SP-property for a pair of C*-algebras

Abstract
Recall that C*-algebra $A$ has the SP-property if every non-zero hereditary C*-subalgebra of $A$ has a non-zero projection. Let $1 \in A \subset B$ be a pair of C*-algebras.

In this paper we investigate a sufficient condition for $B$ to have the SP-property under $A$ holds. As an application, we will present the cancellation property for crossed products of simple C*-algebras by discrete groups.

This paper basically comes from joint works with Ja A Jeong ([7][8]).

1 The SP-Property

In this section we present a sufficient condition for $B$ to have the SP-property under $A$ holds.

The argument in [11, Lemma 10] gives the following general result.

Theorem 1.1 Let $1 \in A \subset B$ be a pair of C*-algebras. Suppose that $A$ has the SP-property and there is a conditional expectation $E$ from $B$ to $A$. If for any non-zero positive element $x$ in $B$ and an arbitrary positive number $\varepsilon > 0$ there is an element $y$ in $B$ such that

$$\|y^*(x - E(x))y\| < \varepsilon,$$
$$\|y^*E(x)y\| \geq \|E(x)\| - \varepsilon$$

then $B$ has the SP-property. Moreover, every non-zero hereditary C*-subalgebra of $B$ has a projection which is a equivalent to some projection in $A$ in the sense of Murray-von Neumann

Next we consider the following stronger assumption on a conditional expectation $E$ from $B$ to $A$.

Definition 1.2 Let $1 \in A \subset B$ be a pair of C*-algebras. A conditional expectation $E$ from $B$ to $A$ is called outer if for any element $x \in B$ with
$E(x) = 0$ and any non-zero hereditary $C^*$-subalgebra $C$ of $A$

\[ \inf\{ ||cxc||; c \in C^+, ||c|| = 1 \} = 0. \]

The following result comes from the same argument as in [10, Lemma 3.2] and Theorem 1.1.

**Corollary 1.3** Let $1 \in A \subset B$ be a pair of $C^*$-algebras. Suppose that $A$ has the SP-property and there is a conditional expectation $E$ from $B$ to $A$. If $E$ is outer, then $B$ has the SP-property.

We present some examples of a pair of $C^*$-algebras with an outer conditional expectations.

**Example 1.4** Let $\rho$ be a corner endomorphism on a unital $C^*$-algebra $A$, and let $E$ be a canonical conditional expectation from a crossed product $A \times_\rho \mathbb{N}$ to $A$. Suppose that

\[ \hat{\mathcal{T}}(\rho) = \{ \lambda \in \mathcal{T} | \hat{\rho}(I) = I \text{ for } \forall I \in \text{Prime}(A \times_\rho \mathbb{N}) \} = \mathcal{T}. \]

Then, $E$ is outer.

*Proof.* See Jeong-Kodaka-Osaka [6]. \qed

**Example 1.5** (Kishimoto[10]) Let $G$ be a discrete group and let $\alpha$ be a representation of $G$ by automorphisms of a simple unital $C^*$-algebra $A$. Suppose $\alpha$ is outer. Then, a canonical conditional expectation from a crossed product $A \times_\alpha G$ to $A$ is outer.

In the case of a crossed product of a simple unital $C^*$-algebra with the SP-property by a finite group $G$, we can deduce the SP-property for the crossed product algebra $A \times_\alpha G$ by any automorphism $\alpha$ on $A$.

**Theorem 1.6** ([7]) Let $A$ be a simple unital $C^*$-algebra with the SP-property, and let $\alpha$ be an action by a finite group $G$. Then, a crossed product algebra $A \times_\alpha G$ has the SP-property.

## 2 C*-Index Theory

In this section, we brief the C*-index theory by Watatani ([16]).

Let $1 \in A \subseteq B$ be a pair of $C^*$-algebras. By a conditional expectation $E : B \to A$ we mean a positive faithful linear map of norm one satisfying

\[ E(aba') = aE(b)a', \quad a, a' \in A, b \in B. \]
A finite family \( \{(u_1, v_1), \cdots, (u_n, v_n)\} \) in \( \mathbb{B} \times \mathbb{B} \) is called a quasi-basis for \( E \) if
\[
\sum_{i=1}^{n} u_i E(v_i b) = \sum_{i=1}^{n} E(b u_i)v_i = b \quad \text{for } b \in \mathbb{B}.
\]
We say that a conditional expectation \( E \) is of index-finite type if there exists a quasi-basis for \( E \). In this case the index of \( E \) is defined by
\[
\text{Index } E = \sum_{i=1}^{n} u_i v_i.
\]
Note that \( \text{Index } E \) does not depend on the choice of a quasi-basis and every conditional expectation \( E \) of index-finite type on a C*-algebra has a quasi-basis of the form \( \{(u_1, u_1^*), \cdots, (u_n, u_n^*)\} \) ([16, Lemma 2.1.6]). Moreover, \( \text{Index } E \) is always contained in the center of \( \mathbb{B} \), so that it is a scalar whenever \( \mathbb{B} \) has the trivial center, in particular when \( \mathbb{B} \) is simple.

Let \( E : \mathbb{B} \to A \) be a conditional expectation. Then \( B_A(=B) \) is a pre-Hilbert module over \( A \) with an \( A \)-valued inner product
\[
\langle x, y \rangle = E(x^* y), \quad x, y \in B_A.
\]
Let \( \mathcal{E} \) be the completion of \( B_A \) with respect to the norm on \( B_A \) defined by
\[
\|x\|_{B_A} = \|E(x^* x)\|_A^{1/2}, \quad x \in B_A.
\]
Then \( \mathcal{E} \) is a Hilbert C*-module over \( A \). Since \( E \) is faithful, the canonical map \( B \to \mathcal{E} \) is injective. Let \( L_A(\mathcal{E}) \) be the set of all (right) \( A \)-module homomorphisms \( T : \mathcal{E} \to \mathcal{E} \) with an adjoint \( A \)-module homomorphism \( T^* : \mathcal{E} \to \mathcal{E} \) such that
\[
\langle T \xi, \zeta \rangle = \langle \xi, T^* \zeta \rangle \quad \xi, \zeta \in \mathcal{E}.
\]
Then \( L_A(\mathcal{E}) \) is a C*-algebra with the operator norm \( \|T\| = \sup\{\|T \xi\| : \|\xi\| = 1\} \). There is an injective *-homomorphism \( \lambda : B \to L_A(\mathcal{E}) \) defined by
\[
\lambda(b)x = bx
\]
for \( x \in B_A, \ b \in B \), so that \( B \) can be viewed as a C*-subalgebra of \( L_A(\mathcal{E}) \).
Note that the map \( e_A : B_A \to B_A \) defined by
\[
e_A x = E(x), \quad x \in B_A
\]
is bounded and thus it can be extended to a bounded linear operator, denoted by \( e_A \) again, on \( \mathcal{E} \). Then \( e_A \in L_A(\mathcal{E}) \) and \( e_A = e_A^2 = e_A^* \), that is, \( e_A \) is a projection in \( L_A(\mathcal{E}) \).
The (reduced) $C^*$-basic construction is a $C^*$-subalgebra of $L_A(E)$ defined to be
\[ C^*(B, e_A) = \text{span}\{\lambda(x)e_A\lambda(y) \in L_A(E) : x, y \in B \} \|\cdot\| \]
see [16, Definition 2.1.2].

Then,

**Lemma 2.1** ([16, Lemma 2.1.4])

1. $e_A C^*(B, e_A)e_A = \lambda(A)e_A$.
2. $\psi : A \rightarrow e_A C^*(B, e_A)e_A, \psi(a) = \lambda(a)e_A$, is a $*$-isomorphism (onto).

**Lemma 2.2** ([16, Lemma 2.1.5]) The following are equivalent:

1. $E : B \rightarrow A$ is of index-finite type
2. $C^*(B, e_A)$ has an identity and there exists a number $c$ with $0 < c < 1$ such that
   \[ E(x^*x) \geq c(x^*x) \quad x \in B. \]

The above inequality was shown first in [13] by Pimsner and Popa for the conditional expectation $E_N : M \rightarrow N$ from a type $\text{II}_1$ factor $M$ onto its subfactor $N$ (c can be taken as the inverse of the Jones index $[M : N]$).

The conditional expectation $E_B : C^*(B, e_A) \rightarrow B$ defined by
\[ E_B(\lambda(x)e_A\lambda(y)) = (\text{Index}E)^{-1}xy, x, y \in B \]
is called the dual conditional expectation of $E : B \rightarrow A$. If $E$ is of index-finite type, so is $E_B$ with a quasi-basis $\{(w_i, w_i^*)\}$, where $w_i = \sqrt{\text{Index}E}u_ie_A$, and $\{(u_i, u_i^*)\}$ are quasis-basis for $E$ ([16, Proposition 2.3.4]).

### 3 The Stable Rank for $C^*$-Crossed Products

Let $\alpha$ be an action of a finite group $G$ on a unital $C^*$-algebra $A$ by automorphisms, and let $A \rtimes_{\alpha} G$ be its crossed product, that is, it is the universal $C^*$-algebra generated by a copy of $A$ and implementing unitaries $\{u_g | g \in G\}$ with $\alpha_g(a) = u_gau_g^*$ for every $g \in G$ and $a \in A$. Then there exists a canonical conditional expectation $E : A \rtimes_{\alpha} G \rightarrow A$ defined by
\[ E(\sum_g a_gu_g) = a_e, \]
for $a_g \in A$ and $g \in G$, where $e$ denotes the identity of the group $G$.

**Lemma 3.1** Under this situation, the canonical conditional expectation $E$ is of index-finite type with a quasi-basis $\{(u_g, u_g^*) : g \in G\}$ and
\[ \text{Index}(E) = \sum_{g \in G} u_gu_g^* = |G|, \] the order of $G$.
Let $B = A \times_{\alpha} G$ and $n = |G|$. Then, a dual conditional expectation $E_B$ is of index-finite type with a quasi-basis $\{(w_g, w_g^*) : g \in G\}$, where $w_g = \sqrt{n}u_ge_A$ (see section 2).

The following fact comes from a simple computation.

**Lemma 3.2** ([8]) The expression $x = \sum_{g \in G} w_gb_g$ ($b_g \in B$) is unique for each $x \in C^*(B, e_A)$.

Let $A$ be a unital $C^*$-algebra and $Lg_n(A)$ denote the $n$-tuples $(x_1, \ldots, x_n)$ in $A^n$ which generate $A$ as a left ideal. The topological stable rank of $A$ ($sr(A)$) is defined to be the least integer for which $Lg_n(A)$ is dense in $A^n$. If there does not exist such an integer then $sr(A)$ is defined to be $\infty$. For a non unital $C^*$-algebra $A$ we define $sr(A) = sr(\tilde{A})$ where $\tilde{A}$ is the unitization of $A$. See [15] for details about stable rank. It is not hard to see that for a unital $C^*$-algebra $A$ $sr(A) = 1$ if and only if the set of invertible elements is dense in $A$.

**Theorem 3.3** ([8]) Let $G$ be a finite group, and $\alpha$ be an action of $G$ on a unital $C^*$-algebra $A$ with $sr(A) = 1$. Then $sr(A \times_{\alpha} G) \leq |G|$.

**Proof.** Let $n = |G|$, and $(b_{g_1}, \ldots, b_{g_n}) \in B^n$, where $B = A \times_{\alpha} G$. Put $y = \sum_{g \in G} w_gb_g \in C^*(B, e_A)$. Since $C^*(B, e_A)$ is strong Morita equivalent to $A$ and $sr(A) = 1$, we have $sr(C^*(B, e_A)) = 1$ ([16, Proposition 1.3.4.]). Approximate $y$ by invertible elements $x$ in $C^*(B, e_A)$, and write $x = \sum_{g \in G} w_gc_g$, $c_g \in B$. Then by Lemma 3.2, $(c_{g_1}, \ldots, c_{g_n})$ is close to $(b_{g_1}, \ldots, b_{g_n})$. Note that

$$x^*x = n \sum_g c_g^*e_A c_g.$$

By Lemma 2.2

$$E_B(x^*x) \geq \frac{1}{n}x^*x, \quad x \in C^*(B, e_A).$$

Since $E_B(x^*x) = \sum_g c_g^* c_g$, it follows that

$$\sum_g c_g^* c_g \geq \frac{1}{n} x^*x$$

which is invertible in $C^*(B, e_A)$. Therefore $\sum_g c_g^* c_g$ is invertible in $B$, that is, $(c_{g_1}, \ldots, c_{g_n}) \in Lg_n(B)$. $\square$
Remark 3.4 If \( sr(A) = m \) then it can be shown that \( sr(A \times_\alpha G) \leq |G|m \)
whenever \( A \) is a simple unital \( C^* \)-algebra. Indeed, it can come from the
following two facts; (i) \( C^*(B, e_A) \) is isomorphic to the matrix algebra \( M_n(A) \)
([16]), (ii) \( sr(M_n(A)) = \{ \frac{sr(A)-1}{n} \} + 1 \), where \( \{ t \} \) denotes the
least integer which is greater than or equal to \( t \) ([15]).

4 The Cancellation Property

A \( C^* \)-algebra \( A \) is said to have cancellation of projections if for any
projections \( p, q, r \) in \( A \) with \( p \perp r, q \perp r, p + r \sim q + r \), we have \( p \sim q \). If
\( M_n(A) \) has cancellation of projections for each \( n = 1, 2, \ldots \), then we simply
say that \( A \) has cancellation. Note that every \( C^* \)-algebra with cancellation
is stably finite, that is, every matrix algebra \( M_n(A) \) with entries from \( A \)
contains no infinite projections for \( n = 1, 2, \ldots \). It can be shown that if \( A \)
is a \( C^* \)-algebra with \( sr(A) = 1 \) then it has cancellation. In the previous
section we proved that the stable rank of the \( C^* \)-crossed product \( A \times_\alpha G \) is
bounded by the order of the group \( G \) if \( sr(A)=1 \), and actually it seems that
the crossed product has stable rank 1, and therefore it would be natural to
ask if it has cancellation.

Theorem 4.1 ([2, Theorem 4.2.2]) Let \( A \) be a simple unital \( C^* \)-algebra.
Suppose \( A \) contains a sequence \((p_k)\) of projections such that
1. for each \( k \) there is a projection \( r_k \) such that \( 2p_{k+1} \oplus r_k \) is equivalent
to a subprojection of \( p_k \oplus r_k \),
2. there is a constant \( K \) such that \( sr(p_kAp_k) \leq K \) for all \( k \).
Then \( A \) has cancellation.

Theorem 4.2 ([8]) Let \( A \) be a simple unital \( C^* \)-algebra with \( sr(A) = 1 \)
and \( SP \)-property. If \( G \) is a finite group and \( \alpha \) is an action of \( G \) on \( A \) then
the crossed product \( A \times_\alpha G \) has cancellation.

Sketch of a proof.
We give a proof in the case that \( A \times_\alpha G \) is simple.
Since the fixed point algebra \( A^\alpha \) can be identified with a hereditary \( C^* \)-
subalgebra of the crossed product it has the \( SP \)-property by Theorem 1.6.
Thus there is a sequence of projections \( \{ p_k \} \in A^\alpha \) such that \( 2[p_{k+1}] \leq [p_k] \)
by [9, Lemma 2.2], where \( [p] \) denotes the equivalence class of \( p \). Since
$p_k \in A^\alpha$, $p_k(A \times_\alpha G)p_k$ is isomorphic to $(p_kAp_k) \times_\alpha G$ for each $k \in N$. Note that each $p_kAp_k$ has stable rank one. By Theorem 3.3 $sr(p_kAp_k \times_\alpha G) \leq |G|$. Therefore, the assertion follows from Theorem 4.1 ($K = |G|$, $r_k = 0$).

Recall that a unital $C^*$-algebra $A$ has real rank zero, $RR(A) = 0$, if the set of invertible self-adjoint elements is dense in $A_{sa}$. It is well known that $RR(A) = 0$ is equivalent to say that every non-zero hereditary $C^*$-subalgebra contains an approximate identity consisting of projections (HP) ([3]). From [2, Section 4] where the HP-property is studied for simple $C^*$-algebras we can deduce the following.

**Corollary 4.3** ([8]) Under the assumptions of the above theorem, if $RR(A \times_\alpha G) = 0$ then its stable rank is one.

For crossed products by the integer group $Z$ we have the following cancellation theorem:

**Theorem 4.4** ([8]) Let $A$ be a simple unital $C^*$-algebra with $sr(A) = 1$ and SP-property. If $\alpha$ is an outer action of the integer group $Z$ on $A$ such that $\alpha_* = id$ on the $K_0$ group $K_0(A)$ of $A$ then the crossed product $A \times_\alpha Z$ has cancellation.

**Example 4.5** If $A$ is a UHF algebra or an irrational rotation algebra then the identity map is the only possible homomorphism on its $K_0$ group. Therefore the theorem says that any crossed product $A \times_\alpha Z$ has cancellation.

**Corollary 4.6** ([8]) Under the same assumption of Theorem 3.5 if $RR(A \times_\alpha Z) = 0$, then its stable rank is one.

参考文献


