$\lim_{\nu \to \infty} \nu^{-1}e^{-i\nu x' \cdot \xi^{0'}} H_{l}^{I}(x) = H_{l \xi^{0}}^{I}(x)$$

in the distribution sense with respect to $x \in \mathbb{R}_{+}^{4}$.

Moreover we have

$$\bigcup_{\xi^{0'} \neq 0} \text{supp} \lim_{x_{3} \to -0} H_{l \xi^{0}}^{I}(x) \subset \text{sing supp} \lim_{x_{3} \to -0} H_{l}^{I}(x),$$

and

$$\text{supp} \lim_{x_{3} \to -0} H_{l \xi^{0}}^{I}(x) = (\Gamma_{St\xi^{0'}})' = \left\{ x' \in \mathbb{R}^{3} : x' \cdot \eta' \geq 0, \eta' \in \Gamma_{St\xi^{0'}} \right\},$$

where

$$\Gamma_{St\xi^{0'}} = \Gamma \left( \eta_{0} - c_{s_{1}}(\xi_{1}^{0} \eta_{1} + \xi_{2}^{0} \eta_{2})/\sqrt{\xi_{1}^{02} + \xi_{2}^{02}}, \vartheta' \right), \quad \vartheta' = (1, 0, 0).$$

(2) For the Riemann function $H_{l}^{II}(x)$ ($l = 1, 2$), we have

$$\lim_{\nu \to \infty} \nu^{-1}e^{-i\nu x' \cdot \xi^{0'}} H_{l}^{II}(x) = H_{l \xi^{0}}^{II}(x)$$

in the distribution sense with respect to $x \in \mathbb{R}_{+}^{4}$.

Moreover we have

$$\bigcup_{\xi^{0'} \neq 0} \text{supp} \lim_{x_{3} \to +0} H_{l \xi^{0}}^{II}(x) \subset \text{sing supp} \lim_{x_{3} \to +0} H_{l}^{II}(x),$$

and

$$\text{supp} \lim_{x_{3} \to +0} H_{l \xi^{0}}^{II}(x) = (\Gamma_{St\xi^{0'}})' = \left\{ x' \in \mathbb{R}^{3} : x' \cdot \eta' \geq 0, \eta' \in \Gamma_{St\xi^{0'}} \right\},$$

where $\Gamma_{\xi^{0'}}$ is the same as (1.18).

**Remark.** If the Lopatinski determinant has no zero, then a singularity corresponding to the Stoneley wave does not appear.
2. Proof of Main Theorem I

In this section, we give the precise proof of Main Theorem I. The procedure of getting explicit expressions of the reflected and refracted Riemann functions $F^I(x, y)$ and $F^{II}(x, y)$ are given in the previous paper.

We prove for the reflected Riemann function $F^I(x, y)$. A similar proof is given for the refracted Riemann function $F^{II}(x, y)$.

The first part of the theorem is derived by the localization method. First we prove the equation (1.5). We consider the case that $j = s_1, k = p_1$, that is, consider $P_1$ reflected wave for $S_1$ incident wave, and consider that the point $\xi^0$ satisfying (3.2) below. We calculate

\[
(2.1) \quad \frac{\nu e^{-i\nu (z' - y') \cdot \xi^0 + z_3 \tau_{s_1}^-(\xi^0) - y_3 \xi_3^0}}{2 \pi^{-4}} \int_{\mathbb{R}^3} e^{-i\nu (z_3' + z_3 + i\eta') \cdot \xi_3} e^{-i\nu (\xi'' + i\eta'')} \cdot \chi(x, y) = 0
\]
Here \( \zeta = \xi + i\eta \). The \( \cdot \) means the same component of the Lopatinski matrices \( \mathcal{R}_1(\zeta') (\zeta' = \xi' + i\eta') \) and \( \mathcal{R}_2(\zeta') \) given below, \( \{P_1^I(\zeta)^{-1}\}_1 \) and \( \{P_1^I(\zeta)^{-1}\}_2 \) are the 1 and 2 columns, respectively, of the inverse matrix of \( P_1^I(\zeta) \) given by

\[
P_1^I(\zeta)^{-1} = (\{P_1^I(\zeta)^{-1}\}_1, \{P_1^I(\zeta)^{-1}\}_2) = \frac{\text{cof}}{\det P_1^I(\zeta)} = \frac{1}{(\zeta_0^2 - c_{p_1}^2|\zeta''|^2)(\zeta_0^2 - c_{s_1}^2|\zeta''|^2)} \\
\times \left( \begin{array}{cc}
-\zeta_0^2 + \{c_{p_1}^2, c_{s_1}^2 + c_{s_1}^2(\zeta_1^2 + \zeta_2^2)\} & - (c_{p_1}^2 - c_{s_1}^2)\sqrt{\zeta_1^2 + \zeta_2^2} \\
-(c_{p_1}^2 - c_{s_1}^2)\sqrt{\zeta_1^2 + \zeta_2^2} & -\zeta_0^2 + \{c_{s_1}^2, c_{s_1}^2 + c_{p_1}^2(\zeta_1^2 + \zeta_2^2)\} 
\end{array} \right),
\]

\( R_1(\zeta') \) and \( R_2(\zeta') \) are the Lopatinski determinants of the systems \( \{P_1^I(\zeta', D_3), P_1^{II}(\zeta', D_3), B_1^I(\zeta', D_3), B_1^{II}(\zeta', D_3)\} \) and \( \{P_2^I(\zeta', D_3), P_2^{II}(\zeta', D_3), B_2^I(\zeta', D_3), B_2^{II}(\zeta', D_3)\} \), respectively, given by

\[
R_1(\zeta') = \det \mathcal{R}_1(\zeta'),
\]

\[
R_2(\zeta') = \det \mathcal{R}_2(\zeta'),
\]

\[
\mathcal{R}_1(\zeta') = \left( \begin{array}{cc}
|\zeta''| & \tau_{s_1}^+(\zeta') \\
-\tau_{p_1}^+(\zeta') & |\zeta''|
\end{array} \right),
\]

\[
\mathcal{R}_2(\zeta') = \left( \begin{array}{cc}
1 & \tau_{s_2}^+(\zeta') \\
\rho_1 c_{s_1}^2 \tau_{s_1}^+(\zeta') & 1
\end{array} \right).
\]

We note that

\[
\tau_{p_1}^+(\zeta') = -\sqrt{\frac{\xi_0^2}{c_{p_1}^2} - (\xi_1^2 + \xi_2^2)} = -\tau_{p_1}^-(\xi')
\]

since \( \tau_{p_1}^+(\zeta') \) is real. Similarly note that \( \tau_{j}^+(\zeta') = -\tau_{j}^-(\xi') (j = \{s_1, p_2, s_2\}) \). Making the change of variable \(-\nu\xi^0 + \xi = \kappa\), then we have

\[
= (2\pi)^{-4} \int_{\mathbb{R}^3} e^{i(x'-y') \cdot (\kappa' + i\eta')} \int_{\mathbb{R}} e^{-i\nu (x^0 + i\eta_0)} U(\nu (\xi'' + \kappa'' + i\eta'')) C \times
\]
$$\left\{ \begin{array}{c} P_1^I(\nu \xi^0 + \kappa + i \eta)^{-1} \quad \cdots \\ B_1^I(\nu \xi^0 + \kappa + i \eta)P_1^I(\nu \xi^0 + \kappa + i \eta)^{-1} \quad \cdots \\ \vdots \end{array} \right\}_{\nu^{-1}R_1(\nu \xi^0 + \kappa + i \eta)} \times e^{-i\zeta_1^+(\nu \xi^0 + \kappa + i \eta)} \times e^{-i\zeta_2^+(\nu \xi^0 + \kappa + i \eta)}$$

and so on

$$U(\nu \xi^0 + \kappa + i \eta)^{-1} d\kappa d\eta.$$
For $3 \times 3$ matrix valued function $\phi(x, y) \in C_0^\infty(\mathbb{R}_-^4 \times \mathbb{R}_+^4)$, we have

\[
\left\langle \nu e^{-i\nu((x' - y') \cdot \xi''') + x_3 r_1^+ (\xi''') - y_3 s_3^2} F^I(x, y), \phi(x, y) \right\rangle_{x, y} = (2\pi)^{-2} \left( U(\xi''' + i\eta''')C \times \right.
\begin{bmatrix}
\{P_1^I(\zeta)^{-1}\}_1 & \cdots & \\
B_1^I(\zeta)\{P_1^I(\zeta)^{-1}\}_1 & \cdots & \\
\nu^{-1}R_1(\xi' + i\eta') & \cdots & \\
\{P_1^I(\zeta)^{-1}\}_1 & \cdots & \\
B_1^I(\zeta)\{P_1^I(\zeta)^{-1}\}_1 & \cdots &
\end{bmatrix}
\left. \frac{\begin{bmatrix}
|\xi''' + i\eta'''| \\
-\tau_1^+ (\xi' + i\eta') \\
\tau_1^+ (\xi' + i\eta') \\
|\xi''' + i\eta'''|
\end{bmatrix} e^{-i\tau_1^+ (\xi' + i\eta') x_3}}{\nu^{-1}R_1(\xi' + i\eta')} \right)
\end{equation}

and so on.

\[
\left( U(\xi''' + i\eta''')C \right)^{-1},
\]

making the change of variable $-\nu\xi^0 + \xi = \kappa$, then we have

(2.13)

\[
(2\pi)^{-2} \left( U(\nu\xi''' + \kappa''' + i\eta''')C \times \right.
\begin{bmatrix}
\{P_1^I(\nu\xi^0 + \kappa + i\eta)^{-1}\}_1 & \cdots & \\
B_1^I(\nu\xi^0 + \kappa + i\eta)\{P_1^I(\nu\xi^0 + \kappa + i\eta)^{-1}\}_1 & \cdots & \\
\nu^{-1}R_1(\nu\xi^0 + \kappa' + i\eta') & \cdots & \\
\{P_1^I(\nu\xi^0 + \kappa + i\eta)^{-1}\}_1 & \cdots & \\
B_1^I(\nu\xi^0 + \kappa + i\eta)\{P_1^I(\nu\xi^0 + \kappa + i\eta)^{-1}\}_1 & \cdots &
\end{bmatrix}
\left. \frac{\begin{bmatrix}
|\nu\xi''' + \kappa''' + i\eta'''| \\
-\tau_1^+ (\nu\xi^0 + \kappa' + i\eta') \\
\tau_1^+ (\nu\xi^0 + \kappa' + i\eta') \\
|\nu\xi''' + \kappa''' + i\eta'''|
\end{bmatrix} e^{-i\tau_1^+ (\nu\xi^0 + \kappa' + i\eta') x_3}}{\nu^{-1}R_1(\nu\xi^0 + \kappa' + i\eta')} \right)
\end{equation}

and so on.

\[
\left( U(\nu\xi''' + \kappa''' + i\eta''')C \right)^{-1},
\]
Here \( < > \) denotes a sum of each component, and

\[
\tilde{\phi}(\zeta', x_3, z) = \mathcal{F}\mathcal{L}_{(x', y)}[\phi(x, y)](\zeta', z),
\]

where \( \mathcal{F}\mathcal{L} \) denotes Fourier-Laplace transformation. If \( \xi^{0'''} \neq 0 \), then for the term including \( e^{-i\mathcal{T}_{p1}^{+}(\nu\xi+i\eta)}x_3 \), we have by using (2.8)

\[
(2.14) \quad \rightarrow (2\pi)^{-2} \left\langle \left\langle \left( U(\xi^{0''''})C \frac{1}{(\det P_{S1})_{\xi^{0}}(\kappa+i\eta)} \left( Q_{1}(\xi^{0}) \left[ \frac{\xi^{0''''}}{R_{1}(\xi^{0})} \right] e^{-i\text{grad}\tau_{p1}^{+}(\xi^{0})} x_3 \right) \right) \phi(-\kappa'-i\eta', x_3, \kappa'+i\eta', \kappa_3 + i\kappa) \right\rangle_{x_3},
\]

where \( Q_{1}(\xi^{0}) \) is defined by (2.19) below. For the term including \( e^{-i\mathcal{T}_{s1}^{+}(\nu\xi+i\eta)}x_3 \), the right-hand side of (2.13) is equal to

\[
(2.15) \quad (2\pi)^{-2} \left\langle \left\langle e^{-i\nu(\tau_{s1}^{+}(\xi^{0'})-\tau_{p1}^{+}(\xi^{0'}))} x_3 f(\kappa + i\eta, \nu), \phi(-\kappa'-i\eta', x_3, \kappa'+i\eta', \kappa_3 + i\kappa) \right\rangle_{x_3},
\]

by using

\[
\tau_{s1}^{+}(\nu\xi^{0'} + \kappa' + i\eta') - \nu \tau_{p1}^{+}(\xi^{0'}) = \nu \{ \tau_{s1}^{+}(\xi^{0'}) - \tau_{p1}^{+}(\xi^{0'}) \} + \text{grad}\tau_{s1}^{+}(\xi^{0'}) \cdot (\kappa' + i\eta') + O(\nu^{-1}).
\]

We put

\[
< >_{x_3} = e^{-i\nu(\tau_{s1}^{+}(\xi^{0'})-\tau_{p1}^{+}(\xi^{0'}))} x_3 g(x_3, \nu).
\]

The \( g(x_3, \nu) \) belongs to \( C_{0}^{\infty} \) with respect to \( x_3 \), and \( \text{supp}_{x_3} g(x_3, \nu) \) is included at a compact set indep of \( \nu \). Moreover

\[
\left| \frac{\partial}{\partial x_3} g(x_3, \nu) \right| \leq 3C \quad \text{indep of } \nu.
\]

So \( < >_{x_3} \) in (2.15) is

\[
(2.16) \quad < >_{x_3} = \int e^{-i\nu(\tau_{s1}^{+}(\xi^{0'})-\tau_{p1}^{+}(\xi^{0'}))} x_3 g(x_3, \nu) dx_3 \to 0 \quad \text{as } \nu \to \infty
\]
because of integral by parts or the Riemann-Lebesgue theorem. By (2.14) and (2.16), we obtain

\begin{equation}
\lim_{\nu \to \infty} \nu e^{-i\nu \cdot \xi} F_s^I(x, y), \phi(x, y) >_{x,y} = < F_s^I, \phi(x, y) >_{x,y} \quad \text{for} \quad \phi(x, y) \in C_0^\infty(\mathbb{R}_-^4 \times \mathbb{R}_-^4).
\end{equation}

Here

\begin{equation}
F_{s_1 \xi_0}^I(x, y) = (2\pi)^{-4} \int_{\mathbb{R}^4} e^{i(x' - y') \cdot \text{grad} \tau_{1}^+(\xi') \cdot (\kappa' + i\eta')} e^{-iy(\kappa + i\eta)} U(\xi') C
\end{equation}

\begin{align*}
&\times \frac{1}{(\det P_{s_1}^I)|\xi|^2} \left( \begin{array}{cc} Q_1(\xi_0^0) & |\xi_0^0| \\ R_1(\xi_0^0) & 0 \end{array} \right) \left( \begin{array}{c} \tau_{1}^+(\xi') \h_1(\xi') \\ 0 \end{array} \right) \left( \begin{array}{c} \tau_{1}^+(\xi') \\ 0 \end{array} \right) \\
&\cdot \left( U(\xi') C \right)^{-1} d\kappa d\kappa',
\end{align*}

where

\begin{align*}
(\det P_{s_1}^I)|\xi|^2 & = 2 \left( \xi_0^2 - c_{p_1}^2 |\xi'|^2 \right) \\
& \times \{\xi_0^0(\kappa_0 + i\eta_0) - c_{s_1}^2(\xi_1^0(\kappa_1 + i\eta_1) + \xi_2^0(\kappa_2 + i\eta_2) + \xi_3^0(\kappa_3 + i\eta_3))\},
\end{align*}

and $Q_1(\xi_0^0)$ and $Q_2(\xi_0^0)$ are given by

\begin{align*}
Q_1(\xi_0^0) & = (\det P_{s_1}^I)|\xi|^2 \left( \begin{array}{c} \{P_1^I(\xi_0^0)^{-1}\} \h_1 \\\nB_1^I(\xi_0^0)\{P_1^I(\xi_0^0)^{-1}\} \end{array} \right) \\
& = \left( \begin{array}{c} -\xi_0^2 + c_{s_1}^2 |\xi'|^2 + c_{p_1}^2 \xi_0^2 \\ -(c_{p_1}^2 - c_{s_1}^2)|\xi'|^2 \xi_0^2 \\ i\rho_1 c_{s_1}^2 \{-(\xi_0^2 - 2c_{s_1}^2)|\xi'|^2 + c_{p_1}^2 \xi_0^2 \} \\
i\rho_1 |\xi'|^2 \{-(c_{p_1}^2 - 2c_{s_1}^2)|\xi'|^2 + c_{p_1}^2 \xi_0^2 \} \} \end{array} \right) \\
Q_2(\xi_0^0) & = (\det P_{s_1}^I)|\xi|^2 \left( \begin{array}{c} \{P_2^I(\xi_0^0)^{-1}\} \h_2 \\
B_2^I(\xi_0^0)\{P_2^I(\xi_0^0)^{-1}\} \end{array} \right) \\
& = \left( \begin{array}{c} -c_{p_1}^2 + c_{s_1}^2 |\xi'|^2 |\xi_0^0| \xi_3^0 \\ -\xi_0^2 + c_{p_1}^2 |\xi'|^2 + c_{s_1}^2 \xi_0^2 \\ i\rho_1 c_{s_1}^2 \{-(\xi_0^2 - 2c_{s_1}^2)|\xi'|^2 + c_{p_1}^2 \xi_0^2 \} \\
i\rho_1 \xi_3^0 \{-(c_{p_1}^2 - 2c_{s_1}^2)|\xi'|^2 + c_{p_1}^2 \xi_0^2 \} \} \end{array} \right).)
\end{align*}
where \( \cdot \) means the same component of the Lopatinski matrix (2.4). If \( \xi^{0'''} = 0 \), then using (2.10) and (2.12), the right-hand side of (2.13) is equal to

\[
(2\pi)^{-2} \left( \frac{1}{(\det P_{p_1}^{I})^{\xi_0}} \right) \left( Q_1(\xi_0) \left( \frac{1}{\nu |\kappa''' + i\eta'''|} \right) \left| -\tau_{p_1}^{+}(\xi_0') \right| \cdot (\kappa' + i\eta') \right)
\]

and so on.

\[
\left( (\kappa''' + i\eta'''^{(3)}) C \right)^{-1}, \left( \phi(-\kappa' + i\eta', x_3, \kappa' + i\eta', \kappa_3 + i\kappa_3) \right) \left. \right|_{x_3}.
\]

Since we could put \( \xi^{0} \) satisfying (3.2) below to \( (1, 0, 0, -\frac{1}{c_0}) \), we obtain (2.17) with

\[
(2.20)
\]

Thus we prove the equation (1.5).

Secondly we prove the equation (1.6). We consider the case that \( j = s_1, k = p_1 \) and \( m = s_2 \) that is, consider that \( S_2 \) lateral wave of \( P_1 \) reflected wave for \( S_1 \) incident wave, and consider that the point \( \xi^{0} \) satisfying (3.7) below.

We calculate

\[
u \frac{3}{2} e^{-i\nu \{ (x - y) \cdot \xi^{0'} + x_{3} \tau_{p_1}^{+}(\xi^{0'}) + y_{3} \xi^{0} \}} F_{s_1, \xi^{0}, p_1}(x, y) = \nu \frac{1}{2} F_{s_1, \xi^{0}, p_1}(x, y)
\]

\[
= \left( 2\pi \right)^{-4} \nu \frac{1}{2} \int_{\mathbb{R}^3} e^{i(x'^{-} - y')} \int_{\mathbb{R}^{}} e^{-iy_{3}(\kappa_3 + i\eta_3)} d\kappa_3 d\kappa'.
\]

and so on.

\[
\left( U(\nu \xi^{0'''} + \kappa''' + i\eta'''^{(3)}) C \right)^{-1} d\kappa_3 d\kappa'.
\]
\[
- (2\pi)^{-4} \nu^{\frac{1}{2}} \int_{\mathbb{R}^3} e^{i(x'-y'-\text{grad} \tau_{p_1}^+(\xi^0') x_3)} (\kappa'+i\eta') \int_{\mathbb{R}} e^{-i\gamma (\kappa_3+i\eta_3)} \ U(\xi^{0''''}) C \\
\times \frac{1}{(\det P_{s_1}^I)_{\xi^0(\kappa+i\eta)}} \begin{pmatrix}
Q_1(\xi^0) & -\tau_{p_1}^+(\xi^0') \\
-\tau_{p_1}^+(\xi^0') & -1
\end{pmatrix} \begin{pmatrix}
Q_2(\xi^0) \\
0
\end{pmatrix} \begin{pmatrix}
|\xi^{0''''}| \\
10
\end{pmatrix} \begin{pmatrix}
0 \\
0
\end{pmatrix} \\
\times \left( U(\xi^{0''''}) C \right)^{-1} d\kappa_3 d\kappa'.
\]

We have for \( \tau_{s_1}, \tau_{p_1}, \tau_{p_2} \):

\[
\nu^{-1} \tau_{s_1}^+(\nu \xi^0' + \kappa' + i\eta') \rightarrow \tau_{l}^+(\xi^0') \quad \text{as} \quad \nu \rightarrow \infty \quad \text{for} \quad l = \{s_1, p_1, p_2\},
\]
and for \( \tau_{s_2} \):

\[
\nu^{\frac{1}{2}} \tau_{s_2}^+(\nu \xi^0' + \kappa' + i\eta') \rightarrow \sqrt{2 \left\{ \frac{\xi_0^{0''}}{c_{s_2}^2} (\kappa_0 + i\eta_0) - \xi_0^{0'''}(\kappa''' + i\eta''') \right\}} \quad \text{as} \quad \nu \rightarrow \infty,
\]

where \( \sqrt{\cdot} \) satisfies \( \text{Im} \sqrt{\cdot} > 0 \). We have

\[
R_1(\nu \xi^0' + \kappa' + i\eta') \\
= \nu^6 \left\{ R_1(\xi^0') + \nu^{-\frac{1}{2}} \sqrt{2 \left\{ \frac{\xi_0^{0''}}{c_{s_2}^2} (\kappa_0 + i\eta_0) - \xi_0^{0'''}(\kappa''' + i\eta''') \right\}} \times \\
\begin{pmatrix}
-\tau_{p_1}^+(\xi^0') \\
-2\rho_1 c_{s_1}^2 (\tau_{s_1}^+(\xi^0') - |\zeta'''|^2) \\
\rho_1 c_{s_1}^2 (\tau_{s_1}^+(\xi^0') - |\zeta'''|^2)
\end{pmatrix} \\
\begin{pmatrix}
|\zeta'''| \\
-\tau_{p_1}^+(\xi^0') \\
0
\end{pmatrix} \\
\begin{pmatrix}
\tau_{s_1}^+(\xi^0') \\
-\rho_1 c_{s_1}^2 (\tau_{s_1}^+(\xi^0') - |\zeta'''|^2) \\
-\rho_1 c_{s_1}^2 (\tau_{s_1}^+(\xi^0') - |\zeta'''|^2)
\end{pmatrix} \\
\begin{pmatrix}
|\zeta'''| \\
|\zeta'''| \\
|\zeta'''|
\end{pmatrix} + O(\nu^{-1}) \right\} \\
= \nu^6 \left\{ R_1(\xi^0') + \nu^{-\frac{1}{2}} \sqrt{2 \left\{ \frac{\xi_0^{0''}}{c_{s_2}^2} (\kappa_0 + i\eta_0) - \xi_0^{0'''}(\kappa''' + i\eta''') \right\}} S(\xi^0') + O(\nu^{-1}) \right\}.
\]

Similarly

\[
\nu \left| \begin{array}{ccc}
\{P_1^I(\nu \xi^0 + \kappa + i\eta)^{-1}\}_1 & \cdots & \nu^6 \\
B_1^I(\nu \xi^0 + \kappa + i\eta) \{P_1^I(\nu \xi^0 + \kappa + i\eta)^{-1}\}_1 & \cdots & \frac{1}{(\det P_{s_1}^I)_{\xi^0(\kappa + i\eta)}} \\
\times \{Q_1(\xi^0) + \nu^{-\frac{1}{2}} \sqrt{2 \left\{ \frac{\xi_0^{0''}}{c_{s_2}^2} (\kappa_0 + i\eta_0) - \xi_0^{0'''}(\kappa''' + i\eta''') \right\}} T_1(\xi^0) + O(\nu^{-1}) \}
\end{array} \right|
\]
\[
\frac{\nu^6}{(\det P_{s_1}^I)^{\xi_0}(\kappa+i\eta)} \times \left\{ Q_2(\xi_0^0) + \nu^{-\frac{1}{2}} \sqrt{2 \left\{ \frac{\xi_0^0}{c_{s_2}^2}(\kappa_0 + i\eta_0) - \xi_0^{0''} \cdot (\kappa'' + i\eta'') \right\}} T_2(\xi_0^0) + O(\nu^{-1}) \right\}.
\]

In a similar manner as the proof of the equation (1.5), we obtain

\[
\left\langle \nu^\frac{3}{2} e^{-i\nu \{x' - y'\} \cdot (\xi_0' - x_3 \tau_{p_1}(\xi_0') + y_3 \xi_0^0} F^I(x, y) - \nu^\frac{1}{2} F^I_{s_1 \xi_0 p_1}(x, y), \phi(x, y) \right\rangle_{x, y} = \langle F_1 \rangle_{x, y}, \phi(x, y) \rangle_{x, y}
\]

where

\[
F^I_{s_1 \xi_0 p_1 s_2}(x, y)
\]

\[
= (2\pi)^{-4} \int_{\mathbb{R}^3} e^{i(x' - y' - \text{grad}_{p_1}(\xi_0') x_3)} \int_{\mathbb{R}} e^{-i y \cdot (\kappa + i\eta)} \frac{1}{(\det P_{s_1}^I)^{\xi_0}(\kappa + i\eta)} \sqrt{2 \left\{ \frac{\xi_0^0}{c_{s_2}^2}(\kappa_0 + i\eta_0) - \xi_0^{0''} \cdot (\kappa'' + i\eta'') \right\}}
\]

Here we remark that

\[
\left( T_1(\xi_0^0) R_1(\xi_0') - Q_1(\xi_0') S(\xi_0'), T_2(\xi_0^0) R_1(\xi_0') - Q_2(\xi_0') S(\xi_0') \right) \neq (0, 0)
\]

since there is at least one non zero point.

If we localize at the point \( \xi_0^0 \) satisfying (3.5) below, that is \( j = s_1, k = s_1, m = p_1 \), then the proof of (1.6) is as follows. For the term including \( e^{-i\tau_{p_1}^+(\nu \xi_0' + \kappa' + i\eta')} \) we prove as same as (2.14). For the term including \( e^{-i\tau_{p_1}^+(\nu \xi_0'' + \kappa' + i\eta')} \), we have

\[
\left\langle \nu^\frac{3}{2} e^{-i\nu \{x' - y'\} \cdot (\xi_0'' + x_3 \tau_{p_1}(\xi_0'') + y_3 \xi_0^0} F^I(x, y) - \nu^\frac{1}{2} F^I_{s_1 \xi_0 p_1}(x, y), \phi(x, y) \right\rangle_{x, y}
\]
\[
\times \tilde{f}(\kappa + i\eta, x_3, \nu) \tilde{\phi}(-\kappa' - i\eta', x_3, \kappa' + i\eta', \kappa_3 + i\eta_3) \, d\kappa dx_3 \\
\equiv I(\nu)
\]
for \( \phi(x, y) \in C_0^\infty(\mathbb{R}_-^4 \times \mathbb{R}_-^4) \), where

\[
\tilde{\phi}(\zeta', X_3, Z) = \mathcal{F}\mathcal{L}_{(x)'y}[\phi(x, y)](\zeta', z),
\]

\[
\overline{f}(\kappa+i\eta, x_3, \mathcal{U}) = e^{\{\text{grad} r_{p_1}^+(\xi^0') \cdot (\kappa'+i\eta') + o(\nu^{-1})\}} x_3 f(\kappa + i\eta, \nu).
\]

If we put
\[
t' L = - \sqrt{2\left\{ \frac{\xi_0^0}{c_{p_1}^2}(\kappa_0 + i\eta_0) - \xi_0''' \cdot (\kappa''' + i\eta''') \right\}} \frac{1}{i} \frac{\partial}{\partial \kappa_0},
\]
then we obtain
\[
I(\nu) = \int e^{i \left\{ \nu r_{x_3}(\xi^0') - \frac{1}{2} \sqrt{2\left\{ \frac{\xi_0^0}{c_{p_1}^2}(\kappa_0 + i\eta_0) - \xi_0''' \cdot (\kappa''' + i\eta''') \right\}} \right\}} x_3
\times \nu^{\frac{1}{2}} L^2 \left[ \tilde{f}(\kappa + i\eta, x_3, \nu) \tilde{\phi}(-\kappa' - i\eta', x_3, \kappa' + i\eta', \kappa_3 + i\eta_3) \right] \, d\kappa dx_3
\rightarrow 0 \quad \text{as} \quad \nu \rightarrow \infty,
\]

since
\[
\left| \left( \frac{\partial}{\partial \kappa_0} \right)^j \left[ \tilde{f}(\kappa + i\eta, x_3, \nu) \tilde{\phi}(-\kappa' - i\eta', x_3, \kappa' + i\eta', \kappa_3 + i\eta_3) \right] \right| \leq C(\eta)(|\kappa| + |\eta|)^{-6} \quad \text{if} \quad \nu \geq 1, \quad j \leq 2,
\]

and

\(-\exists C_1 \leq x_3 \leq -\exists C_2 < 0 \quad \text{on supp } \tilde{\phi}.
\]

Thus we prove the equation (1.6).

The inclusion relation (1.7) is proved in the previous paper, so we omit the proof of (1.7).

Finally we prove the formula (1.8) and (1.9). If \((Q_1(\xi^0), Q_2(\xi^0)) \neq (0, 0)\) in (2.18) or if \(Q_1(\xi^0) \neq 0\) in (2.20), then we could put

\[
F^I_{s_1 \xi^0 p_1}(x, y) = \text{Const.}(2\pi)^{-4} \int_{\mathbb{R}^4} \frac{e^{i(x' \cdot y') - \text{grad} r_{p_1}^+(\xi^0')x_3 \cdot (\kappa'+i\eta') - y_5(\kappa_3+i\eta_3)}}{(\kappa_0 + i\eta_0) - \frac{c_{s_1}^2}{\xi_0^0} \xi^0 \cdot (\kappa'' + i\eta'')} \, d\kappa,
\]

and would like to obtain \(\text{supp} F^I_{s_1 \xi^0 p_1}(x, y)\). If we put

\[
G_1(x) = (2\pi)^{-4} \int_{\mathbb{R}^4} \frac{e^{ix \cdot (\kappa + i\eta)}}{(\kappa_0 + i\eta_0) - \frac{c_{s_1}^2}{\xi_0} \xi^0 \cdot (\kappa'' + i\eta'')} \, d\kappa,
\]
then
\[ F^I_{s_1, \xi_0 p_1}(x, y) = G_1(x' - y' - \text{grad} r^+_{p_1}(\xi_0') x_3, -y_3). \]

So it is sufficient that we consider \( \text{supp} G_1 \). From the Paley-Wiener-Schwartz theorem,
\[ ch[\text{supp} G_1] = \{ x \in \mathbb{R}^4 | x \cdot \eta \geq 0 \text{ for } \forall \eta \in \Gamma_{s_1, \xi_0} \}, \]
where \( ch \) denotes a convex hull, and
\[
(2.22) \quad \Gamma_{s_1, \xi_0} = \left\{ \eta \in \mathbb{R}^4_\eta | \eta_0 - \frac{c^2}{\xi_0} \xi_0' \cdot \eta'' > 0 \right\}.
\]

By (2.22), we have
\[ ch[\text{supp} G_1] = \left\{ x \in \mathbb{R}^4 | x = \lambda \left( 1, -\frac{c^2}{\xi_0} \xi_0' \right), \ \lambda \geq 0 \right\} \]
and it is half-line. So we obtain
\[ \text{supp} G_1 = \text{ch}[\text{supp} G_1] = \left\{ x \in \mathbb{R}^4 | x = \lambda \left( 1, -\frac{c^2}{\xi_0} \xi_0' \right), \ \lambda \geq 0 \right\}, \]
since \( G_1 \) is a homogeneous distribution. Thus we prove the formula (1.8). Next we prove the formula (1.9). If
\[ \left( T_1(\xi_0) R_1(\xi_0') - Q_1(\xi_0) S(\xi_0'), \ T_2(\xi_0) R_1(\xi_0') - Q_2(\xi_0) S(\xi_0') \right) \neq (0, 0) \]
in (2.21), then we could put
\[
F^I_{s_1, \xi_0 p_1 s_2}(x, y) = \text{Const.} (2\pi)^{-4} \int_{\mathbb{R}^4} \frac{e^{i(x' - y' - \text{grad} r^+_{p_1}(\xi_0') x_3, \eta_0 - \xi_0' \cdot \eta'' - y_3)} \cdot \xi'''}{(\kappa_0 + i\eta_0) - \frac{c^2}{\xi_0} \xi_0' \cdot (\kappa'' + i\eta'')} \times \sqrt{\frac{\xi_0'}{c^2 s_2}} (\kappa_0 + i\eta_0) - \xi_0'' \cdot (\kappa'' + i\eta'') \ d\kappa,
\]
and would like to obtain \( \text{supp} F^I_{s_1, \xi_0 p_1 s_2}(x, y) \). If we put
\[ G_2(x) = (2\pi)^{-4} \int_{\mathbb{R}^4} \frac{e^{ix \cdot (\kappa + i\eta)}}{(\kappa_0 + i\eta_0) - \frac{c^2}{\xi_0} \xi_0' \cdot (\kappa'' + i\eta'')} \times \sqrt{\frac{\xi_0'}{c^2 s_2}} (\kappa_0 + i\eta_0) - \xi_0'' \cdot (\kappa'' + i\eta'') \ d\kappa,
\]
then
\[ F^I_{s_1, \xi_0 p_1 s_2}(x, y) = G_2(x' - y' - \text{grad} r^+_{p_1}(\xi_0') x_3, -y_3). \]
So it is sufficient that we consider $\text{supp}G_2$. From the Paley-Wiener-Schwartz theorem,

$$\text{ch}[\text{supp}G_2] = \{ x \in \mathbb{R}^4 | x \cdot \eta \geq 0 \quad \forall \eta \in \Gamma_{s_1 \xi^0_{s_2}} \},$$

where $\text{ch}$ denotes a convex hull, and

$$\Gamma_{s_1 \xi^0_{s_2}} = \left\{ \eta \in \mathbb{R}^4 \bigg| \eta_0 - \frac{c^2_{s_1}}{\xi^0_{s_2}} \xi^0_0 \cdot \eta'' > 0, \quad \eta_0 - \frac{c^2_{s_2}}{\xi^0_{s_2}} \xi^0_0 \cdot \eta'' > 0 \right\}.$$  \hspace{1cm} (2.23)

By (2.19), we have

$$\text{ch}[\text{supp}G_2] = \left\{ x \in \mathbb{R}^4 \bigg| x = k_1 \left( 1, -\frac{c^2_{s_1}}{\xi^0_{s_2}} \xi^0_0 \right) + k_2 \left( 1, -\frac{c^2_{s_2}}{\xi^0_{s_2}} \xi^0_0, 0 \right), \quad k_1, k_2 \geq 0 \right\}.$$  \hspace{1cm} (2.24)

We would like to verify $\text{ch}[\text{supp}G_2] = \text{supp}G_2$. We take the change of coordinates such as

$$p = A \kappa, \quad A = \begin{pmatrix} 1 & -\frac{c^2_{s_1}}{\xi^0_{s_2}} \xi^0_1 & 0 & 0 \\ -\frac{c^2_{s_2}}{\xi^0_{s_2}} \xi^0_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where we note that $A$ is holomorphic matrix by the assumption (1.2). Then $G_2(x)$ is given by

$$G_2(x) = (2\pi)^{-4} \frac{1}{|\det A|} \int_{\mathbb{R}^4} e^{iA^{-1}x \cdot (p-i\vartheta)} \frac{1}{c_{s_2} \xi^0_{s_0}} \frac{\sqrt{\xi^0_{s_0}(p_0-i)}}{p_1-i} dp,$$

where $\vartheta = (1, 0, 0, 0)$ and $\sqrt{\xi^0_{s_0}(p_0-i)}$ is taken the branch such that $\text{Im} \sqrt{\xi^0_{s_0}(p_0-i)} > 0$. By $A \vartheta = (1, 1, 0, 0)$ and the Cauchy integral theorem, we obtain

$$G_2(x) = (2\pi)^{-4} \frac{1}{|\det A|} \int_{\mathbb{R}^4} e^{iA^{-1}x \cdot (p-i(1,1,0,0))} \frac{1}{c_{s_2} \xi^0_{s_0}} \frac{\sqrt{\xi^0_{s_0}(p_0-i)}}{p_1-i} dp$$

$$= (2\pi)^{-4} \frac{1}{|\det A|} \int_{\mathbb{R}^4} e^{iA^{-1}x \cdot p} \frac{1}{c_{s_2} \xi^0_{s_0}} \frac{\sqrt{\xi^0_{s_0}(p_0-i0)}}{p_1-i0} dp.$$  \hspace{1cm} (2.24)

By the Fourier transform formula (cf. [Hö, Example 7.1.17]), we deduce

$$\mathcal{F}^{-1} \left[ \frac{1}{\xi - i0} \right] (x) = iH(x),$$

$$\mathcal{F}^{-1} \left[ (\xi - i0)^{\frac{1}{2}} \right] (x) = -\frac{e^{-\frac{1}{2}\pi}}{2\sqrt{\pi}} x_+^{-\frac{3}{2}}.$$  \hspace{1cm} (2.25)\hspace{1cm} (2.26)

where $H(x)$ denotes the Heaviside function and

$$x_+^a = \begin{cases} x^a & \text{for } x > 0, \\
0 & \text{for } x \leq 0 \end{cases}$$
for $a \in \mathbb{C}$. By (2.25) and (2.26), the right-hand side of (2.24) is equal to

$$
\left\{ \begin{array}{ll}
\frac{-1}{|\det A|c_2\sqrt{\xi_0^0}} \frac{e^{-\frac{1}{2}\pi}}{2\sqrt{\pi}} (z_0)_+ \otimes iH(z_1) \otimes \delta(z_2,z_3) & \text{for } \xi_0^0 > 0, \\
\frac{-1}{|\det A|c_2\sqrt{\xi_0^0}} \frac{e^{-\frac{1}{2}\pi}}{2\sqrt{\pi}} (z_0)_+ \otimes iH(z_1) \otimes \delta(z_2,z_3) & \text{for } \xi_0^0 < 0,
\end{array} \right.
$$

where $z = tA^{-1}x$. Thus we get

$$
\text{supp } G_2 = \{ x \in \mathbb{R}^4 | z_0 \geq 0, z_1 \geq 0, z_2 = z_3 = 0 \}
$$

$$
= \left\{ x \in \mathbb{R}^4 \mid x = tA \begin{pmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{pmatrix}, \quad k_1, k_2 \geq 0 \right\}
$$

$$
= \left\{ x \in \mathbb{R}^4 \mid x = k_1 \begin{pmatrix} 1, -\frac{c_{s_1}^2}{\xi_0^0} \xi_0^{0''} \\ \xi_0^0 \end{pmatrix} + k_2 \begin{pmatrix} 1, -\frac{c_{s_2}^2}{\xi_0^0} \xi_0^{0''}, 0 \end{pmatrix}, \quad k_1, k_2 \geq 0 \right\}
$$

$$
= ch[\text{supp } G_2],
$$

thereby we prove the formula (1.9).

This completes the proof of Main Theorem I.

3. The Location of Singularities

By using Main Theorem I, we find an inner estimate of the location of singularities of reflected and refracted Riemann functions $F^I(x, y)$ and $F^{II}(x, y)$.

In the expressions $F^I(x, y)$ and $F^{II}(x, y)$, the parts put between $U(\xi'' + i\eta'')C$ and $(U(\xi'' + i\eta'')C)^{-1}$ are decomposed into $2 \times 2$ and $1 \times 1$ matrices valued Riemann functions $F_{2 \times 2}^I(x, y)$ and $F_{1 \times 1}^I(x, y)$ for $F^I(x, y)$, and $F_{2 \times 2}^{II}(x, y)$ and $F_{1 \times 1}^{II}(x, y)$ for $F^{II}(x, y)$. The displacement vector of $F_{2 \times 2}^I(x, y)$ ($i = \{I, II\}$) lies in $(x_1 - y_1)(x_3 - y_3)$-plane and that of $F_{1 \times 1}^I(x, y)$ ($i = \{I, II\}$) lies in $(x_2 - y_2)$-axis. Thus we can treat $F_{2 \times 2}^I(x, y)$ and $F_{1 \times 1}^I(x, y)$ ($i = \{I, II\}$) independently.

For $F_{2 \times 2}^I(x, y)$, we have the following 4 sets of $\xi^0$ that are roots of $\det P^I_1(\xi^0) = 0$ and are not zeros of $\tau^+ m(\xi^0)$ ($m \in \{p_1, p_2, s_2\}$): roots of $\det P^I_{s_1}(\xi^0) = \xi_0^{0^2} - c_{s_1}^2 |\xi_0^{0''}|^2 = 0$ are

(3.1) $\xi^0 = (1, \xi_1^0, \xi_2^0, \tau^+ s_1(\xi^0'))$ for $\tau^+ s_1(\xi^0')$ in (1.5) with $|\xi^0'''| < \frac{1}{c_{s_1}}$,

(3.2) $\xi^0 = (1, \xi_1^0, \xi_2^0, \tau^+ s_1(\xi^0'))$ for $\tau^+ s_1(\xi^0')$ in (1.5) with $|\xi^0'''| < \frac{1}{c_{p_1}}$,

and roots of $\det P^I_{p_1}(\xi^0) = \xi_0^{0^2} - c_{p_1}^2 |\xi_0^{0''}|^2 = 0$ are

(3.3) $\xi^0 = (1, \xi_1^0, \xi_2^0, \tau^+ p_1(\xi^0'))$ for $\tau^+ p_1(\xi^0')$ in (1.5) with $|\xi^0'''| < \frac{1}{c_{p_1}}$. 

\( \xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^0')) \) for \( \tau_{p_1}^-(\xi^0') \) in (1.5) with \( |\xi^0'| < \frac{1}{c_{p_1}} \).

(3.1) and (3.2) (resp. (3.3) and (3.4)) correspond to \( P_1 \) and \( S_1 \) reflected waves for \( S_1 \) (resp. \( P_1 \)) incident wave, respectively.

We have the following 9 sets of \( \xi^0 \) that are roots of \( \det P_I^1(\xi^0) = 0 \) and are not zeros of \( \tau_m^+(\xi^0) \) (\( m \in \{p_1, s_2, p_2\} \)); (3.5)-(3.9) are roots of \( \det P_{s_1}^I(\xi^0) = 0 \) and

(3.5) \( \xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^0')) \) for \( \tau_{s_1}^-(\xi^0') \) in (1.6) with \( |\xi^0'| = \frac{1}{c_{p_1}} \)

are zeros of \( \tau_{s_1}^+(\xi^0') \),

(3.6) \( \xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_2}^+(\xi^0')) \) for \( \tau_{s_2}^-(\xi^0') \) in (1.6) with \( |\xi^0'| = \frac{1}{c_{s_2}} \),

(3.7) \( \xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^0')) \) for \( \tau_{s_1}^-(\xi^0') \) in (1.6) with \( |\xi^0'| = \frac{1}{c_{s_2}} \)

are zeros of \( \tau_{s_2}^+(\xi^0') \),

(3.8) \( \xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^0')) \) for \( \tau_{p_1}^-(\xi^0') \) in (1.6) with \( |\xi^0'| = \frac{1}{c_{s_2}} \),

(3.9) \( \xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_2}^+(\xi^0')) \) for \( \tau_{p_2}^-(\xi^0') \) in (1.6) with \( |\xi^0'| = \frac{1}{c_{s_2}} \)

are zeros of \( \tau_{s_2}^+(\xi^0') \),

(3.10) \( \xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^0')) \) for \( \tau_{p_1}^-(\xi^0') \) in (1.6) with \( |\xi^0'| = \frac{1}{c_{s_2}} \),

(3.11) \( \xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^0')) \) for \( \tau_{p_1}^-(\xi^0') \) in (1.6) with \( |\xi^0'| = \frac{1}{c_{s_2}} \)

are zeros of \( \tau_{s_2}^+(\xi^0') \),

(3.12) \( \xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^0')) \) for \( \tau_{p_1}^-(\xi^0') \) in (1.6) with \( |\xi^0'| = \frac{1}{c_{p_2}} \),

(3.13) \( \xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{p_1}^+(\xi^0')) \) for \( \tau_{p_1}^-(\xi^0') \) in (1.6) with \( |\xi^0'| = \frac{1}{c_{p_2}} \)

are zeros of \( \tau_{p_2}^+(\xi^0') \).

(3.5) corresponds to \( S_1 \) lateral or glancing wave for \( S_1 \) incident wave with \( P_1 \) influence. (3.6) and (3.7) (resp. (3.8) and (3.9)) correspond to \( P_1 \)
and $S_1$ lateral waves for $S_1$ incident wave with $S_2$ (resp. $P_2$) influence, respectively. (3.10) and (3.11) (resp. (3.12) and (3.13)) correspond to $P_1$ and $S_1$ lateral waves for $P_1$ incident wave with $S_2$ (resp. $P_2$) influence, respectively.

For $F_{2\times 2}^{I}(x, y)$, we have the following the 4 sets of $\xi^0$ that are roots of $\det P_1^I(\xi^0) = 0$ and are not zeros of $\tau_{p_2}^+(\xi^0)$; roots of $\det P_1^I(\xi^0) = 0$ are

\begin{equation}
\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^0')) \text{ for } \tau_{s_2}^+(\xi^0') \text{ in (1.10) with } |\xi^0''| < \frac{1}{c_{s_2}},
\end{equation}

(3.14)

and roots of $\det P_1^I(\xi^0) = 0$ are

\begin{equation}
\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^0')) \text{ for } \tau_{s_2}^+(\xi^0') \text{ in (1.10) with } |\xi^0''| < \frac{1}{c_{s_2}},
\end{equation}

(3.15)

\begin{equation}
\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^0')) \text{ for } \tau_{p_2}^+(\xi^0') \text{ in (1.10) with } |\xi^0''| < \frac{1}{c_{p_2}}.
\end{equation}

(3.16)

(3.14) and (3.15) (resp. (3.16) and (3.17)) correspond to $S_2$ and $P_2$ reflected waves for $S_1$ (resp. $P_1$) incident wave, respectively.

We have the following 2 sets of $\xi^0$ that are roots of $\det P_1^I(\xi^0) = 0$ and are not zeros of $\tau_{p_2}^+(\xi^0)$;

\begin{equation}
\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^0')) \text{ for } \tau_{s_2}^+(\xi^0') \text{ in (1.11) with } |\xi^0''| = \frac{1}{c_{p_2}}.
\end{equation}

(3.18)

are roots of $\det P_1^I(\xi^0) = 0$ and zeros of $\tau_{p_2}^+(\xi^0)$,

\begin{equation}
\xi^0 = (1, \xi_1^0, \xi_2^0, \tau_{s_1}^+(\xi^0')) \text{ for } \tau_{s_2}^+(\xi^0') \text{ in (1.11) with } |\xi^0''| = \frac{1}{c_{p_2}}.
\end{equation}

(3.19)

are roots of $\det P_1^I(\xi^0) = 0$ and zeros of $\tau_{p_2}^+(\xi^0)$.

Remark. It is sufficient to consider only the case $\xi^0 = 1$ since $(\Gamma_{j,\xi^0})_k^I = (\Gamma_{j,\xi^0})_k^I$,

\begin{equation}
(\Gamma_{j,\xi^0})_k^I = (\Gamma_{j,\xi^0})_k^I (j = \{p_1, s_1\}, k = \{p_1, p_2, s_2\}), \text{ and } (\Gamma_{j,\xi^0})_k^I
\end{equation}

(3.10)

for $t \in \mathbb{R} \setminus \{0\}$.

The figures of inner estimate of the location of singularities of the reflected and refracted Riemann functions $F_{2\times 2}^{I}(x, y)$ and $F_{2\times 2}^{II}(x, y)$ are given as in Figure 2 and Figure 3 in the previous paper. However they are wrong. Here we show the correct figure of the inner estimate of the location of singularities of $F_{2\times 2}^{I}(x, y)$ and $F_{2\times 2}^{II}(x, y)$ with pass of time as in Figure 2.
\begin{align*}
x_0 &< \frac{-y_3}{c_{p_1}} \\
-\frac{y_3}{c_{p_1}} &\leq x_0 < \frac{c_{p_2}(-y_3)}{c_{p_1}\sqrt{c_{p_2}^2 - c_{p_1}^2}}
\end{align*}
Figure 2  Inner estimate of the location of singularities of $F_{2 \times 2}^I(x, y)$ and $F_{2 \times 2}^{II}(x, y)$
4. The Stoneley Wave

In this section, we consider the elastic wave propagation if unit impulse Dirac's delta is put on the interface $x_3 = 0$.

The Riemann functions of the elastic mixed problem (1.3) is given by

\[
H_l(x) = \begin{cases} 
H_l^I(x) & \text{for } x_3 < 0, \\
H_l^{II}(x) & \text{for } x_3 > 0, \quad l = 1, 2,
\end{cases}
\]

that are obtained to solve the following interface problems:

\[
\begin{align*}
&\text{(4.2)} \\
&\begin{cases}
P^I(D_x)H_1^I(x) = 0, \quad x \in \mathbb{R}^4, \\
P^{II}(D_x)H_1^{II}(x) = 0, \quad x \in \mathbb{R}^4_+,
\end{cases} \\
&\begin{cases}
H_1^I(x)|_{x_3=-0} - H_1^{II}(x)|_{x_3=+0} = \delta(x',0)E, \quad x' \in \mathbb{R}^3, \\
B^I(D_x)H_1^I(x)|_{x_3=-0} = B^{II}(D_x)H_1^{II}(x)|_{x_3=+0},
\end{cases}
\]

\[
\begin{align*}
&\text{(4.3)} \\
&\begin{cases}
P^I(D_x)H_2^I(x) = 0, \quad x \in \mathbb{R}^4, \\
P^{II}(D_x)H_2^{II}(x) = 0, \quad x \in \mathbb{R}^4_+,
\end{cases} \\
&\begin{cases}
H_2^I(x)|_{x_3=-0} = H_2^{II}(x)|_{x_3=+0}, \\
B^I(D_x)H_1^I(x)|_{x_3=-0} - B^{II}(D_x)H_1^{II}(x)|_{x_3=+0} = \delta(x',0)E, \quad x' \in \mathbb{R}^3,
\end{cases}
\]

where $E$ is a $3 \times 3$ identity matrix.

As the same procedure to get the explicit expression of the reflected and refracted Riemann functions $F^I(x, y)$ and $F^{II}(x, y)$, we obtain the expression of Riemann functions $H_1^I(x)$ and $H_1^{II}(x)$. $H_2^I(x)$ and $H_2^{II}(x)$ are given by similar expression.

\[
\begin{align*}
&\text{(4.4)} \\
&H_1^I(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix' \cdot (\xi' + i\eta')} U(\zeta''' + i\eta''')C \\
&\times \begin{pmatrix}
1 & \cdots & 1 \\
0 & \cdots & 0 \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
|\xi''' + i\eta'''| & -i\tau_1^+(\xi' + i\eta') \\
-i\tau_1^+(\xi' + i\eta') & \zeta''' + i\eta'''
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\times \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
|\xi''' + i\eta'''| & -i\tau_1^+(\xi' + i\eta') \\
-i\tau_1^+(\xi' + i\eta') & \zeta''' + i\eta'''
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\times \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
|\xi''' + i\eta'''| & -i\tau_1^+(\xi' + i\eta') \\
-i\tau_1^+(\xi' + i\eta') & \zeta''' + i\eta'''
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\end{align*}
\]
\[
\begin{bmatrix}
0 & \cdots & 0 \\
1 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix}
\begin{pmatrix}
|\xi'' + i\eta'''| \\
-\tau_{p_{1}}^{+}(\xi' + i\eta') \\
R_{1}(\xi' + i\eta')
\end{pmatrix}
\begin{pmatrix}
e^{-i\tau_{p_{1}}^{+}(\xi' + i\eta')x_{3}} \\
\end{pmatrix}
\]

\[
+ \begin{bmatrix}
0 \\
1 \\
0 \\
\end{bmatrix}
\begin{pmatrix}
\tau_{s_{1}}^{+}(\xi' + i\eta') \\
\xi'' + i\eta''' \\
R_{1}(\xi' + i\eta')
\end{pmatrix}
\begin{pmatrix}
e^{-i\tau_{s_{1}}^{+}(\xi' + i\eta')x_{3}} \\
\end{pmatrix}
\]

\[
0
\]

\[
\begin{bmatrix}
1 \\
0 \\
\end{bmatrix}
\begin{pmatrix}
\tau_{s_{1}}^{+}(\xi' + i\eta') \\
\xi'' + i\eta''' \\
R_{1}(\xi' + i\eta')
\end{pmatrix}
\begin{pmatrix}
e^{-i\tau_{s_{1}}^{+}(\xi' + i\eta')x_{3}} \\
\end{pmatrix}
\]

\[
= (2\pi)^{-3} \int_{\mathbb{R}^{3}} e^{ix' \cdot (\xi' + i\eta')} U(\xi'' + i\eta''' \mathcal{C})
\]

(4.5) \[
H_{1}^{II}(x) = (2\pi)^{-3} \int_{\mathbb{R}^{3}} e^{ix' \cdot (\xi' + i\eta')} U(\xi'' + i\eta''' \mathcal{C})
\]
Here the Lopatinski determinants $R_1(\zeta') (\zeta' = \xi' + i\eta')$ and $R_2(\zeta')$ are defined $(2.3)$-$(2.4)$ and $(2.5)$-$(2.6)$, respectively. $\cdot$ means the same component of the Lopatinski matrices $\mathcal{R}_1(\zeta')$ and $\mathcal{R}_2(\zeta')$ defined by $(2.4)$ and $(2.6)$.

We obtained the following conclusion for the Lopatinski determinant $R_1(\xi')$ (cf.[Sh, Section 3]). Here $c_{St}$ is the speed of the Stoneley wave corresponding to the interface wave and is less than or equal to most minimum speed $c_{s_1}$. The zero does not necessarily exist. For the discriminant

\[
\text{Dis}(c_{s_1}^2) = \left( \mu_1 - 2\mu_2 + \frac{c_{s_1}^2}{c_{s_2}^2} \mu_2 \right)^2 + \sqrt{\left( \frac{c_{s_1}^2}{c_{p_2}^2} - 1 \right) \left( -1 \right)} \left( \mu_1 \right) + \frac{c_{s_1}^2}{c_{s_2}^2} - 1 \mu_1 \mu_2 \frac{c_{s_1}^2}{c_{s_2}^2},
\]

we get

(i) \quad \text{Dis}(c_{s_1}^2) > 0 \quad \implies \quad \text{The zero } \xi_0 = c_{St}|\xi''| \text{ of } R_1(\xi') \text{ exists in } [0, c_{s_1}|\xi''|) \text{ with order 1.}

(ii) \quad \text{Dis}(c_{s_1}^2) = 0 \quad \implies \quad c_{St} = c_{s_1} \text{ and we shall consider this case under some restricted conditions (cf. [Sh, Lemma 6.4]).}

(iii) \quad \text{Dis}(c_{s_1}^2) < 0 \quad \implies \quad R_1(\xi') \text{ has no zero.}

If \text{Dis}(c_{s_1}^2) > 0$, then we find an inner estimate of the location of singularities of the Riemann functions $H^I(x)$ and $H^{II}(x)$ by using Main Theorem II. And if \text{Dis}(c_{s_1}^2) < 0$, then a singularity corresponding to the Stoneley wave does not appear. For the Lopatinski determinant $R_2(\xi')$, it has no zeros.

The Lopatinski determinant $R_1(\xi')$ has the expression

\[
R_1(\xi') = \left\{ \frac{1}{c_{p_1} c_{s_1}} \mu_2^2 + \frac{1}{c_{p_2} c_{s_2}} \mu_1^2 \right\} \left( \frac{1}{c_{s_1} c_{s_2}} + \frac{1}{c_{s_2} c_{s_1}} \mu_1 \mu_2 \right) \left( \xi_0 - c_{St}|\xi''| \right) \left( \xi_0^{5} + \text{lower term of } \xi_0 \right).
\]

The first factor of the right-hand side of (4.6) is constant, and the third factor of that does not have the factor of zeros of $R_1(\xi')$. A point causes a singularity corresponding to the Stoneley wave is

\[
\xi^{0'} = (1, \xi_1^0, \xi_2^0), \quad |\xi''^0| = \frac{1}{c_{St}}.
\]
We note that there are not any real $\xi'$ that are zeros of $\xi_0 - c_{ST}|\xi'''|$ and some of $\tau^+_{s1}(\xi')$, $\tau^+_{p1}(\xi')$, $\tau^+_{s2}(\xi')$, and $\tau^+_{p2}(\xi')$ are real because of $0 < c_{ST} \leq c_s$ and the assumption (1.2).

We consider that

$$\nu^{-1} e^{-i\nu x \cdot \xi'''} H_1^I(x)$$

$$= (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix' \cdot (-i\nu \xi'' - \xi' + i\eta')} U(\xi''' + i\eta''') C \times \nu^{-1}$$

$$\times \left( \frac{1}{R_1(\xi' + i\eta')} \begin{vmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \end{vmatrix} \begin{vmatrix} |\xi''' + i\eta'''| \\ -\tau^+_{p1}(\xi' + i\eta') \\ \end{vmatrix} e^{-ir^+_{p1}(\xi' + i\eta')x_3} \right) \times (U(\xi''' + i\eta''') C)^{-1} d\xi'.$$

Making the change of variable $\nu \xi_0' + \xi' = \kappa'$, then we have

$$= (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix' \cdot (\kappa' + i\eta')} U(\nu \xi''' + \kappa''' + i\eta''') C$$

$$\times \nu^{-1} \left( \frac{1}{R_1(\nu \xi_0' + \kappa' + i\eta')} \begin{vmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \end{vmatrix} \begin{vmatrix} |\nu \xi''' + \kappa''' + i\eta'''| \\ -\tau^+_{p1}(\nu \xi_0' + \kappa' + i\eta') \\ \end{vmatrix} e^{-ir^+_{p1}(\nu \xi_0' + \kappa' + i\eta')x_3} \right) \times (U(\nu \xi''' + \kappa''' + i\eta''') C)^{-1} d\kappa'.$$

By (2.8), we have for the point (4.7)

$$\nu^{-1} \tau^+_k(\nu \xi_0' + \kappa' + i\eta') = \tau^+_k(\xi_0') + O(\nu^{-1}), \quad k = \{p_1, s_1\},$$

where

$$\tau^+_p(\xi_0') = i \sqrt{c^2 - c_{ST}^2}, \quad \tau^+_s(\xi_0') = i \sqrt{c^2 - c_{ST}^2}.$$
\[
= \text{Const.}(\xi^{0'})\left( (\kappa_{0} + i\eta_{0}) - c_{S\ell}(\kappa_{1} + i\eta_{1}) + (\kappa_{2} + i\eta_{2}) \right) + O(\nu^{-1})
\]
\[
= \mathcal{R}_{1}(\kappa' + i\eta') + O(\nu^{-1}).
\]

Moreover if we put
\[
\begin{vmatrix}
1 & \tau_{s_{1}}^{+}(\xi^{0'}) \\
0 & \left| \xi^{0''} \right|
\end{vmatrix}
\]
\[
\begin{vmatrix}
0 & -\rho_{1}c_{s_{1}}^{2}(\tau_{s_{1}}^{+}(\xi^{0'})^{2} - \left| \xi^{0''} \right|^{2}) \\
0 & -2\rho_{1}c_{s_{1}}^{2}\tau_{s_{1}}^{+}(\xi^{0'})\left| \xi^{0''} \right|
\end{vmatrix}
\]
\[
\begin{vmatrix}
0 & -\tau_{s_{2}}^{+}(\xi^{0'}) \\
0 & \left| \xi^{0''} \right|
\end{vmatrix}
\]
\[
\begin{vmatrix}
2\rho_{2}c_{s_{2}}^{2}\tau_{s_{2}}^{+}(\xi^{0'})\left| \xi^{0''} \right| & -\rho_{2}c_{s_{2}}^{2}(\tau_{s_{2}}^{+}(\xi^{0'})^{2} - \left| \xi^{0''} \right|^{2}) \\
\rho_{2}c_{s_{2}}^{2}(\tau_{s_{2}}^{+}(\xi^{0'})^{2} - \left| \xi^{0''} \right|^{2}) & 2\rho_{2}c_{s_{2}}^{2}\tau_{s_{2}}^{+}(\xi^{0'})\left| \xi^{0''} \right|
\end{vmatrix}
\]

is equal to \(V_{1}(\xi^{0'})\) (resp.
\[
\begin{vmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{vmatrix}
\]

\(= V_{2}(\xi^{0'}), \quad V_{3}(\xi^{0'}), \quad V_{4}(\xi^{0'})\),

then
\[
(4.11) \quad \nu^{-5}V_{n}(\nu\xi^{0'} + \kappa' + i\eta') = V_{n}(\xi^{0'}) + O(\nu^{-1}), \quad n = 1, 2, 3, 4.
\]

Thus we get from (2.9), (2.10), (4.8), (4.10), and (4.11),
\[
\lim_{\nu \to \infty} \nu^{-1} e^{-i\nu \xi' \cdot \xi^{0'}} H_{1}(x) = (2\pi)^{-3} \int_{\mathbb{R}^{3}} e^{ix' \cdot (\xi^{0'} + \kappa' + i\eta')} U(\xi^{0''}) C_{x} \times
\]
\[
\begin{vmatrix}
\frac{V_{1}(\xi^{0'})}{R_{1}(\xi^{0'} + \kappa' + i\eta')} & \frac{\sqrt{c_{1}^{2} - c_{St}^{2}}}{c_{s_{1}} e_{s_{1}} e_{St}} & \frac{V_{3}(\xi^{0'})}{R_{3}(\xi^{0'} + \kappa' + i\eta')} & \frac{\sqrt{c_{3}^{2} - c_{St}^{2}}}{c_{s_{3}} e_{s_{3}} e_{St}} \\
0 & e_{s_{1}} e_{St} & 0 & 0 \\
0 & e_{s_{3}} e_{St} & 0 & 0 \\
0 & 0 & 0 & 0
\end{vmatrix}
\times (U(\xi^{0''}) C_{x})^{-1} d\kappa' = H_{1}(x).
\]

Therefore we can prove Main Theorem II in the same way as Main Theorem I.

Finally, we calculate the singularity of \(\lim_{x_{3} \to 0} H_{1}(x)\) caused by the point (4.7) as the Stoneley wave.
\[
\Gamma_{St^{0'}} = \{ \eta \in \mathbb{R}^{3} | \eta_{0} - c_{St}^{2}(\xi^{0}_{1}\eta_{1} + \xi^{0}_{2}\eta_{2}) > 0 \} \times \mathbb{R}_{\eta},
\]
and

\[
\supp \lim_{x_3 \to -0} H_{1}^{I}(x) = (\Gamma_{St\xi^{0}})' = \{x' \in \mathbb{R}^{3} : x' \cdot \eta' \geq 0, \ \eta' \in \Gamma_{St\xi^{0}}'\} = \{x' \in \mathbb{R}^{3} : x' = u(1, -c_{St}^{2}t_{1}^{0}, -c_{St}^{2}t_{1}^{0}), \ u \geq 0\}.
\]

Thus an inner estimate of the location of singularities of \(\lim_{x_3 \to -0} H_{1}^{I}(x)\), corresponding to the Stoneley wave on the interface \(x_3 = 0\) is given by

\[
(4.12) \bigcup_{|\xi'''| = \frac{1}{c_{St}}} (\Gamma_{St\xi^{0}})' = \{x' \in \mathbb{R}^{3} : x_{1}^{2} + x_{2}^{2} = c_{St}^{2}t_{0}^{2}, \ x_{0} \geq 0\}.
\]

Thus an inner estimate of the location of singularities of \(\lim_{x_3 \to +0} H_{1}^{II}(x)\) is also given by (4.12) in a similar manner.

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**REFERENCES**


