数理解析研究所講究録
1050巻 1998年 100-106



# 余次元2ではめ込まれた多様体の多重点と同境類について

東大数理 鈴岡 啓一 ( KEIICHI SUZUOKA )

## 1. Introduction and Known Results

Throughout this paper, we will work in the smooth category.
Any immersion can be approximated by a self-transverse one. So we suppose that all immersions are self-transverse.

We will study multiple points and cobordism classes of orientable $4m$-manifolds which are immersed into $\mathbf{R}^{4m+2}$.

**Notation** :

$f : M^{k(r-1)} \looparrowright \mathbf{R}^{kr}$ is an immersion of an oriented closed $k(r-1)$-manifold in $kr$-space.$(r > 2)$

$\nu$ is the normal bundle of $f$.

$e$ is the Euler class of $\nu$.

$w_i$ is the $i$-th Stiefel-Whitney class of $M$.

$\overline{w}_i$ is the $i$-th normal Stiefel-Whitney class of $M$.

$P_i$ is the $i$-th Pontryagin class of $M$.

$\overline{P}_i$ is the $i$-th normal Pontryagin class of $M$.

$[M^{k(r-1)}]$ (resp.$[M^{k(r-1)}]_2$) is the fundamental homology class of $M^{k(r-1)}$ with $\mathbf{Z}$ (resp. $\mathbf{Z}_2$) coefficient.

$\Theta_r(f)$ is the set of r-tuple points of $f$ in $\mathbf{R}^{kr}$.

$\Delta_r(f) = f^{-1}(\Theta_r(f))$.

For $f$ is a self transverse immersion, $\Theta_r(f)$ and $\Delta_r(f)$ are finite point sets. If $k$ is even, a sign can be attached to each point in $\Theta_r(f)$ by comparing the standard orientation of $\mathbf{R}^{kr}$ with that provided by the orientation of the $r$ normal planes at that point. We attach the same sign to each point $p \in \Delta_r(f)$ as $f(p) \in \Theta_r(f)$.

**Definition 1.1.** The *algebraic number of r-tuple points* of $f$ is the number of $\Theta_r$ counted in a signed way. We write $[\Theta_r(f)]$ for it. The *algebraic number* $[\Delta_r(f)]$ is defined in the same way.

We write $[\Theta_r(f)]_2$ (resp. $[\Delta_r(f)]_2$) for the mod 2 reduction of the number of $\Theta_r(f)$ (resp. $\Delta_r(f)$).
In case $k$ is odd, however, we cannot attach a sign to an $r$-tuple point, and we do not define the algebraic number of $r$-tuple points. In this case, the only $[\Theta_r(f)]_2$ and $[\Delta_r(f)]_2$ make sense.

In [7], Herbert proved the following;

**Theorem 1.2.**

$$[\Delta_r(f)]_2 = \langle \overline{w}_k^{(r-1)}, [M^{k(r-1)}]_2 \rangle_2. \tag{1.1}$$

*In case k is even,*

$$[\Delta_r(f)] = (-1)^{r-1} \langle e^{(r-1)}, [M^{k(r-1)}] \rangle. \tag{1.2}$$

$\langle , \rangle$ (resp. $\langle , \rangle_2$ ) is the Kronecker product with $\mathbf{Z}$ (resp. $\mathbf{Z_2}$ ) coefficient.
These are very simple versions of his beautiful formulae.

By definition, it is easy to see that

$$[\Delta_r(f)] = r[\Theta_r(f)]. \tag{1.3}$$

So if $r$ is even,

$$[\Delta_r(f)]_2 = 0. \tag{1.4}$$

In [6], Felali proved the following (cf.[4]);

**Theorem 1.3.** *There exist an orientable* $2(r-1)$*-manifold* $M^{2(r-1)}$ *and an immersion* $f : M^{2(r-1)} \looparrowright \mathbf{R}^{2r}$ *with* $[\Theta_r(f)] = d$ *if and only if* $d$ *can be divided by* $(r-1)!$.

## 2. Multiple Points of Codimension 2 Immersions

In this section, we consider the case $k = 2$ and $r$ is odd ( $r = 2m+1$ ).Our aim is to prove the following theorem;

**Theorem 2.1.** *Let* $M^{4m}$ *be a closed* $4m$*-manifold and* $f : M^{4m} \looparrowright \mathbf{R}^{4m+2}$ *be an immersion. If* $M^{4m}$ *is a spin manifold (i.e.* $M^{4m}$ *is oriented and* $w_2 = 0$*), then the algebraic number of* $(2m+1)$*-tuple points of the immersion* $[\Theta_{2m+1}(f)]$ *can be divided by* $2^{2m}(2m)!$*. Moreover, if* $m$ *is odd, then* $[\Theta_{2m+1}(f)]$ *can be divided by* $2^{2m+1}(2m)!$

To prove Theorem2.1 we need two lemmas.
In [1], Atiyah and Hirzebruch proved the following lemma.

**Lemma 2.2.** *If* $M^{4m}$ *is a spin manifold ,then* $\hat{A}(M^{4m})$ *( the* $\hat{A}$*-genus of* $M^{4m}$*) is an integer. Moreover, if* $m$ *is odd, then* $\hat{A}(M^{4m})$ *is an even integer.*

The total Pontryagin class of $M^{4m}$ can be written in the form of

$$P(M^{4m}) = 1 + P_1 + P_1^2 + \cdots + P_1^m + \text{elements of order } 2, \qquad (2.1)$$

because

$$T(M^{4m}) \oplus \nu = \varepsilon^{4m+2} \qquad (2.2)$$

is the trivial $(4m+2)$-bundle.
Therefore, $\hat{A}(M^{4m})$ can be represented by the $P_1^m$ only.

The following lemma was proved in [2].

**Lemma 2.3.** *If $M^{4m}$ can be immersed into $\mathbf{R}^{4m+2}$, then*

$$\hat{A}(M^{4m}) = \frac{(-1)^m}{2^{2m}(2m+1)!}\langle P_1^m, [M^{4m}]\rangle. \tag{2.3}$$

Now we prove our main theorem.

*Proof of Theorem2.1.*

The relation between $e$ (the Euler class of $\nu$) and $\overline{P}_1$ (the first normal Pontryagin class of $M^{4m}$) is

$$e^2 = \overline{P}_1. \tag{2.4}$$

By Theorem1.2 (in this case $k = 2$ and $r = 2m+1$), we have

$$\begin{aligned}[\Delta_{2m+1}(f)] &= \langle e^{2m}, [M^{4m}]\rangle \\ &= \langle \overline{P}_1^m, [M^{4m}]\rangle \\ &= (-1)^m\langle P_1^m, [M^{4m}]\rangle.\end{aligned} \tag{2.5}$$

By (1.3)

$$[\Delta_{2m+1}(f)] = (2m+1)[\Theta_{2m+1}(f)]. \tag{2.6}$$

Thus the algebraic number of $(2m+1)$-tuple points is

$$[\Theta_{2m+1}(f)] = \frac{(-1)^m}{2m+1}\langle P_1^m, [M^{4m}]\rangle. \tag{2.7}$$

By Lemma2.2 and Lemma2.3, we can easily see that $\langle P_1^m, [M^{4m}]\rangle$ can be divided by $2^{2m}(2m+1)!$. Together with (2.7), we obtain that $[\Theta_{2m+1}(f)]$ can be divided by $2^{2m}(2m)!$. In case $m$ is odd, we can obtain the result in the same way.
This completes the proof of Theorem2.1. □

## 3. Cobordism Classes of Codimension 2 Immersed Manifolds

**Theorem 3.1.** *Let $f : M^{4m} \looparrowright \mathbf{R}^{4m+2}$ and $g : N^{4m} \looparrowright \mathbf{R}^{4m+2}$ be immersions of oriented closed $4m$-manifolds.*
*Then $M^{4m}$ and $N^{4m}$ are oriented cobordant if and only if $[\Theta_{2m+1}(f)] = [\Theta_{2m+1}(g)]$ .*
*In particular, $M^{4m}$ is oriented cobordant to $0$ if and only if $[\Theta_{2m+1}(f)] = 0$.*

*Proof of Theorem3.1.*

At first we want to show that $M^{4m}$ and $N^{4m}$ are unoriented cobordant to 0.

By (2.2), the total Stiefel-Whitney class of $M^{4m}$ is

$$w(M^{4m}) = 1 + w_2 + w_2^2 + \cdots + w_2^{2m}. \tag{3.1}$$

Thus the only non-trivial Stiefel-Whitney number of $M^{4m}$ is $\langle w_2^{2m}, [M^{4m}]_2 \rangle_2$.

For $2m+1$ is odd,

$$[\Delta_{2m+1}(f)]_2 = [\Theta_{2m+1}(f)]_2.$$

By (1.1)

$$\begin{aligned}[\Delta_{2m+1}(f)]_2 &= \langle \overline{w}_2^{2m}, [M^{4m}]_2 \rangle_2 \\ &= \langle w_2^{2m}, [M^{4m}]_2 \rangle_2.\end{aligned}$$

Theorem1.3 implies that $[\Theta_{2m+1}(f)]$ is even, so

$$\langle w_2^{2m}, [M^{4m}]_2 \rangle_2 = 0. \tag{3.2}$$

Therefore, $M^{4m}$ is unoriented cobordant to 0. And so is $N^{4m}$.

By (2.1), the only non trivial Pontryagin number of $M^{4m}$ (resp. $N^{4m}$) is $\langle P_1^m, [M^{4m}] \rangle$ (resp. $\langle P_1^m, [N^{4m}] \rangle$). Thus we can see that $M^{4m}$ and $N^{4m}$ are oriented cobordant if and only if $\langle P_1^m, [M^{4m}] \rangle = \langle P_1^m, [N^{4m}] \rangle$. By (2.7), the latter condition is equivalent to saying that the algebraic number of $(2m+1)$-tuple points of $f$ and $g$ attain the same value (i.e.$[\Theta_{2m+1}(f)] = [\Theta_{2m+1}(g)]$).

In particular, $M^{4m}$ is oriented cobordant to 0 if and only if $[\Theta_{2m+1}(f)] = 0$. This completes the proof of Theorem3.1. □

**Remark 3.2.** Stong [12] proved that if $M^n$ is an oriented closed $n$-manifold immersed in $\mathbf{R}^{n+2}$, then $M^n$ is unoriented cobordant to 0.

Moreover, he proved that if $n \not\equiv 0 \pmod 4$, then $M^n$ is oriented cobordant to 0.Here we gave a proof to the first assertion for completeness.

**Corollary 3.3.** *$M^{4m}$ is as in Theorem3.1.*
*If $M^{4m}$ satisfies the following conditions (1) or (2), then $M^{4m}$ is oriented cobordant to 0.*
*(1) $M^{4m}$ can be immersed in $\mathbf{R}^{4m+2}$ with less than $(2m)!$ $(2m+1)$-tuple points.*
*(2) There exists an integer $i$ such that $0 < i < 2m$ and $H_{2i}(M^{4m};\mathbf{Z})$ has no free part.*

*Proof of Corollary3.3.*
Case (1). $[\Theta_{2m+1}(f)]$ is divided by $(2m)!$. Thus if the number of $2m+1$-tuple points is less than $(2m)!$, then $[\Theta_{2m+1}(f)] = 0$. Therefore, $M^{4m}$ is oriented cobordant to 0 by Theorem3.1.

Case (2). If such an $i$ exists, then $e^{2m}$ is a torsion element.
Thus

$$[\Theta_{2m+1}(f)] = \frac{1}{2m+1}\langle e^{2m}, [M^{4m}]\rangle$$

must be 0.Therefore , $M^{4m}$ is oriented cobordant to 0 by Theorem3.1. □

**Remark 3.4.** If $M^{4m}$ is not oriented cobordant to 0, then the number of $(2m+1)$-tuple points is more than or equal to $(2m)!$.

**Acknowledgment.** The author would like to express sincere gratitude to Professor Yukio Matsumoto for his instructive advice and encouragement.

## References

[1] M. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281
[2] ———, *Charakteristische Klassen und Anwendungen*, Enseignement Math. **7** (1961), 188–213
[3] N. Boudriga and S. Zarati, *Points multiples isolés d'immersion de codimension* 2, C. R. Acad. Sci. Paris Ser. I Math. **296** no.14 (1983), 573–576
[4] P.J. Eccles, *Characteristic numbers of immersions and self-intersection manifolds*, Topology with Applications,(Szekszárd 1993), Bolyai Soc. Math. Stud **4** (1995), 197–216

[5] P.J. Eccles and W.P.R. Mitchell, *Triple points of immersed orientable 2n-manifolds in 3n-space*, J. London Math. Soc. **39**(2) (1989), 335–346

[6] K.S. Felali, *Intersection points of immersed manifolds*, Ph. D. thesis, University of Manchester (1982)

[7] R.J. Herbert, *Multiple points of immersed manifolds*, Memoirs AMS no.250 **34** (1981)

[8] J. F. Hughes, *Triple points of immersed 2n-manifolds in 3n-space*, Quart. J. Math. Oxford (2) **34** (1983), 427–431

[9] H.B. Lawson JR. and L. Michelson, *Clifford bundles ,immersions of manifolds and the vector field problem* , J. Differential Geom. **15** (1980), 237–267

[10] ———, *Spin Geometry* , Princeton Mathematical Series **38** (1989), Princeton University Press

[11] J.W. Milnor and J. Stasheff, *Characteristic Classes*, Ann. of Math. Studies **no.76** (1974) Princeton University Press

[12] R.E. Stong, *Manifolds which immerse in small codimension*, Illinois J. Math. **27** (1983) ,182–223

[13] J.H. White, *Twist invariants and the Pontryagin numbers of immersed manifolds*, Proc. Sympos. Pure Math. **27** (1975), 429–437



Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153 Japan