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双対ファジィ動的計画について

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1 Introduction

Since Bellman and Zadeh's seminal paper [4], a large amount of efforts has been devoted to the study of decision-making in a fuzzy environemnt ([5],[12],[13],[14],[20]). Bellman and Zadeh have originated three kinds of systems in fuzzy environment; deterministic, stochastic and fuzzy systems. Of the three, they give a detailed analysis on both deterministic and stochastic systems in [4]. Further Iwamoto and Fujita [9] have analyzed *stochastic system* by use of the *regular* (i.e., multiplication-addition) expectation operator.

However, as for the terminology *fuzzy system*, Bellman and Zadeh only touched it. This has motivated further researches. Baldwin and Pilsworth [1] have derived a dynamic programming functional equation for a fuzzy system defined by fuzzy automata. Recently Iwamoto and Sniedovich [10] have proposed a decision process with fuzzy system where a fuzzy expectation is taken by use of *minimum-maximum* operator. Both papers [9],[10] have applied an invariant imbedding method ([3],[15]).

In this paper, we are concerned with a large class of fuzzy dynamic programs. We focus our attention on a duality between optimal value functions in the class. In a few typical environments, we optimize a fuzzy-like expected value of the associatively combined aggregation (fuzzy variable) of stage-wise memberships.

In §2 we give notations and definitions used in the paper. In §3 we formulate a fuzzy dyamic program in a general environment. By imposing two additional parameters on associative aggregations, we derive a parametrized recursive equation for the fuzzy dynamic program. Further, we show that a substitution of left-identity elements for the two parameters yields the desired optimum value. This is an invariant imbedding technique (Lee [15], see also [3]). In §4 we define a dual of fuzzy dyamic program. Two duality theorems between primal and dual optimal value functions are shown. In §5 we introduce two typical environments; fuzzy environment and quasi-stochastic environment. Further we illustrate both maximum-minimum process and minimum-minimum process in fuzzy environment and multiplicative-multiplicative process in quasi-stochastic environment. Specifying their dual fuzzy dynamic programs, we verify that the duality relation holds between primal and dual ones.

2 Notations and Definitions

Throughout the paper, we use the following notations and definitions. Let a binary relation $\odot : [0,1] \times [0,1] \to [0,1]$ be *associative* :

$$(x \odot y) \odot z = x \odot (y \odot z). \tag{1}$$

The common value is denoted by $x \odot y \odot z$. We use the multiple notation $x_1 \odot x_2 \odot \cdots \odot x_n$. Further we assume that it is *commutative* :

$$x \odot y = y \odot x. \tag{2}$$

Any $\tilde{x}$ satisfying

$$\tilde{x} \odot x = x \qquad \forall x \in [0,1]$$

is called a *left-identity element* for $\odot$. We say that the binary relation $\odot$ is *monotone* if

$$y < z \implies x \odot y \leq x \odot z \qquad \forall x \in [0,1]. \tag{3}$$

Both commutativity and associativity enable us to define the operator $\bigodot$ for any function $g : V \to [0,1]$ as follows :

$$\bigodot_{v \in V} g(v) := g(v_1) \odot g(v_2) \odot \cdots \odot g(v_k). \tag{4}$$

where $V = \{v_1, v_2, \ldots, v_k\}$ is a finite set. Just like the summation

$$\sum_{v \in V} g(v) := g(v_1) + g(v_2) + \cdots + g(v_k) \tag{5}$$

we use similar notations ; $\bigoplus_{v \in V} g(v)$, $\bigotimes_{v \in V} g(v)$, $\cdots$.

We define the following operations :

$$\overline{\mathrm{Opt}} := \begin{cases} \min \\ \mathrm{Max} \end{cases} \quad \text{for} \quad \mathrm{Opt} = \begin{cases} \mathrm{Max} \\ \min \end{cases}$$

$$\bar{a} := 1 - a, \qquad \overline{f}(x) := 1 - f(x) \quad \text{for} \quad f : X \to [0,1]$$

$$a \overline{\odot} b := \overline{\bar{a} \odot \bar{b}}.$$

We say that $\overline{\odot}$ is the *dual* binary relation of $\odot$. Thus we see the dual operation preserves (inherits) commutativity, associativity and monotonicity. Therefore, we define

$$\overline{\bigoplus_{v \in V}} g(v) := g(v_1) \overline{\oplus} g(v_2) \overline{\oplus} \cdots \overline{\oplus} g(v_k). \tag{6}$$

We say that an ordered pair of binary relations $(\diamond, \star)$ is *dual* if

$$\overline{\diamond} = \star. \tag{7}$$

We also say that a pair of functions $f,\ F : X \to R^1$ is *dual* if

$$F = \overline{f} \tag{8}$$

that is

$$f(x) + F(x) = 1 \quad \forall x \in X. \tag{9}$$

3 Fuzzy Dynamic Program

A fuzzy dynamic program (FDP) is specified by a six-tuple:

$$\mathcal{F} = < \text{Opt},\ \{S_n\}_1^{N+1},\ \{A_n\}_1^N,\ (\{\nu_n\}_1^N,\ \bullet,\ \xi),\ (\{\mu_n\}_1^N,\ \circ),\ (\otimes, \oplus) >$$

where the compoments are specified as follows.

(i) Let N be a positive integer; *total number of stages.* The subscript n ranges $1 \leq n \leq N$ (or $N+1$). It specifies the current number of stage.

(ii) Let S_n be a nonempty finite set; *n-th state space.* Its element $s_n \in S_n$ is called an *n-th states.* s_1 is an *initial* state. s_{N+1} is a *terminal* state.

(iii) Let A_n be a nonempty finite set; *n-th action space.* Let $A_n(s_n) \subset A_n$ be a nonempty subset; *n-th feasible action space at state* s_n. Its element $a_n \in A_n(s_n)$ is called an *n-th action at state* s_n.

(iv) Let $\nu_n : S_n \times A_n \to [0,1]$ be a membership function of *n-th fuzzy set* R_n on $S_n \times A_n$:

$$\nu_n(s_n, a_n) = \mu_{R_n}(s_n, a_n). \tag{10}$$

We call ν_n an *n-th membership function.* Let $\xi : S_{N+1} \to [0,1]$ be a membership function of *terminal fuzzy set* T on state space S_{N+1}:

$$\xi(s_{N+1}) = \mu_T(s_{N+1}). \tag{11}$$

We call ξ a *terminal membership function.* Let $\bullet : [0,1] \times [0,1] \to [0,1]$ be an associative binary relation with a left-identity element $\tilde{\lambda}$. The relation $\bullet$ combines assocatively membership degrees between two adjacent fuzzy sets, *objective-membership generator.* The three-tuple $(\{\nu_n\}_1^N, \bullet, \xi)$ is called a *membership system.* This system induces the *objective membership* of the aggregated fuzzy set $R_1 * R_2 * \cdots * R_N * T$ on history (direct) space $H = S_1 \times A_1 \times S_2 \times A_2 \times \cdots \times A_N \times S_{N+1}$

$$\begin{aligned} & \mu_{R_1 * R_2 * \cdots * R_N * T}(s_1, a_1, s_2, a_2, \cdots, s_N, a_N, s_{N+1}) \\ = \ & \nu_1(s_1, a_1) \bullet \nu_2(s_2, a_2) \bullet \cdots \bullet \nu_N(s_N, a_N) \bullet \xi(s_{N+1}). \end{aligned} \tag{12}$$

Here, the operation $*$ between fuzzy sets corresponds to the binary relation $\bullet$ between their memberships:

$$\mu_{R_n * R_{n+1}} = \mu_{R_n} \bullet \mu_{R_{n+1}} \quad 1 \leq n \leq N \ \ (R_{N+1} = T,\ \ \mu_{R_{N+1}} = \mu_T). \tag{13}$$

(v) Let $\mu_n = \mu_n(s_{n+1} \mid s_n, a_n)$ be an *n-th fuzzy transition law* from s_n onto S_{n+1} depending on the current action a_n. When the system is in state s_n on stage n and action a_n is chosen, the next state will become s_{n+1} with membership degree $0 \leq \mu_n(s_{n+1} \mid s_n, a_n) \leq 1$. Symbolically we express this kind of transition as follows: $s_{n+1} \simeq \mu_n(\cdot \mid s_n, a_n)$. Let $\circ : [0,1] \times [0,1] \to [0,1]$ be also associative binary operation with a left-identity element $\tilde{\kappa}$, *system-membership composer.* Combining membership degrees between two adjacent transitions, it generates a system-membership on the history space H

$$\mu_1(s_2|s_1, a_1) \circ \mu_2(s_3|s_2, a_2) \circ \cdots \circ \mu_N(s_{N+1}|s_N, a_N). \tag{14}$$

The pair $(\{\mu_n\}_1^N, \circ)$ is called a *fuzzy transition system.*
(vi) Let $\otimes, \oplus : [0,1] \times [0,1] \to [0,1]$ be commutative, associative and monotone binary relations. The operation $\otimes$ connects objective membership (12) and system-membership (14), *connector.* The operation $\oplus$ integrates in a wide sense all the connected memberships over the history space, *integrator.* We call the ordered pair $(\otimes, \oplus)$ an *expectation-generating environment.*
(vii) Let Opt denote either Max or min; *optimizer.* It means that FDP $\mathcal{F}$ represents the fuzzy optimization problem :

$$\begin{array}{ll} \text{Optimize} & F^\sigma[\,\nu_1(s_1,a_1) \bullet \nu_2(s_2,a_2) \bullet \cdots \bullet \nu_N(s_N,a_N) \bullet \xi(s_{N+1})\,] \\ \text{subject to} & (\mathrm{i})_\mathrm{n} \quad s_{n+1} \simeq \mu_n(\cdot|\, s_n,a_n) \quad 1 \le n \le N \\ & (\mathrm{ii})_\mathrm{n} \quad a_n \in A_n(s_n) \quad 1 \le n \le N \end{array} \tag{15}$$

where F^σ denotes a *fuzzy-like expectation operator* on $S_1 \times S_2 \times \cdots \times S_{N+1}$ induced from the fuzzy transition system $(\{\mu_n\}_1^N, \circ)$, a general policy $\sigma = \{\sigma_1, \sigma_2, \cdots, \sigma_N\}$, and an initial state $s_1 \in S_1$. Thus we have a "fuzzy-like expected" value of objective membership function (12)

$$\begin{aligned} & F^\sigma[\,\nu_1(s_1,a_1) \bullet \nu_2(s_2,a_2) \bullet \cdots \bullet \nu_N(s_N,a_N) \bullet \xi(s_{N+1})\,] \\ = & \bigoplus_{s \in S^N} \{\,[\,\nu_1(s_1,a_1) \bullet \nu_2(s_2,a_2) \bullet \cdots \bullet \nu_N(s_N,a_N) \bullet \xi(s_{N+1})\,] \\ & \qquad \otimes [\,\mu_1(s_2|s_1,a_1) \circ \mu_2(s_3|s_2,a_2) \circ \cdots \circ \mu_N(s_{N+1}|s_N,a_N)\,]\,\} \end{aligned} \tag{16}$$

where

$$a_1 = \sigma_1(s_1),\ a_2 = \sigma_2(s_1,s_2),\ \ldots,\ a_N = \sigma_N(s_1,\ldots,s_N)$$
$$s = (s_2,\cdots,s_{N+1}), \qquad S^N = S_2 \times \cdots \times S_{N+1}.$$

When the general policy σ reduces to the *Markov* polcy, we write F^π instead of F^σ. In this case, the sequence of actions are chosen as follows :

$$a_1 = \pi_1(s_1),\ a_2 = \pi_2(s_2),\ \ldots,\ a_N = \pi_N(s_N).$$

We note that the Markov policy is not always enough. That is, sometimes there does not exist an optimal policy in the Markov class ([11]).

Throughout the paper, we use the short notation

$$\nu_n \bullet \nu_{n+1} \bullet \cdots \bullet \nu_N \bullet \xi$$

for the "fuzzy" variable

$$\nu_n(s_n,a_n) \bullet \nu_{n+1}(s_{n+1},a_{n+1}) \bullet \cdots \bullet \nu_N(s_N,a_N) \bullet \xi(s_{N+1})$$

where

$$\nu_n = \nu_n(s_n,a_n) \qquad 1 \le n \le N, \qquad \xi = \xi(s_{N+1}).$$

Thus, problem (15) has the following simple form:

$$\begin{array}{ll} \text{Optimize} & F^{\sigma}[\,\nu_1 \bullet \nu_2 \bullet \cdots \bullet \nu_N \bullet \xi\,] \\ \text{subject to} & (\mathrm{i})_{\mathrm{n}}, \ \ (\mathrm{ii})_{\mathrm{n}} \qquad 1 \le n \le N. \end{array}$$

Now, let us define for any given $1 \le n \le N$ and $s_n \in S_n$ the subproblem :

$$v^{N-n+1}(s_n) := \operatorname*{Opt}_{\sigma} F^{\sigma}[\,\nu_n \bullet \cdots \bullet \nu_N \bullet \xi \,|\, (\mathrm{i})_{\mathrm{m}}, \ (\mathrm{ii})_{\mathrm{m}} \quad n \le m \le N\,] \tag{17}$$

where optimization is taken for all general policies $\sigma = \{\sigma_n, \sigma_{n+1}, ..., \sigma_N\}$. We remark that *general policy* σ satisfies

$$\sigma_n : S_n \to A_n, \ \ \sigma_{n+1} : S_n \times S_{n+1} \to A_{n+1}, \ \ ..., \ \sigma_N : S_n \times \cdots \times S_N \to A_N$$

together with the feasibility

$$\sigma_m(s_n, \ldots, s_m) \in A_m(s_m) \quad (s_n, \ldots, s_m) \in S_n \times \cdots \times S_m, \ \ n \le m \le N.$$

Then we are concerned with a derivation of recursive equation between the value $v^{N-n+1}(s)$ and the function $v^{N-n}(\cdot)$.

We have two conceivable "formal candidates":

$$v^{N-n+1}(s) = \operatorname*{Opt}_{a} \bigoplus_{t} [\,(\nu_n(s,a) \bullet v^{N-n}(t)) \otimes \mu_n(t|s,a)\,] \tag{18}$$

$$s \in S_n \qquad n = 1, 2, \cdots, N \tag{19}$$

and

$$v^{N-n+1}(s) = \operatorname*{Opt}_{a} [\,\nu_n(s,a) \bullet \bigoplus_{t} (v^{N-n}(t) \otimes \mu_n(t|s,a)\,)\,] \tag{20}$$

where

$$v^0(s) = \xi(s) \qquad s \in S_{N+1}. \tag{21}$$

Here optimizations are taken for all a in $A_n(s)$:

$$\operatorname*{Opt}_{a} = \operatorname*{Opt}_{a \in A_n(s)}$$

and the wide integrations for all t in S_{n+1} :

$$\bigoplus_{t} = \bigoplus_{t \in S_{n+1}} .$$

These two simplified notations are also used throughout the paper.

In general, neither (18) nor (20) holds. It is too general to derive such recursive equations. To do so, we take concrete forms for binary relations $\bullet$, $\circ$, $\otimes$, $\oplus$. In the last section we specify fuzzy environment where minimum-minimum process enjoys the validity of these two equations.

In this section, we rather apply an invariant imbedding technique ([3],[15]). We imbed problem (15) into a family of two-parameter problems. Let us consider for any given $s_n \in S_n$ and $\lambda, \kappa \in [0,1]$ the optimization problem :

$$u^{N-n+1}(s_n; \lambda, \kappa) = \mathop{\mathrm{Opt}}_{\pi} F^{\pi}_{\kappa}[\, \lambda \bullet \nu_n \bullet \cdots \bullet \nu_N \bullet \xi \,|\, (\mathrm{i})_{\mathrm{m}},\ (\mathrm{ii})_{\mathrm{m}} \quad n \le m \le N \,] \tag{22}$$

where the fuzzy-like expetation operator F^{π}_{κ} with an *augmented Markov* policy $\pi = \{\pi_n, \pi_{n+1}, ..., \pi_N\}$ and a starting system-membership degree κ is defined as follows:

$$\begin{aligned} & F^{\pi}_{\kappa}[\, \lambda \bullet \nu_n \bullet \cdots \bullet \nu_N \bullet \xi \,|\, (\mathrm{i})_{\mathrm{m}},\ (\mathrm{ii})_{\mathrm{m}} \quad n \le m \le N \,] \\ = & \bigoplus_{s \in S^{N-n+1}} \{ [\, \lambda \bullet \nu_n(s_n, a_n) \bullet \nu_{n+1}(s_{n+1}, a_{n+1}) \bullet \cdots \bullet \nu_N(s_N, a_N) \bullet \xi(s_{N+1}) \,] \qquad (23) \\ & \otimes [\, \kappa \circ \mu_n(s_{n+1}|s_n, a_n) \circ \mu_{n+1}(s_{n+2}|s_{n+1}, a_{n+1}) \circ \cdots \circ \mu_N(s_{N+1}|s_N, a_N) \,] \}. \end{aligned}$$

Here we note that

$$a_n = \pi_n(\lambda_n, \kappa_n, s_n),\ a_{n+1} = \pi_{n+1}(\lambda_{n+1}, \kappa_{n+1}, s_{n+1}),\ \ldots,\ a_N = \pi_N(\lambda_N, \kappa_N, s_N)$$

$$\lambda_n = \lambda,\ \lambda_{n+1} = \lambda_n \bullet \nu_n(s_n, a_n),\ \ldots,\ \lambda_{N+1} = \lambda_N \bullet \nu_N(s_N, a_N)$$

$$\kappa_n = \kappa,\ \kappa_{n+1} = \kappa_n \circ \mu_n(s_{n+1}|s_n, a_n),\ \ldots,\ \kappa_{N+1} = \kappa_N \circ \mu_N(s_{N+1}|s_N, a_N)$$

$$s = (s_{n+1}, \cdots, s_{N+1}), \qquad S^{N-n+1} = S_{n+1} \times \cdots \times S_{N+1}.$$

Then, the substitution of two left-identity elements yields an optimal value

$$u^{N-n+1}(s; \tilde{\lambda}, \tilde{\kappa}) = v^{N-n+1}(s) \quad s \in S_n, \quad 1 \le n \le N. \tag{24}$$

(This fact is justified by considering a correspondence between the class of all *general* policies and the class of the *augmented Markov* ones. Since we are concerned with recusiveness and duality for optimal value functions, we do not discuss the justification. For the details, see [11]).

At the same time, we have the following recursive equation for the value $u^{N-n+1}(s; \lambda, \kappa)$ and the three-variable function $u^{N-n}(\cdot\,; \cdot, \cdot)$:

THEOREM 3. 1 *(Two-parametric Recursive Equation)*

$$u^{N-n+1}(s; \lambda, \kappa) = \mathop{\mathrm{Opt}}_{a} \bigoplus_{t} u^{N-n}(t; \lambda \bullet \nu_n(s, a),\ \kappa \circ \mu_n(t|s, a)) \tag{25}$$

$$s \in S_n \qquad \lambda, \kappa \in [0,1] \qquad n = 1, 2, \cdots, N$$

$$u^0(s; \lambda, \kappa) = [\lambda \bullet \xi(s)] \otimes \kappa \qquad s \in S_{N+1} \quad \lambda, \kappa \in [0,1]. \tag{26}$$

Here is a separation problem whether the identity

$$u^{N-n+1}(s; \lambda, \kappa) = [\lambda \bullet v^{N-n+1}(s)] \otimes \kappa \tag{27}$$

holds or not. According to the commuatativity, associativity and others in $\bullet, \circ, \otimes$ and $\oplus$, this problem is also solved in the following sections.

4 Dual Fuzzy Dynamic Program

Given a (primal) fuzzy dynamic program (FDP)

$$\mathcal{F} = < \text{Opt},\ \{S_n\}_1^{N+1},\ \{A_n\}_1^N,\ (\{\nu_n\}_1^N,\ \bullet,\ \xi),\ (\{\mu_n\}_1^N,\ \circ),\ (\otimes,\oplus) >,$$

we define its *dual FDP* $\mathcal{G}$ by the following six-tuple:

$$\mathcal{G} = < \overline{\text{Opt}},\ \{S_n\}_1^{N+1},\ \{A_n\}_1^N,\ (\{\overline{\nu_n}\}_1^N,\ \bar{\bullet},\ \bar{\xi}),\ (\{\overline{\mu_n}\}_1^N,\ \bar{\circ}),\ (\overline{\otimes},\overline{\oplus}) > .$$

Thus $\mathcal{G}$ represents the fuzzy optimization problem :

$$\begin{array}{lll} \overline{\text{Optimize}} & F^\sigma[\,\overline{\nu_1}(s_1,a_1)\,\bar{\bullet}\,\overline{\nu_2}(s_2,a_2)\,\bar{\bullet}\,\cdots\,\bar{\bullet}\,\overline{\nu_N}(s_N,a_N)\,\bar{\bullet}\,\bar{\xi}(s_{N+1})\,] & \\ \text{subject to} & (\mathrm{i})_{\mathrm{n}}'\ \ s_{n+1} \simeq \overline{\mu_n}(\cdot\,|\,s_n,a_n) \quad 1 \le n \le N & (28) \\ & (\mathrm{ii})_{\mathrm{n}}\ \ a_n \in A_n(s_n) \quad 1 \le n \le N & \end{array}$$

where the "fuzzy-like expected" value is in turn

$$\begin{aligned} & F^\sigma[\,\overline{\nu_1}(s_1,a_1)\,\bar{\bullet}\,\overline{\nu_2}(s_2,a_2)\,\bar{\bullet}\,\cdots\,\bar{\bullet}\,\overline{\nu_N}(s_N,a_N)\,\bar{\bullet}\,\bar{\xi}(s_{N+1})\,] \\ = & \overline{\bigoplus_{s\in S^N}}\{\,[\,\overline{\nu_1}(s_1,a_1)\,\bar{\bullet}\,\overline{\nu_2}(s_2,a_2)\,\bar{\bullet}\,\cdots\,\bar{\bullet}\,\overline{\nu_N}(s_N,a_N)\,\bar{\bullet}\,\bar{\xi}(s_{N+1})\,] \qquad (29) \\ & \overline{\otimes}[\,\overline{\mu_1}(s_2|s_1,a_1)\,\bar{\circ}\,\overline{\mu_2}(s_3|s_2,a_2)\,\bar{\circ}\,\cdots\,\bar{\circ}\,\overline{\mu_N}(s_{N+1}|s_N,a_N)\,]\,\}. \end{aligned}$$

We consider two corresponding families of subproblems. One has no parameter:

$$V^{N-n+1}(s_n) \;=\; \overline{\text{Opt}}_{\sigma}\, F^\sigma[\,\overline{\nu_n}\,\bar{\bullet}\,\cdots\,\bar{\bullet}\,\overline{\nu_N}\,\bar{\bullet}\,\bar{\xi}\,|\,(\mathrm{i})_{\mathrm{m}}',\ (\mathrm{ii})_{\mathrm{m}} \quad n \le m \le N\,] \qquad (30)$$
$$s_n \in S_n.$$

where the optimization in (30) is taken for all general policies $\{\sigma\}$. The other is two-parametric:

$$U^{N-n+1}(s_n;\lambda,\kappa) \;=\; \overline{\text{Opt}}_{\pi}\, F_\kappa^\pi[\,\lambda\,\bar{\bullet}\,\overline{\nu_n}\,\bar{\bullet}\,\cdots\,\bar{\bullet}\,\overline{\nu_N}\,\bar{\bullet}\,\bar{\xi}\,|\,(\mathrm{i})_{\mathrm{m}}',\ (\mathrm{ii})_{\mathrm{m}} \quad n \le m \le N\,] \qquad (31)$$
$$s_n \in S_n, \quad \lambda,\kappa \in [0,1].$$

We note that the optimization in (31) is taken for all augmented Markov policies $\{\pi\}$. Then we have the recursive equation for two-parametric subproblems:

COROLLARY 4. 1 *(Recursive Equation)*

$$U^{N-n+1}(s;\lambda,\kappa) = \overline{\text{Opt}}_{a}\,\overline{\bigoplus_{t}}\, U^{N-n}(t;\lambda\,\bar{\bullet}\,\overline{\nu_n}(s,a),\ \kappa\,\bar{\circ}\,\overline{\mu_n}(t|s,a)) \qquad (32)$$
$$s \in S_n \qquad \lambda,\kappa \in [0,1] \qquad n = 1,2,\cdots,N$$
$$U^0(s;\lambda,\kappa) = [\lambda\,\bar{\bullet}\,\bar{\xi}(s)]\overline{\otimes}\kappa \qquad s \in S_{N+1} \qquad \lambda,\kappa \in [0,1]. \qquad (33)$$

Further we have the following dual relations between $\mathcal{F}$ and $\mathcal{G}$:

THEOREM 4.1 *(Duality Theorem 1) The pair of optimal membership functions* $\{v^n\}_1^{N+1}$ *for* $\mathcal{F}$ *and optimal membership functions* $\{V^n\}_1^{N+1}$ *for* $\mathcal{G}$ *is dual:*

$$v^n(s) + V^n(s) = 1 \quad \forall s \in S_n,\ 1 \le n \le N+1 \tag{34}$$

that is

$$V^n = \overline{v^n} \quad 1 \le n \le N+1. \tag{35}$$

THEOREM 4.2 *(Duality Theorem 2) The pair of optimal membership functions* $\{u^n\}_1^{N+1}$ *for* $\mathcal{F}$ *and optimal membership functions* $\{U^n\}_1^{N+1}$ *for* $\mathcal{G}$ *is dual in the following sense:*

$$\overline{U^n}(s;\lambda,\kappa) = u^n(s;\overline{\lambda},\overline{\kappa}) \quad \forall s \in S_n, \quad \lambda,\kappa \in [0,1], \quad 1 \le n \le N+1. \tag{36}$$

5 Fuzzy and Quasi-Stochastic Environments

In this section we consider two typical environments. One is *fuzzy* environment ($\otimes := \wedge,\ \oplus := \vee$). The fuzzy environment has the *minimum* connector and the *maximum* integrator. The other is *quasi-stochastic* environment ($\otimes := \times,\ \oplus := \sqcup$), where $a \sqcup b = (a+b) \wedge 1$. The quasi-stochastic environment has the *multiplicative* connector and the *bounded-additive* integrator. We illustrate two primal fuzzy dynamic programs in fuzzy environment and a primal fuzzy dynamic program in quasi-stochastic one. We give their dual fuzzy dynamic programs.

Let X, U be two nonempty finite sets throughout this section. We take both sets X, Uas state space and action space, respectively:

$$S_n = X, \qquad A_n = A = U.$$

5.1 Maxi-mini Process in Fuzzy Environment

As a primal primal FDP, we consider the *maximum objective* ($\bullet := \vee$) and the *minimum system* ($\circ := \wedge$) in fuzzy environment :

$$\mathcal{F} = <\text{Max},\ \{S_n\}_1^{N+1},\ \{A_n\}_1^N,\ (\{\nu_n\}_1^N,\ \vee,\ \xi),\ (\{\mu_n\}_1^N,\ \wedge),\ (\wedge,\vee) > .$$

Then $\mathcal{F}$ represents the fuzzy maximization problem:

$$\begin{array}{ll} \text{Maximize} & F^\sigma[\,\nu_1(x_1,u_1) \vee \nu_2(x_2,u_2) \vee \cdots \vee \nu_N(x_N,u_N) \vee \xi(x_{N+1})\,] \\ \text{subject to} & \text{(i)}_\text{n} \;\; x_{n+1} \simeq \mu_n(\cdot\,|\, x_n,u_n) \quad 1 \le n \le N \\ & \text{(ii)}_\text{n} \;\; u_n \in U \quad 1 \le n \le N. \end{array} \tag{37}$$

Note that the fuzzy-expected value becomes

$$\bigvee_{x \in X^N} \{\,[\,\nu_1(x_1,u_1) \vee \nu_2(x_2,u_2) \vee \cdots \vee \nu_N(x_N,u_N) \vee \xi(x_{N+1})\,] \wedge [\,\mu_1(x_2|x_1,u_1) \wedge \mu_2(x_3|x_2,u_2) \wedge \cdots \wedge \mu_N(x_{N+1}|x_N,u_N)\,]\,\} \tag{38}$$

where

$$x = (x_2, \cdots, x_{N+1}), \qquad X^N = X \times \cdots \times X.$$

In this subsection, we imbed problem (37) into a family of one-parameter problems. Let us consider for any given $s_n \in S_n$ and $\lambda \in [0,1]$ the optimization problem :

$$u^{N-n+1}(s_n; \lambda) = \operatorname*{Opt}_{\pi} F^{\pi}[\lambda \vee \nu_n \vee \cdots \vee \nu_N \vee \xi \mid (\mathrm{i})_{\mathrm{m}},\ (\mathrm{ii})_{\mathrm{m}} \quad n \leq m \leq N] \tag{39}$$

where the fuzzy-like expetation operator F^{π} with a *one-dimensionally* augmented Markov policy $\pi = \{\pi_n, \pi_{n+1}, \ldots, \pi_N\}$. Then the corresponding one-parametric recursive equation holds:

THEOREM 5. 1 *(One-parametric Recursive Equation)*

$$u^{N-n+1}(x; \lambda) = \operatorname*{Max}_{u \in U} \bigvee_{y \in X} [u^{N-n}(y; \lambda \vee \nu_n(x,u)) \wedge \mu_n(y|x,u)] \tag{40}$$

$$n = 1, 2, \cdots, N$$

$$u^0(x; \lambda) = \lambda \vee \xi(x) \qquad \lambda \in [0,1], \ x \in X. \tag{41}$$

Proof Note that $\vee$ is distributive over $\wedge$. Then the proof is the same as for the two-parametric recursive equation. □

On the other hand, its dual FDP

$$\mathcal{G} = < \min,\ \{S_n\}_1^{N+1},\ \{A_n\}_1^N,\ (\{\overline{\nu_n}\}_1^N,\ \wedge,\ \overline{\xi}),\ (\{\overline{\mu_n}\}_1^N,\ \vee),\ (\vee, \wedge) >$$

represents the fuzzy minimization problem:

$$\begin{aligned} &\text{minimize} \quad F^{\sigma}[\overline{\nu_1}(x_1,u_1) \wedge \overline{\nu_2}(x_2,u_2) \wedge \cdots \wedge \overline{\nu_N}(x_N,u_N) \wedge \overline{\xi}(x_{N+1})] \\ &\text{subject to} \quad (\mathrm{i})_{\mathrm{n}} \quad x_{n+1} \simeq \overline{\mu_n}(\cdot \mid x_n, u_n) \quad 1 \leq n \leq N \\ &\qquad\qquad\quad (\mathrm{ii})_{\mathrm{n}} \quad u_n \in U \quad 1 \leq n \leq N \end{aligned} \tag{42}$$

where the objective function is

$$\begin{aligned} \bigwedge_{x \in X^N} \{ & [\overline{\nu_1}(x_1,u_1) \wedge \overline{\nu_2}(x_2,u_2) \wedge \cdots \wedge \overline{\nu_N}(x_N,u_N) \wedge \overline{\xi}(x_{N+1})] \\ & \vee [\overline{\mu_1}(x_2|x_1,u_1) \vee \overline{\mu_2}(x_3|x_2,u_2) \vee \cdots \vee \overline{\mu_N}(x_{N+1}|x_N,u_N)] \}. \end{aligned} \tag{43}$$

The dual FDP $\mathcal{G}$ admits the following one-parametric recursive equation:

COROLLARY 5. 1

$$U^{N-n+1}(x; \lambda) = \min_{u \in U} \bigwedge_{y \in X} [U^{N-n}(y; \lambda \wedge \overline{\nu_n}(x,u)) \vee \overline{\mu_n}(y|x,u)] \tag{44}$$

$$n = 1, 2, \cdots, N$$

$$U^0(x; \lambda) = \lambda \wedge \overline{\xi}(x) \qquad x \in X \quad \lambda \in [0,1]. \tag{45}$$

We see that the duality relation

$$\overline{U^n}(x; \lambda) = u^n(x; \overline{\lambda}) \quad \forall x \in X, \ \ \lambda \in [0,1], \ \ 1 \leq n \leq N+1 \tag{46}$$

holds between $\mathcal{F}$ and $\mathcal{G}$.

5.2 Bellman and Zadeh's Fuzzy System

Recently Iwamoto and Sniedovich ([10]) have proposed a sequential decision process with fuzzy dynamics, which is viewed as a FDP

$$\mathcal{H} = < \text{Max},\ \{S_n\}_1^{N+1},\ \{A_n\}_1^N,\ (\{\nu_n\}_1^N,\ \wedge,\ \xi),\ (\{\mu_n\}_1^N,\ \wedge),\ (\wedge, \vee) > .$$

We see that $\mathcal{H}$ is the *mini-mini process* in fuzzy environment. It has the minimum objective ($\bullet := \wedge$) and the minimum system ($\circ := \wedge$). This process represents the fuzzy maxmization problem:

$$\begin{array}{ll} \text{Maximize} & F^\pi[\,\nu_1(x_1,u_1)\wedge\nu_2(x_2,u_2)\wedge\cdots\wedge\nu_N(x_N,u_N)\wedge\xi(x_{N+1})\,] \\ \text{subject to} & (\mathrm{i})_\mathrm{n}\quad x_{n+1}\simeq\mu_n(\cdot|\,x_n,u_n)\qquad 1\le n\le N \\ & (\mathrm{ii})_\mathrm{n}\quad u_n\in U\qquad 1\le n\le N. \end{array} \tag{47}$$

We remark that it suffices to restrict the fuzzy-like expected value to the class of all the *regular (nonaugmented) Markov* policies $\{\pi\}$ ([10]) :

$$\begin{aligned} \bigvee_{x\in X^N} \{\,&[\,\nu_1(x_1,u_1)\wedge\nu_2(x_2,u_2)\wedge\cdots\wedge\nu_N(x_N,u_N)\wedge\xi(x_{N+1})\,] \\ &\wedge[\,\mu_1(x_2|x_1,u_1)\wedge\mu_2(x_3|x_2,u_2)\wedge\cdots\wedge\mu_N(x_{N+1}|x_N,u_N)\,]\,\} \end{aligned} \tag{48}$$

where

$$u_1=\pi_1(x_1),\ u_2=\pi_2(x_2),\ \ldots,\ u_N=\pi_N(x_N).$$

The corresponding non-parametric recursive equation holds:

COROLLARY 5. 2 *(Non-parametric Recursive Equation [10])*

$$v^{N-n+1}(x) = \operatorname*{Max}_{u\in U} \bigvee_{y\in X} [\,\nu_n(x,u)\wedge(v^{N-n}(y)\wedge\mu_n(y|x,u))\,] \tag{49}$$

$$n=1,2,\cdots,N$$

$$v^0(x)=\xi(x)\qquad x\in X. \tag{50}$$

Note that Eq (49) has the regular expression :

$$v^{N-n+1}(x) = \operatorname*{Max}_{u\in U}[\,\nu_n(x,u)\wedge\bigvee_{y\in X}(v^{N-n}(y)\wedge\mu_n(y|x,u))\,]. \tag{51}$$

On the other hand, its dual FDP $\mathcal{K}$ has the following six-tuple:

$$\mathcal{K} = < \min,\ \{S_n\}_1^{N+1},\ \{A_n\}_1^N,\ (\{\overline{\nu_n}\}_1^N,\ \vee,\ \overline{\xi}),\ (\{\overline{\mu_n}\}_1^N,\ \vee),\ (\vee,\wedge) > .$$

Then $\mathcal{K}$ represents the fuzzy minimization problem:

$$\begin{array}{ll} \text{minimize} & F^\pi[\,\overline{\nu_1}(x_1,u_1)\vee\overline{\nu_2}(x_2,u_2)\vee\cdots\vee\overline{\nu_N}(x_N,u_N)\vee\overline{\xi}(x_{N+1})\,] \\ \text{subject to} & (\mathrm{i})_\mathrm{n}\quad x_{n+1}\simeq\overline{\mu_n}(\cdot|\,x_n,u_n)\qquad 1\le n\le N \\ & (\mathrm{ii})_\mathrm{n}\quad u_n\in U\qquad 1\le n\le N. \end{array} \tag{52}$$

This objective value for Markov policy π is

$$\bigwedge_{x\in X^N} \{ [\overline{\nu_1}(x_1,u_1) \vee \overline{\nu_2}(x_2,u_2) \vee \cdots \vee \overline{\nu_N}(x_N,u_N) \vee \overline{\xi}(x_{N+1})] \\ \vee[\overline{\mu_1}(x_2|x_1,u_1) \vee \overline{\mu_2}(x_3|x_2,u_2) \vee \cdots \vee \overline{\mu_N}(x_{N+1}|x_N,u_N)]\}. \tag{53}$$

The corresponding non-parametric recursive equation holds:

COROLLARY 5. 3

$$V^{N-n+1}(x) = \min_{u\in U} \bigwedge_{y\in X} [\overline{\nu_n}(x,u) \vee (V^{N-n}(y) \vee \overline{\mu_n}(y|x,u))] \tag{54}$$

$$n = 1,2,\cdots,N$$

$$V^0(x) = \overline{\xi}(x) \qquad x \in X. \tag{55}$$

We have the regular expression for Eq (54) as follows :

$$V^{N-n+1}(x) = \min_{u\in U}[\overline{\nu_n}(x,u) \vee \bigwedge_{y\in X} (V^{N-n}(y) \vee \overline{\mu_n}(y|x,u))]. \tag{56}$$

We see that the dual relation

$$\overline{V^n} = v^n \quad 1 \le n \le N+1 \tag{57}$$

is valid for $\mathcal{H}$ and $\mathcal{K}$.

5.3 Multi-multi Process in Quasi-Stochastic Environment

We consider the *multiplicative-multiplicative process* in quasi-stochastic environment. It has multiplicative objective ($\bullet := \times$) and the multiplicative system ($\circ := \times$). This process is representeted by FDP

$$\mathcal{F} = < \text{Max},\ \{S_n\}_1^{N+1},\ \{A_n\}_1^N,\ (\{\nu_n\}_1^N,\ \times,\ \xi),\ (\{\mu_n\}_1^N,\ \times),\ (\times,\sqcup) >,$$

where

$$a \sqcup b = (a+b) \wedge 1, \quad a \sqcup b \sqcup c = (a+b+c) \wedge 1.$$

Note that $\sqcup$ is not distributive over $\times$. The $\mathcal{F}$ represents the quasi-stochastic maximization problem:

$$\begin{array}{lll} \text{Maximize} & F^\sigma[\nu_1(x_1,u_1)\nu_2(x_2,u_2)\cdots\nu_N(x_N,u_N)\xi(x_{N+1})] & \\ \text{subject to} & \text{(i)}_\text{n} \quad x_{n+1} \simeq \mu_n(\cdot|\, x_n,u_n) \quad 1 \le n \le N & \\ & \text{(ii)}_\text{n} \quad u_n \in U \quad 1 \le n \le N. & \end{array} \tag{58}$$

The expected value for general policy σ is

$$\bigsqcup_{x\in X^N} \{ [\nu_1(x_1,u_1)\nu_2(x_2,u_2)\cdots\nu_N(x_N,u_N)\xi(x_{N+1})] \\ \times[\mu_1(x_2|x_1,u_1)\mu_2(x_3|x_2,u_2)\cdots\mu_N(x_{N+1}|x_N,u_N)]\}. \tag{59}$$

We note that

$$\bigsqcup_{v\in V} f(v) = (\sum_{v\in V} f(v)) \wedge 1 = (f(v_1) + f(v_2) + \cdots + f(v_k)) \wedge 1 \tag{60}$$

where $V = \{v_1, v_2, \ldots, v_k\}$. Then the corresponding two-parametric recursive equation holds:

COROLLARY 5. 4

$$u^{N-n+1}(x;\lambda,\kappa) = \operatorname*{Max}_{u\in U} \bigsqcup_{y\in X} [\, u^{N-n}(y; \lambda\nu_n(x,u), \kappa\mu_n(y|x,u))\,] \tag{61}$$

$$n = 1, 2, \cdots, N$$

$$u^0(x;\lambda) = \lambda\xi(x)\kappa \qquad \lambda, \kappa \in [0,1], \;\; x \in X. \tag{62}$$

In general, neither the non-parametric recursive equation

$$v^{N-n+1}(x) = \operatorname*{Max}_{u\in U} \bigsqcup_{y\in X} [\, \nu_n(x,u) v^{N-n}(y)\mu_n(y|x,u)\,] \tag{63}$$

nor the corresponding regular expression

$$v^{N-n+1}(x) = \operatorname*{Max}_{u\in U}[\, \nu_n(x,u) \bigsqcup_{y\in X} (v^{N-n}(y)\mu_n(y|x,u))\,] \tag{64}$$

holds.

On the other hand, the dual FDP $\mathcal{G}$ has the following six-tuple:

$$\mathcal{G} = <\min,\; \{S_n\}_1^{N+1},\; \{A_n\}_1^N,\; (\{\overline{\nu_n}\}_1^N,\; \overline{\times},\; \overline{\xi}),\; (\{\overline{\mu_n}\}_1^N,\; \overline{\times}),\; (\overline{\times}, \sqcap) >$$

where

$$a\overline{\times}b = a + b - ab$$

$$a \sqcap b = a \overline{\sqcup} b = (a + b - 1) \vee 0$$

$$a \sqcap b \sqcap c = a \overline{\sqcup} b \overline{\sqcup} c = (a + b + c - 2) \vee 0.$$

Then the $\mathcal{G}$ represents the fuzzy minimization problem:

$$\begin{array}{ll} \text{minimize} & F^{\sigma}[\,\overline{\nu_1}(x_1,u_1)\overline{\times}\overline{\nu_2}(x_2,u_2)\overline{\times}\cdots\overline{\times}\overline{\nu_N}(x_N,u_N)\overline{\times}\overline{\xi}(x_{N+1})\,] \\ \text{subject to} & (\mathrm{i})_\mathrm{n} \;\; x_{n+1} \simeq \overline{\mu_n}(\cdot|\, x_n, u_n) \quad\; 1 \le n \le N \\ & (\mathrm{ii})_\mathrm{n} \;\; u_n \in U \quad\; 1 \le n \le N. \end{array} \tag{65}$$

This objective value for general policy σ is

$$\begin{array}{l} \overline{\bigsqcup}_{x\in X^N} \{\, [\,\overline{\nu_1}(x_1,u_1)\overline{\times}\overline{\nu_2}(x_2,u_2)\overline{\times}\cdots\overline{\times}\overline{\nu_N}(x_N,u_N)\overline{\times}\overline{\xi}(x_{N+1})\,] \\ \overline{\times}[\,\overline{\mu_1}(x_2|x_1,u_1)\overline{\times}\overline{\mu_2}(x_3|x_2,u_2)\overline{\times}\cdots\overline{\times}\overline{\mu_N}(x_{N+1}|x_N,u_N)\,]\,\}. \end{array} \tag{66}$$

Here we rematk that

$$\overline{\bigsqcup}_{v\in V} g(v) = (\sum_{v\in V} g(v) - k) \vee 0 = (g(v_1) + g(v_2) + \cdots + g(v_{k+1}) - k) \vee 0 \tag{67}$$

where $k+1$ is the cardinal number of the finite set $V = \{v_1, v_2, \ldots, v_{k+1}\}$.

The corresponding two-parametric recursive equation holds:

COROLLARY 5. 5

$$U^{N-n+1}(x;\lambda,\kappa) = \min_{u\in U} \bigsqcup_{y\in X} [\,U^{N-n}(y;\lambda\overline{\times}\overline{\nu_n}(x,u), \kappa\overline{\times}\overline{\mu_n}(y|x,u))\,] \tag{68}$$

$$n = 1, 2, \cdots, N$$

$$U^0(x;\lambda,\kappa) = \lambda\overline{\times}\overline{\xi}(x)\overline{\times}\kappa \qquad \lambda,\kappa \in [0,1], \ \ x \in X. \tag{69}$$

We also see the dual relation

$$\overline{U^n}(s;\lambda,\kappa) = u^n(s;\overline{\lambda},\overline{\kappa}) \quad \forall\lambda,\kappa \in [0,1], \ \ s \in S_n, \ \ 1 \leq n \leq N+1 \tag{70}$$

is still valid for $\mathcal{F}$ and $\mathcal{G}$.

Concluding Remarks on Stochastic Environment

In this paper we have proposed *quasi-stochastic* environment by taking the multiplicative connector ($\otimes := \times$) and the bounded-additive integrator ($\oplus := \sqcup$), where $a \sqcup b = (a+b) \wedge 1$. In the same way we can define *stochastic* environment by taking the multiplicative connector ($\otimes := \times$) and the (regular) *additive* integrator ($\oplus := +$). Further we impose that the stochastic environment takes a *Markov* transition system $(\{\mu_n\}_1^N, \circ)$:

$$\circ := \times$$

$$\sum_{y\in X} \mu_n(y|x,u) = 1 \quad \forall(x,u) \in X{\times}U, \ \ n = 1, 2, \cdots, N$$

However the addition $+$ does not map $[0,1] \times [0,1]$ into $[0,1]$. This is the main reason why we have developped a duality not in the stochastic environment but in the quasi-stochastic one.

References

[1] J.F. Baldwin and B.W. Pilsworth, Dynamic programming for fuzzy systems with fuzzy environment, J. Math. Anal. Appl. **85**(1982), 1-23

[2] R.E. Bellman, *Dynamic Programming*, Princeton Univ. Press, NJ, 1957.

[3] R.E. Bellman and E.D. Denman, *Invariant Imbedding*, Lect. Notes in Operation Research and Mathematical Systems, Vol. 52, Springer-Verlag, Berlin, 1971.

[4] R.E. Bellman and L.A. Zadeh, Decision-making in a fuzzy environment, Management Sci. **17**(1970), B141-B164.

[5] A.O. Esogbue and R.E. Bellman, Fuzzy dynamic programming and its extensions, TIMS/Studies in the Management Sciences **20**(1984), 147-167.

[6] R. A. Howard, *Dynamic Programming and Markov Processes*, MIT Press, Cambridge, Mass., 1960.

[7] S. Iwamoto, Associative dynamic programs, J. Math. Anal. Appl. **201**(1996), 195-211.

[8] S. Iwamoto, Fuzzy decision processes, under consideration.

[9] S. Iwamoto and T. Fujita, Stochastic decision-making in a fuzzy environment, J. Oper. Res. Soc. Japan, **38**(1995), 467-482.

[10] S. Iwamoto and M. Sniedovich, Sequential decision-making in fuzzy environment, under preparation.

[11] S. Iwamoto, Y. Tsurusaki and T. Fujita, On Markov policies for minimax decision processes, under preparation.

[12] J. Kacprzyk, Decision-making in a fuzzy environment with fuzzy termination time, Fuzzy Sets and Systems **1**(1978), 169-179.

[13] J. Kacprzyk and A.O. Esogbue, Fuzzy dynamic programming: Main developments and applications, Fuzzy Sets and Systems **81**(1996), 31-45.

[14] J. Kacprzyk and P. Staniewski, A new approach to the control of stochastic systems in a fuzzy environment, Archiwum Automatyki i Telemechaniki **XXV**(1980), 443-444.

[15] E. S. Lee, *Quasilinearization and Invariant Imbedding*, Academic Press, New York, 1968.

[16] M. L. Puterman, *Markov Decision Processes : discrete stochastic dynamic programming*, Wiley & Sons, New York, 1994.

[17] M. Sniedovich, A class of nonseparable dynamic programming problems, J. Opt. Theo. Anal. **52**(1987), 111-121.

[18] M. Sniedovich, Analysis of a class of fractional programming problems, Math. Prog. **43**(1989), 329-347.

[19] M. Sniedovich, *Dynamic Programming*, Marcel Dekker, Inc. NY, 1992.

[20] W.E. Stein, Optimal stopping in a fuzzy environment, Fuzzy Sets and Systems. **3**(1980), 253-259.