数理解析研究所講究録
1052 巻 1998 年 112-127



# MATRIX COEFFICIENTS OF THE MIDDLE DISCRETE SERIES OF $SU(2,2)$

東京大学数理科学 早田孝博 (TAKAHIRO HAYATA),
三重大学教育 古関春隆 (HARUTAKA KOSEKI),
東京大学数理科学 織田孝幸 (TAKAYUKI ODA)

## 1. Introduction

Among the discrete series representation of the non-compact real unitary group $SU(2,2)$ of signature $(2+,2-)$, there are representations whose Gel'fand-Kirillov dimensions are 5. Because other discrete series representations have Gel'fand-Kirillov dimension 6 (the large discrete series), or 4 (holomorphic or anti-holomorphic discrete series), and also because the $(\mathfrak{g},\mathfrak{k})$-cohomology of these representations have "Hodge type" $(2,2)$ (others (3,1), (1,3), (4,0), (0,4)), we call them *the middle discrete series.*

We determine the $A$-radial part of the matrix coefficients with minimal $K$-type of a representation belonging to the middle discrete series in this paper. It is written in terms of Gaussian hypergeometric series (Main Theorem 5.5). Our method of proof is a direct computation of the $A$-radial part of the Schmid operator, a gradient-type operator which characterize the minimal $K$-type vectors in the representation space of a discrete series representation (§3). The obtained operators constitute a holonomic system of 2 variables with rank 2 (§4). It is rather complicated difference-differential equations. We honestly solve this system step by step.

## 2. The group $SU(2,2)$ and its discrete series

2.1. **Structure of $SU(2,2)$ and its Lie algebra.** Let $G$ be the special unitary group $SU(2,2)$ realized as

$$G=\{g\in SL_4(\mathbb{C})\mid g^*I_{2,2}\,g=I_{2,2}\},\qquad I_{2,2}=\mathrm{diag}(1,1,-1,-1),$$

where $g^*={}^t\overline{g}$ denotes the adjoint of a matrix $g$. Let $U(n)$ be the unitary group of degree $n$. Take a maximal compact subgroup $K=G\cap U(4)=S(U(2)\times U(2))$. We denote by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebra of $G$, $K$, respectively. Let $\theta(X)=-{}^t\overline{X}$ be a Cartan involution and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$.

We set $\mathfrak{a}=\mathbb{R}H_1+\mathbb{R}H_2$ with $H_1=X_{23}+X_{32}$, $H_2=X_{14}+X_{41}$, where the $X_{ij}$'s are elementary matrices given by

$$X_{ij}=(\delta_{ip}\delta_{jq})_{1\le p,q\le 4}\quad\text{with Kronecker's delta }\delta_{ip}.$$

Then $\mathfrak{a}$ is a maximally $\mathbb{R}$-split abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Then the restricted root system $\Delta=\Delta(\mathfrak{g},\mathfrak{a})$ is expressed as

$$\Delta=\Delta(\mathfrak{g},\mathfrak{a})=\{\pm\lambda_1\pm\lambda_2,\ \pm 2\lambda_1,\ \pm 2\lambda_2\}.$$

where $\lambda_j$ is the dual of $H_j$. We choose a positive system $\Delta^+$ and a fundamental system $\Delta_{\mathrm{fund}}$ of $\Delta$:

$$\Delta^+=\{\lambda_1\pm\lambda_2,\ 2\lambda_1,\ 2\lambda_2\},\qquad \Delta_{\mathrm{fund}}=\{\lambda_1-\lambda_2,\ 2\lambda_2\}.$$

We also denote the corresponding nilpotent subalgebra by $\mathfrak{n} = \sum_{\beta \in \Delta^+} \mathfrak{g}_\beta$. Here $\mathfrak{g}_\beta$ is the root subspace of $\mathfrak{g}$ corresponding to $\beta \in \Delta^+$. Then one obtains an Iwasawa decomposition of $\mathfrak{g}$ and $G$:

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}, \quad G = NAK,$$

with $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$. Now let

$$\begin{aligned}
&E_1 = H_{13} - \sqrt{-1}X_{13} + \sqrt{-1}X_{31}, \quad E_2 = H_{24} - \sqrt{-1}X_{24} + \sqrt{-1}X_{42},\\
&E_3 = 1/2\,(X_{12} - X_{21} - X_{14} + X_{23} + X_{32} - X_{41} - X_{34} + X_{43}),\\
&E_4 = \sqrt{-1}/2\,(X_{12} + X_{21} - X_{14} - X_{23} + X_{32} + X_{41} - X_{34} - X_{43}),\\
&E_5 = 1/2\,(X_{12} - X_{21} + X_{14} + X_{23} + X_{32} + X_{41} + X_{34} - X_{43}),\\
&E_6 = \sqrt{-1}/2\,(X_{12} + X_{21} + X_{14} - X_{23} + X_{32} - X_{41} + X_{34} + X_{43}),
\end{aligned}$$

where $H_{ij} = \sqrt{-1}(X_{ii} - X_{jj})$ for $1 \le i < j \le 4$. Then it is easy to see that

$$\mathfrak{g}_{2\lambda_j} = \mathbb{R}E_j \quad (j = 1, 2), \quad \mathfrak{g}_{\lambda_1+\lambda_2} = \mathbb{R}E_3 + \mathbb{R}E_4, \quad \mathfrak{g}_{\lambda_1-\lambda_2} = \mathbb{R}E_5 + \mathbb{R}E_6.$$

2.2. **Parametrization of the discrete series.** Let us now parametrize the discrete series of $SU(2,2)$. Take a compact Cartan subalgebra $\mathfrak{t}$ defined by

$$\mathfrak{t} = \mathbb{R}\sqrt{-1}h^1 + \mathbb{R}\sqrt{-1}h^2 + \mathbb{R}\sqrt{-1}I_{2,2} \quad \text{with } h^1 = X_{11} - X_{22},\, h^2 = X_{33} - X_{44}$$

and let $\mathfrak{t}_{\mathbb{C}}$ be its complexification. Then the absolute root system, of type $A_3$, is given by

$$\widetilde{\Delta} = \widetilde{\Delta}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\,[\pm 2, 0; 0], [0, \pm 2; 0], [\pm 1, \pm 1; \pm 2]\,\}.$$

where $\beta = [r, s; u]$ means $r = \beta(h^1)$, $s = \beta(h^2)$ and $u = \beta(I_{2,2})$. We write the set of compact positive roots by $\widetilde{\Delta}_c^+ = \{\,[2, 0; 0], [0, 2; 0]\,\}$ and we fix it hereafter. The Weyl group $\widetilde{W} = \widetilde{W}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is generated by $s_1, s_2, s_3$ where

$$\begin{aligned}
&s_1[r, s; u] = [-r, s; u],\\
&s_2[r, s; u] = [(r - s + u)/2, (-r + s + u)/2; r + s],\\
&s_3[r, s; u] = [r, -s; u].
\end{aligned}$$

We identify $\widetilde{W}$ and the symmetric group $\mathfrak{S}_4$ of degree 4 by the map: $s_j \mapsto (j, j+1)$. The compact Weyl group is given by $\widetilde{W}_c = \langle s_1, s_3 \rangle$, also identified canonically with the subgroup $\mathfrak{S}_2 \times \mathfrak{S}_2$.

There are exactly six positive systems $\widetilde{\Delta}_{\mathrm{I}}^+, \widetilde{\Delta}_{\mathrm{II}}^+, \ldots, \widetilde{\Delta}_{\mathrm{VI}}^+$ containing $\widetilde{\Delta}_c^+$, defined by $\widetilde{\Delta}_J^+ = w_J \widetilde{\Delta}^+$, where

$$\widetilde{\Delta}^+ = \{\,[2, 0; 0], [0, 2; 0], [\pm 1, \pm 1; 2]\,\}$$

and the elements $w_J \in \widetilde{W}$ are given by

$$w_{\mathrm{I}} = 1,\ w_{\mathrm{II}} = s_2,\ w_{\mathrm{III}} = s_2 s_3,\ w_{\mathrm{IV}} = s_2 s_1,\ w_{\mathrm{V}} = s_2 s_3 s_1,\ w_{\mathrm{VI}} = s_2 s_1 s_3 s_2.$$

We denote by $\widetilde{\Delta}_{J,n}^+$ the noncompact positive roots in $\widetilde{\Delta}_J^+$.

By definition, the space of the Harish-Chandra parameters $\Xi_c$ is given by

$$\Xi_c = \{\Lambda \in \mathfrak{t}_{\mathbb{C}}^* \mid \Lambda \text{ is } \widetilde{\Delta}\text{-regular, } K\text{-analytically integral and } \widetilde{\Delta}_c^+\text{-dominant}\}.$$

Put $\Xi_J = \{\Lambda \in \Xi_c \mid \widetilde{\Delta}_J^+\text{-dominant}\}$. We also put $\rho_{G,J} = 2^{-1}\sum_{\beta\in\widetilde{\Delta}_J^+}\beta$, $\rho_c = 2^{-1}\sum_{\beta\in\widetilde{\Delta}_c^+}\beta$ and $\rho_{J,n} = 2^{-1}\sum_{\beta\in\widetilde{\Delta}_{J,n}^+}\beta$ the half sum of positive roots, the half sum of compact positive roots and the the half sum of noncompact positive roots, respectively. The space $\Xi_c \subset \mathfrak{t}_{\mathbb{C}}^*$ are divided into six parts: $\Xi_c = \bigcup_{\mathrm{I}\le J\le \mathrm{V}} \Xi_J$. For $\Lambda \in \bigcup_{\mathrm{I}\le J\le \mathrm{VI}} \Xi_J$, we denote the corresponding discrete series by $\pi_\Lambda$. We say that $\pi_\Lambda$ is the middle discrete series representation if $\Lambda \in \Xi_{\mathrm{III}} \cup \Xi_{\mathrm{IV}}$. Particularly in this case, we have

$$\Delta_{\mathrm{III}}^+ = \{[1,\pm1;\pm2]\},\ \rho_{\mathrm{III},n} = [2,0;0],\ \Delta_{\mathrm{IV}}^+ = \{[\pm1,1;\pm2]\},\ \rho_{\mathrm{IV},n} = [0,2;0].$$

2.3. **Representations of the maximal compact subgroup.** Let $d_1, d_2 \in \mathbb{Z}_{\ge 0}$ and $d_3 \in \mathbb{Z}$. For $d = [d_1, d_2; d_3] \in \mathfrak{t}_{\mathbb{C}}^*$, define $\tau_d \in \widehat{K}$ by the following rule ($j = 1, 2$):

$$(1)\qquad \begin{aligned} \tau_d(h^j) f_{k_1k_2}^{(d)} &= (2k_j - d_j) f_{k_1k_2}^{(d)}, & \tau_d(I_{2,2}) f_{k_1k_2}^{(d)} &= d_3 f_{k_1k_2}^{(d)},\\ \tau_d(e_+^j) f_{k_1k_2}^{(d)} &= (d_j - k_j) f_{k_1+\delta_{1j},k_2+\delta_{2j}}^{(d)}, & \tau_d(e_-^j) f_{k_1k_2}^{(d)} &= k_j f_{k_1-\delta_{1j},k_2-\delta_{2j}}^{(d)}. \end{aligned}$$

Here, $V_d = \{\, f_{k_1k_2}^{(d)} \mid 0 \le k_j \le d_j \,\}_{\mathbb{C}}$ is the standard basis (see [2, §3]) and

$$h^1,\quad h^2,\quad e_+^1 = X_{12},\quad e_+^2 = X_{34},\quad e_-^j = {}^t e_+^j$$

are the generators of $\mathfrak{k}_{\mathbb{C}}$. Then according to [2, Prop. 3.1], $\widehat{K}$ is exhausted by

$$\{(\tau_d, V_d) \mid d = [d_1, d_2; d_3],\ d_1 + d_2 + d_3 \text{ is even}\}.$$

The adjoint representation $\mathrm{Ad} = \mathrm{Ad}_{\mathfrak{p}_{\mathbb{C}}}$ of $K$ on $\mathfrak{p}_{\mathbb{C}}$ is decomposed into a direct sum of two irreducible subrepresentations: $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$, where,

$$\mathfrak{p}_+ = \mathbb{C}X_{13} + \mathbb{C}X_{14} + \mathbb{C}X_{23} + \mathbb{C}X_{24}, \qquad \mathfrak{p}_- = {}^t\mathfrak{p}_+.$$

In fact, $\mathrm{Ad}_\pm = \mathrm{Ad}|_{\mathfrak{p}_\pm}$ is isomorphic to $\tau_{[1,1;\pm2]}$, respectively. For later use, we fix the $K$-isomorphisms $\iota_\pm : \mathfrak{p}_\pm \to V_{[1,1;\pm2]}$ (write $f_{kl} = f_{kl}^{[1,1;\pm2]}$):

$$\iota_+ : (X_{23}, X_{13}, X_{24}, X_{14}) \mapsto (f_{00}, f_{10}, -f_{01}, -f_{11}),$$
$$\iota_- : (X_{41}, X_{31}, X_{42}, X_{32}) \mapsto (f_{00}, f_{01}, -f_{10}, -f_{11}), \quad \text{([2, Prop. 3.10].)}$$

The irreducible decomposition of $\mathfrak{t}_{\mathbb{C}}$-module $V_d \otimes \mathfrak{p}_{\mathbb{C}}$ is given as

$$V_d \otimes \mathfrak{p}_{\mathbb{C}} = V_d \otimes \mathfrak{p}_- \oplus V_d \otimes \mathfrak{p}_-, \quad V_d \otimes \mathfrak{p}_\pm \simeq \bigoplus_{\epsilon_1,\epsilon_2\in\{\pm1\}} V_{[r+\epsilon_1, s+\epsilon_2; u\pm2]}.$$

The projectors

$$P_{rs}^{(\epsilon_1,\epsilon_2)} : V_d \otimes \mathfrak{p}_+ \to V_{[r+\epsilon_1,s+\epsilon_2;u+2]}, \qquad \overline{P}_{rs}^{(\epsilon_1,\epsilon_2)} : V_d \otimes \mathfrak{p}_- \to V_{[r+\epsilon_1,s+\epsilon_2;u-2]},$$

are explicitly given by [2, Lemma 3.12].

2.4. ***K*-types of the middle discrete series representations.** Let $\pi_\Lambda$ be the discrete series representation of $G$ with Harish-Chandra parameter $\Lambda \in \Xi_J$. Then the Blatter parameter of $\pi_\Lambda$ becomes $\lambda = \Lambda + \rho_{G,J} - 2\rho_c$. In the following we put $d = [r, s; u]$.

If $\Lambda \in \Xi_{\mathrm{III}}$ (resp. $\Xi_{\mathrm{IV}}$), then the $K$-types $\tau_\lambda$ of $\pi_\Lambda$ are parametrized by

$$[r, s; u] \text{ with } r > s + 2 + |u|,\ r \in \mathbb{Z}_{>0},\ s \in \mathbb{Z}_{\ge0},\ r + s + u \in 2\mathbb{Z},$$
$$(\text{resp. } [r, s; u] \text{ with } s > r + 2 + |u|,\ r \in \mathbb{Z}_{\ge0},\ s \in \mathbb{Z}_{>0},\ r + s + u \in 2\mathbb{Z}.)$$

## 3. Schmid's differential operator

**3.1. $(\tau, \tau^*)$-matrix coefficient of the middle discrete series.** Let $(\pi_\Lambda, H_\Lambda)$ be a middle discrete series representation, and $(\tau_d, V_d)$ its minimal $K$-type. Put $d = [r, s; u]$. Then the contragredient representation $\tau_d^*$ of $\tau_d$ is isomorphic to $\tau_{[r,s;-u]}$. We identify the representation spaces $V_d$, $V_d^*$ with their unique images in $H_\Lambda$, $H_\Lambda^*$ respectively. Then the matrix coefficient of $\pi$ is defined by

$$\langle \pi_\Lambda(g)v, w^* \rangle$$

for $v \in V_d$, $w^* \in V_d^* \subset H_\pi^*$. Here we consider a more convenient vector-valued function:

$$\Phi_{\pi,\tau}(g) = \sum_{i,j,k,l} \langle \pi_\Lambda(g) f_{kl}^*, f_{ij}^* \rangle f_{ij} \otimes f_{kl}$$

where $\{f_{ij} = f_{ij}^{[r,s;u]}\}_{ij}$ (resp. $\{f_{kl} = f_{kl}^{[r,s;-u]}\}_{kl}$) is a standard basis of $V_d$ (resp. $V_d^*$). Then we find that $\Phi_{\pi,\tau}$ belongs to the following function space:

$$C^\infty_{\tau,\tau^*}(K\backslash G/K) = \left\{\phi\colon G \to V_d \otimes V_d{}^* \mid \phi(k_1 g k_2) = \tau_d(k_1) \otimes \tau_d^*(k_2^{-1})\phi(g), \quad k_j \in K\right\}.$$

For simplicity, we write the index $M = (i, j; k, l)$ and coefficients $c_M(g) = \langle \pi_\Lambda(g) f_{ij}^*, f_{kl}^* \rangle$.

Due to the Cartan decomposition $G = KAK$, $c_M(g)$ is determined uniquely by its restriction to $A$.

**Lemma 3.1.** *If $c_M$ is not zero, it satisfies the condition*:

$$k_1 + l_1 + k_2 + l_2 = r + s.$$

*Proof.* The centralizer of $A$ in $K$ is

$$\{m = \mathrm{diag}(u, \bar{u}\epsilon, u, \bar{u}\epsilon) \mid |u| = 1,\ \epsilon = \pm 1\}.$$

Therefore $\phi \in C^\infty_{\tau,\tau^*}(K\backslash G/K)$ satisfies

$$\phi(mam^{-1}) = \phi(a) \qquad a \in A, m \in Z_K(A),$$

which implies the assertion. □

We can construct two intertwing operators $\Phi_\pi^R$, $\Phi_{\pi^*}^L$ using the matrix coefficients:

$$\Phi_\pi^R \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda, C^\infty_{\tau_d}(K\backslash G)),$$
$$\Phi_{\pi^*}^L \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, C^\infty_{\tau_d^*}(G/K)),$$

by

$$\Phi_\pi^R(v)(g) = \sum_{ij} \langle \pi(g)v, f_{ij}^* \rangle f_{ij},$$

$$\Phi_{\pi^*}^L(w)(g) = \sum_{kl} \langle f_{kl}^*, \pi^*(g^{-1})w \rangle f_{kl}.$$

If we put

$$\Phi_{\pi,\tau}^R(g) = \sum_{kl} \Phi_\pi^R(f_{kl})(g) \otimes f_{kl},$$

$$\Phi_{\pi,\tau}^L(g) = \sum_{ij} f_{ij} \otimes \Phi_{\pi^*}^L(f_{ij})(g),$$

then $\Phi^R_{\pi,\tau}(g)$ and $\Phi^L_{\pi^*,\tau}(g)$ are identical to $\Phi_{\pi,\tau}(g)$.

3.2. **Some functions on $A$.** We put $a_i = \exp(t_i)$ for the element $a = \exp(t_1 H_1 + t_2 H_2)$ of the $\mathbb{R}$-split torus $A$, We use for notation the following symbols:

$$\mathrm{sh}(x) = (x - x^{-1})/2, \quad \mathrm{ch}(x) = (x + x^{-1})/2, \quad \mathrm{cth}(x) = \mathrm{ch}(x)/\mathrm{sh}(x),$$
$$D = D(a) = \mathrm{sh}(a_1^2) - \mathrm{sh}(a_2^2), \ p = p(a) = \mathrm{ch}(a_1)\,\mathrm{ch}(a_2), \ t = t(a) = (\mathrm{ch}(a_1)/\mathrm{ch}(a_2))^2,$$
$$z_\pm(t) = (\mathrm{ch}(a_1)/\mathrm{ch}(a_2) \pm \mathrm{ch}(a_2)/\mathrm{ch}(a_1)), \ \partial_j = a_j\frac{\partial}{\partial a_j}, \ \partial_t = t\frac{\partial}{\partial t}, \ \partial_p = p\frac{\partial}{\partial p}.$$

3.3. **The Schmid operator.** Let $\tau_{d_1}, \tau_{d_2}$ be representations of $K$. For $F(g) \in C^\infty_{d_1,d_2}(K\backslash G/K)$ and orthonormal basis $\{X_k\}$ of $\mathfrak{p}$,

$$\nabla^R_{d_1,d_2} F(g) = \sum_k R_{X_k} F(g) \otimes X_k,$$
$$\nabla^L_{d_1,d_2} F(g) = \sum_k L_{X_k} F(g) \otimes X_k,$$

are called the Schmid operator. Here $R_g$, (resp. $L_g$) is a right (resp. left) translation. Put $\mathcal{D}^{(J),R}_{d_1,d_2} = P^{(J)}_{d_2} \circ \nabla^R_{d_1,d_2}$, $\mathcal{D}^{(J),L}_{d_1,d_2} = P^{(J)}_{d_1} \circ \nabla^L_{d_1,d_2}$ with defining the projectors

$$P^{(J)}_d : V_d \otimes \mathfrak{p}_{\mathbb{C}} \to V^-_d = \bigoplus_{\beta\in\tilde{\Delta}^+_{n,J}} V_{d-\beta}.$$

**Theorem 3.2** ([7]). *Let $\Lambda \in \Xi_J$. Then,*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda, C^\infty_{\tau_d}(K\backslash G)) \simeq \ker(\mathcal{D}^{(J),R}_{d,d^*}),$$
$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi^*_\Lambda, C^\infty_{\tau^*_d}(G/K)) \simeq \ker(\mathcal{D}^{(J),L}_{d,d^*}),$$

*where $d$ is the Blattner parameter of $\pi_\Lambda$.*

We see that $\nabla^{L/R}$ is also decomposed into $\nabla^{L/R}_+ + \nabla^{L/R}_-$ along the decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$ (see [2, §6]). The following formula is found in [5]:

**Theorem 3.3** (Koseki-Oda). *Let $\nabla^{R/L}_\pm$ be the Schmid operators and $\rho_A(\nabla^{R/L}_\pm)$ their restriction to $A$. Put $Z_{13} = 2^{-1}(I_{2,2} + h^1 - h^2)$, $Z_{24} = 2^{-1}(I_{2,2} - h^1 + h^2)$ and $\tau^{(*)}_\pm = \tau^{(*)} \otimes \mathrm{Ad}_\pm$. Then, we have*

$$\begin{aligned}
\rho_A(\nabla^R_+)\phi &= \frac{1}{2}\left(\partial_1 - \mathrm{sh}(a_1^2)^{-1}\tau(Z_{13}) - \mathrm{cth}(a_1^2)\tau^*_+(Z_{13}) + 2\,\mathrm{cth}(a_1^2) + \frac{2}{D}\mathrm{sh}(a_1^2)\right)(\phi\otimes X_{13})\\
&+ \frac{1}{2}\left(\partial_2 - \mathrm{sh}(a_2^2)^{-1}\tau(Z_{24}) - \mathrm{cth}(a_2^2)\tau^*_+(Z_{24}) + 2\,\mathrm{cth}(a_2^2) - \frac{2}{D}\mathrm{sh}(a_2^2)\right)(\phi\otimes X_{24})\\
&+ \frac{1}{D}\left(\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau(e^1_-) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau(e^2_-) + \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau^*_+(e^1_-)\right.\\
&\left. + \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau^*_+(e^2_-)\right)(\phi\otimes X_{14})\\
&- \frac{1}{D}\left(\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau(e^1_+) + \mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau(e^2_+) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau^*_+(e^1_+)\right.\\
&\left. + \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau^*_+(e^2_+)\right)(\phi\otimes X_{23}),
\end{aligned}$$

$$\rho_A(\nabla^R_-)\phi = \frac{1}{2}\left(\partial_1 + \mathrm{sh}(a_1^2)^{-1}\tau(Z_{13}) + \mathrm{cth}(a_1^2)\tau_2^-(Z_{13}) + 2\,\mathrm{cth}(a_1^2) + \frac{2}{D}\mathrm{sh}(a_1^2)\right)(\phi\otimes X_{31})$$
$$+\frac{1}{2}\left(\partial_2 + \mathrm{sh}(a_2^2)^{-1}\tau(Z_{24}) + \mathrm{cth}(a_2^2)\tau_2^-(Z_{24}) + 2\,\mathrm{cth}(a_2^2) - \frac{2}{D}\mathrm{sh}(a_2^2)\right)(\phi\otimes X_{42})$$
$$-\frac{1}{D}\left(\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau(e^1_+) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau(e^2_+) + \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau_2^-(e^1_+)\right.$$
$$\left.+\,\mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau_2^-(e^2_+)\right)(\phi\otimes X_{41})$$
$$+\frac{1}{D}\left(\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau(e^1_-) + \mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau(e^2_-) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau_2^-(e^1_-)\right.$$
$$\left.+\,\mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau_2^-(e^2_-)\right)(\phi\otimes X_{32}),$$
$$\rho_A(\nabla^L_+)\phi = -\frac{1}{2}\left(\partial_1 - \mathrm{sh}(a_1^2)^{-1}\tau^*(Z_{13}) - \mathrm{cth}(a_1^2)\tau_+(Z_{13}) + 2\,\mathrm{cth}(a_1^2) + \frac{2}{D}\mathrm{sh}(a_1^2)\right)(\phi\otimes X_{13})$$
$$-\frac{1}{2}\left(\partial_2 - \mathrm{sh}(a_2^2)^{-1}\tau^*(Z_{24}) - \mathrm{cth}(a_2^2)\tau_+(Z_{24}) + 2\,\mathrm{cth}(a_2^2) - \frac{2}{D}\mathrm{sh}(a_2^2)\right)(\phi\otimes X_{24})$$
$$-\frac{1}{D}\left(\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau^*(e^1_-) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau^*(e^2_-) + \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau_+(e^1_-)\right.$$
$$\left.+\,\mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau_+(e^2_-)\right)(\phi\otimes X_{14})$$
$$+\frac{1}{D}\left(\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau^*(e^1_+) + \mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau^*(e^2_+) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau_+(e^1_+)\right.$$
$$\left.+\,\mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau_+(e^2_+)\right)(\phi\otimes X_{23}),$$
$$\rho_A(\nabla^L_-)\phi = -\frac{1}{2}\left(\partial_1 + \mathrm{sh}(a_1^2)^{-1}\tau^*(Z_{13}) + \mathrm{cth}(a_1^2)\tau^-(Z_{13}) + 2\,\mathrm{cth}(a_1^2) + \frac{2}{D}\mathrm{sh}(a_1^2)\right)(\phi\otimes X_{31})$$
$$-\frac{1}{2}\left(\partial_2 + \mathrm{sh}(a_2^2)^{-1}\tau^*(Z_{24}) + \mathrm{cth}(a_2^2)\tau^-(Z_{24}) + 2\,\mathrm{cth}(a_2^2) - \frac{2}{D}\mathrm{sh}(a_2^2)\right)(\phi\otimes X_{42})$$
$$+\frac{1}{D}\left(\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau^*(e^1_+) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau^*(e^2_+) + \mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau^-(e^1_+)\right.$$
$$\left.+\,\mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau^-(e^2_+)\right)(\phi\otimes X_{41})$$
$$-\frac{1}{D}\left(\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)\tau^*(e^1_-) + \mathrm{ch}(a_1)\,\mathrm{sh}(a_2)\tau^*(e^2_-) + \mathrm{sh}(a_1)\,\mathrm{ch}(a_1)\tau^-(e^1_-)\right.$$
$$\left.+\,\mathrm{sh}(a_2)\,\mathrm{ch}(a_2)\tau^-(e^2_-)\right)(\phi\otimes X_{32}).$$

## 4. Holonomic system for the spherical functions

We treat the case of $\Lambda \in \Xi_{\mathrm{III}} \cup \Xi_{\mathrm{IV}}$. Then, the Blattner parameter of $\pi_\Lambda$ in $\Lambda \in \Xi_{\mathrm{III}}$ (resp. $\Lambda \in \Xi_{\mathrm{IV}}$) is $d = \Lambda + [1,-1;0]$ (resp. $\Lambda + [-1,1;0]$).

**Lemma 4.1.** *The projector $P_d^{(\mathrm{III})}$ decomposes into four projectors as follows*:

$$P_d^{(\mathrm{III})} = P^{(-,+)} \oplus P^{(-,-)} \oplus \overline{P}^{(-,+)} \oplus \overline{P}^{(-,-)},$$
$$P_d^{(\mathrm{IV})} = P^{(+,-)} \oplus P^{(-,-)} \oplus \overline{P}^{(+,-)} \oplus \overline{P}^{(-,-)}.$$

*Proof.* We find that

$$\widetilde{\Delta}^+_{n,\mathrm{III}} = \{[1,1;\pm 2],\ [1,-1;\pm 2]\},\ \ \widetilde{\Delta}^+_{n,\mathrm{IV}} = \{[1,1;\pm 2],\ [-1,1;\pm 2]\}$$

Thus the lemma follows. □

According to Theorem 3.2, spherical functions are characterized by the differential equations derived by the composition of the Schmid operator and projectors which appears in the decomposition of $P^{(\mathrm{III})}_d$. Let $\Phi_{\pi_\Lambda,\tau_d}(a) = \sum_M c_M(a) f_{k_1,l_1} \otimes f_{k_2,l_2}$ for $M = (k_1, l_1; k_2, l_2)$. Then $c_M$'s satisfy the following system which is equivalent to $\mathcal{D}^{(\mathrm{III}),\,R/L}_{d_1,d_2} \Phi_{\pi_\Lambda,\tau_d} = 0$:

**Lemma 4.2.**

$$(2)\quad \begin{aligned}&(r_2-k_2)\{\partial_1 - \frac{1}{2}(u_1+2k_1-r_1-2l_1+s_1)\frac{1}{\mathrm{sh}(a_1^2)}\\&-\frac{1}{2}(u_2+2k_2-r_2-2l_2+s_2)\,\mathrm{cth}(a_1^2) + (k_2+1)\frac{\mathrm{sh}(a_1^2)}{D}\}c_{k_1,l_1;k_2,l_2}\\&+(k_2+1)(s_2-l_2+1)\frac{\mathrm{sh}(a_2^2)}{D}c_{k_1,l_1;k_2+1,l_2-1}\\&+2(k_2+1)(r_1-k_1+1)\frac{\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}\\&+2(k_2+1)(s_1-l_1+1)\frac{\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2} = 0,\end{aligned}$$

$$(3)\quad \begin{aligned}&(k_2+1)\{\partial_2 - \frac{1}{2}(u_1-2k_1+r_1+2l_1-s_1)\frac{1}{\mathrm{sh}(a_2^2)}\\&-\frac{1}{2}(u_2-2k_2+r_2+2l_2-s_2-4)\,\mathrm{cth}(a_2^2) - (r_2-k_2)\frac{\mathrm{sh}(a_2^2)}{D}\}c_{k_1,l_1;k_2+1,l_2-1}\\&-(r_2-k_2)l_2\frac{\mathrm{sh}(a_1^2)}{D}c_{k_1,l_1;k_2,l_2}\\&-2(r_2-k_2)(k_1+1)\frac{\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1}\\&-2(r_2-k_2)(l_1+1)\frac{\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)}{D}c_{k_1,l_1+1;k_2,l_2-1} = 0,\end{aligned}$$

$$(4)\quad \begin{aligned}&(k_2+1)\{\partial_1 + \frac{1}{2}(u_1+2k_1-r_1-2l_1+s_1)\frac{1}{\mathrm{sh}(a_1^2)}\\&+\frac{1}{2}(u_2+2k_2-r_2-2l_2+s_2+4)\,\mathrm{cth}(a_1^2) + (r_2-k_2)\frac{\mathrm{sh}(a_1^2)}{D}\}c_{k_1,l_1;k_2+1,l_2-1}\\&+(r_2-k_2)l_2\frac{\mathrm{sh}(a_2^2)}{D}c_{k_1,l_1;k_2,l_2}\\&+2(r_2-k_2)(k_1+1)\frac{\mathrm{sh}(a_1)\,\mathrm{ch}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1}\\&+2(r_2-k_2)(l_1+1)\frac{\mathrm{ch}(a_1)\,\mathrm{sh}(a_2)}{D}c_{k_1,l_1+1;k_2,l_2-1} = 0,\end{aligned}$$

$$(5)\quad (r_2-k_2)\{\partial_2 + \frac{1}{2}(u_1-2k_1+r_1+2l_1-s_1)\frac{1}{\mathrm{sh}(a_2^2)}$$

$$+\frac{1}{2}(u_2-2k_2+r_2+2l_2-s_2)\operatorname{cth}(a_2^2)-(k_2+1)\frac{\operatorname{sh}(a_2^2)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$-(k_2+1)(s_2-l_2+1)\frac{\operatorname{sh}(a_1^2)}{D}c_{k_1,l_1;k_2+1,l_2-1}$$
$$-2(k_2+1)(r_1-k_1+1)\frac{\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}$$
$$-2(k_2+1)(s_1-l_1+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2}=0.$$

*As for left equation systems, we have the following system*:

$$(6)\quad (r_1-k_1)\{\partial_1-\frac{1}{2}(u_2+2k_2-r_2-2l_2+s_2)\frac{1}{\operatorname{sh}(a_1^2)}$$
$$-\frac{1}{2}(u_1+2k_1-r_1-2l_1+s_1)\operatorname{cth}(a_1^2)+(k_1+1)\frac{\operatorname{sh}(a_1^2)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$+(k_1+1)(s_1-l_1+1)\frac{\operatorname{sh}(a_2^2)}{D}c_{k_1+1,l_1-1;k_2,l_2}$$
$$+2(k_1+1)(r_2-k_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}$$
$$+2(k_1+1)(s_2-l_2+1)\frac{\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1}=0,$$

$$(7)\quad (k_1+1)\{\partial_2-\frac{1}{2}(u_2-2k_2+r_2+2l_2-s_2)\frac{1}{\operatorname{sh}(a_2^2)}$$
$$-\frac{1}{2}(u_1-2k_1+r_1+2l_1-s_1-4)\operatorname{cth}(a_2^2)-(r_1-k_1)\frac{\operatorname{sh}(a_2^2)}{D}\}c_{k_1+1,l_1-1;k_2,l_2}$$
$$-(r_1-k_1)l_1\frac{\operatorname{sh}(a_1^2)}{D}c_{k_1,l_1;k_2,l_2}$$
$$-2(r_1-k_1)(k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2}$$
$$-2(r_1-k_1)(l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1,l_1-1;k_2,l_2+1}=0,$$

$$(8)\quad (k_1+1)\{\partial_1+\frac{1}{2}(u_2+2k_2-r_2-2l_2+s_2)\frac{1}{\operatorname{sh}(a_1^2)}$$
$$+\frac{1}{2}(u_1+2k_1-r_1-2l_1+s_1+4)\operatorname{cth}(a_1^2)+(r_1-k_1)\frac{\operatorname{sh}(a_1^2)}{D}\}c_{k_1+1,l_1-1;k_2,l_2}$$
$$+(r_1-k_1)l_1\frac{\operatorname{sh}(a_2^2)}{D}c_{k_1,l_1;k_2,l_2}$$
$$+2(r_1-k_1)(k_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2}$$
$$+2(r_1-k_1)(l_2+1)\frac{\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1,l_1-1;k_2,l_2+1}=0,$$

$$(9)\quad (r_1-k_1)\{\partial_2+\frac{1}{2}(u_2-2k_2+r_2+2l_2-s_2)\frac{1}{\operatorname{sh}(a_2^2)}$$

$$+\frac{1}{2}(u_1-2k_1+r_1+2l_1-s_1)\operatorname{cth}(a_2^2)-(k_1+1)\frac{\operatorname{sh}(a_2^2)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$-(k_1+1)(s_1-l_1+1)\frac{\operatorname{sh}(a_1^2)}{D}c_{k_1+1,l_1-1;k_2,l_2}$$
$$-2(k_1+1)(r_2-k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}$$
$$-2(k_1+1)(s_2-l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1}=0.$$

4.1. **Going up/down equations.** We can reduce the obtained equations to the following going up system (10), (11), (12), (13) as follows:

**Lemma 4.3.**

(10)
$$\begin{aligned}&(r-k_2)\{\operatorname{cth}(a_1)\partial_1-(s-l_1-l_2)\operatorname{cth}^2(a_1)\\&\quad-\frac{1}{2}(-u-k_1+k_2+l_1-l_2)+2(k_2+1)\frac{\operatorname{ch}^2(a_1)}{D}\}c_{k_1,l_1;k_2,l_2}\\&\quad+2(k_2+1)(r-k_1+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}\\&\quad=-2(k_2+1)(s-l_2+1)\frac{\operatorname{cth}(a_1)\operatorname{sh}(a_2)\operatorname{ch}(a_2)}{D}c_{k_1,l_1;k_2+1,l_2-1}\\&\quad-2(k_2+1)(s-l_1+1)\frac{\operatorname{cth}(a_1)\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2},\end{aligned}$$

(11)
$$\begin{aligned}&(r-k_2)\{\operatorname{cth}(a_2)\partial_2-(s-l_1-l_2)\operatorname{cth}^2(a_2)\\&\quad-\frac{1}{2}(u-k_1+k_2+l_1-l_2)-2(k_2+1)\frac{\operatorname{ch}^2(a_2)}{D}\}c_{k_1,l_1;k_2,l_2}\\&\quad-2(k_2+1)(r-k_1+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}\\&\quad=2(k_2+1)(s-l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_1)\operatorname{cth}(a_2)}{D}c_{k_1,l_1;k_2+1,l_2-1}\\&\quad+2(k_2+1)(s-l_1+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)\operatorname{cth}(a_2)}{D}c_{k_1,l_1-1;k_2+1,l_2},\end{aligned}$$

(12)
$$\begin{aligned}&(r-k_1)\{\operatorname{cth}(a_1)\partial_1-(s-l_1-l_2)\operatorname{cth}^2(a_1)\\&\quad-\frac{1}{2}(u+k_1-k_2-l_1+l_2)+2(k_1+1)\frac{\operatorname{ch}^2(a_1)}{D}\}c_{k_1,l_1;k_2,l_2}\\&\quad+2(k_1+1)(r-k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}\\&\quad=-2(k_1+1)(s-l_1+1)\frac{\operatorname{cth}(a_1)\operatorname{sh}(a_2)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1-1;k_2,l_2}\\&\quad-2(k_1+1)(s-l_2+1)\frac{\operatorname{cth}(a_1)\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1},\end{aligned}$$

(13)
$$\begin{aligned}&(r-k_1)\{\operatorname{cth}(a_2)\partial_2-(s-l_1-l_2)\operatorname{cth}^2(a_2)\\&\quad-\frac{1}{2}(-u+k_1-k_2-l_1+l_2)-2(k_1+1)\frac{\operatorname{ch}^2(a_2)}{D}\}c_{k_1,l_1;k_2,l_2}\end{aligned}$$

$$- 2(k_1+1)(r-k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}$$
$$= 2(k_1+1)(s-l_1+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_1)\operatorname{cth}(a_2)}{D}c_{k_1+1,l_1-1;k_2,l_2}$$
$$+ 2(k_1+1)(s-l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)\operatorname{cth}(a_2)}{D}c_{k_1+1,l_1;k_2,l_2-1}.$$

*Going down equations are as follows*:

(14)
$$k_2\{\operatorname{cth}(a_1)\partial_1 + (s-l_1-l_2)\operatorname{cth}^2(a_1)$$
$$+ \frac{1}{2}(-u-k_1+k_2+l_1-l_2) + 2(r-k_2+1)\frac{\operatorname{ch}^2(a_1)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$+ 2(r-k_2+1)(k_1+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}$$
$$= -2(r-k_2+1)(l_2+1)\frac{\operatorname{cth}(a_1)\operatorname{sh}(a_2)\operatorname{ch}(a_2)}{D}c_{k_1,l_1;k_2-1,l_2+1}$$
$$- 2(r-k_2+1)(l_1+1)\frac{\operatorname{cth}(a_1)\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1,l_1+1;k_2-1,l_2},$$

(15)
$$k_2\{\operatorname{cth}(a_2)\partial_2 + (s-l_1-l_2)\operatorname{cth}^2(a_2)$$
$$+ \frac{1}{2}(u-k_1+k_2+l_1-l_2) - 2(r-k_2+1)\frac{\operatorname{ch}^2(a_2)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$- 2(r-k_2+1)(k_1+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1+1,l_1;k_2-1,l_2}$$
$$= 2(r-k_2+1)(l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_1)\operatorname{cth}(a_2)}{D}c_{k_1,l_1;k_2-1,l_2+1}$$
$$+ 2(r-k_2+1)(l_1+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)\operatorname{cth}(a_2)}{D}c_{k_1,l_1+1;k_2-1,l_2},$$

(16)
$$k_1\{\operatorname{cth}(a_1)\partial_1 + (s-l_1-l_2)\operatorname{cth}^2(a_1)$$
$$+ \frac{1}{2}(u+k_1-k_2-l_1+l_2) + 2(r-k_1+1)\frac{\operatorname{ch}^2(a_1)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$+ 2(r-k_1+1)(k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}$$
$$= -2(r-k_1+1)(l_1+1)\frac{\operatorname{cth}(a_1)\operatorname{sh}(a_2)\operatorname{ch}(a_2)}{D}c_{k_1-1,l_1+1;k_2,l_2}$$
$$- 2(r-k_1+1)(l_2+1)\frac{\operatorname{cth}(a_1)\operatorname{ch}(a_1)\operatorname{sh}(a_2)}{D}c_{k_1-1,l_1;k_2,l_2+1},$$

(17)
$$k_1\{\operatorname{cth}(a_2)\partial_2 + (s-l_1-l_2)\operatorname{cth}^2(a_2)$$
$$+ \frac{1}{2}(-u+k_1-k_2-l_1+l_2) - 2(r-k_1+1)\frac{\operatorname{ch}^2(a_2)}{D}\}c_{k_1,l_1;k_2,l_2}$$
$$- 2(r-k_1+1)(k_2+1)\frac{\operatorname{ch}(a_1)\operatorname{ch}(a_2)}{D}c_{k_1-1,l_1;k_2+1,l_2}$$
$$= 2(r-k_1+1)(l_1+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_1)\operatorname{cth}(a_2)}{D}c_{k_1-1,l_1+1;k_2,l_2}$$

$$+ 2(r-k_1+1)(l_2+1)\frac{\operatorname{sh}(a_1)\operatorname{ch}(a_2)\operatorname{cth}(a_2)}{D}c_{k_1-1,l_1;k_2,l_2+1}.$$

To make equations more "symmetric", we consider (10) $\pm$ (11), etc, and rewrite them using $p$ and $t$. Put

$$c_{k_1,l_1;k_2,l_2}(a) = (\operatorname{sh}(a_1)\operatorname{sh}(a_2))^{|s-l_1-l_2|}(\operatorname{ch}(a_1)\operatorname{ch}(a_2))^{-(r+s+2)/2}\tilde{c}_{k_1,l_1;k_2,l_2}(a).$$

In the following, we assume that $0 \leq l_1 + l_2 \leq s$. We remark that

$$\begin{aligned} 2\partial_p &= \operatorname{cth}(a_1)\partial_1 + \operatorname{cth}(a_2)\partial_2, \\ 4\partial_t &= \operatorname{cth}(a_1)\partial_1 - \operatorname{cth}(a_2)\partial_2. \end{aligned}$$

Then, we have

**Lemma 4.4.**

$$(18) \qquad \begin{aligned} &(r-k_2)(\partial_p - l_1)\tilde{c}_{k_1,l_1;k_2,l_2} = (k_2+1)(s-l_2+1)p\tilde{c}_{k_1,l_1;k_2+1,l_2-1} \\ &+ (k_2+1)(s-l_1+1)\tilde{c}_{k_1,l_1-1;k_2+1,l_2}, \end{aligned}$$

$$(19) \qquad \begin{aligned} &(r-k_2)\left(2\partial_t + \frac{u}{2} + (k_2+1)\frac{t+1}{t-1}\right)\tilde{c}_{k_1,l_1;k_2,l_2} \\ &+ 2(k_2+1)(r-k_1+1)z_-(t)^{-1}\tilde{c}_{k_1-1,l_1;k_2+1,l_2} \\ &= (k_2+1)(s-l_2+1)z_-(t)^{-1}(2-pz_+)\tilde{c}_{k_1,l_1;k_2+1,l_2-1} \\ &+ (k_2+1)(s-l_1+1)z_-(t)^{-1}(z_+ - 2p)\tilde{c}_{k_1,l_1-1;k_2+1,l_2}, \end{aligned}$$

$$(20) \qquad \begin{aligned} &k_2\big\{(p^2 - z_+(t)p+1)(\partial_p + l_1 - s) \\ &+ (s-l_1-l_2)(2p - z_+(t))p\big\}\tilde{c}_{k_1,l_1;k_2,l_2} \\ &= (r-k_2+1)(l_2+1)p\tilde{c}_{k_1,l_1;k_2-1,l_2+1} \\ &+ (r-k_2+1)(l_1+1)\tilde{c}_{k_1,l_1+1;k_2-1,l_2}, \end{aligned}$$

$$(21) \qquad \begin{aligned} &(r-k_1)(\partial_p - l_2)\tilde{c}_{k_1,l_1;k_2,l_2} = (k_1+1)(s-l_1+1)p\tilde{c}_{k_1+1,l_1-1;k_2,l_2} \\ &+ (k_1+1)(s-l_2+1)\tilde{c}_{k_1+1,l_1;k_2,l_2-1}, \end{aligned}$$

$$(22) \qquad \begin{aligned} &(r-k_1)\left(2\partial_t - \frac{u}{2} + (k_1+1)\frac{t+1}{t-1}\right)\tilde{c}_{k_1,l_1;k_2,l_2} \\ &+ 2(k_1+1)(r-k_2+1)z_-(t)^{-1}\tilde{c}_{k_1+1,l_1;k_2-1,l_2} \\ &= (k_1+1)(s-l_1+1)z_-(t)^{-1}(2-pz_+)\tilde{c}_{k_1+1,l_1-1;k_2,l_2} \\ &+ (k_1+1)(s-l_2+1)z_-(t)^{-1}(z_+ - 2p)\tilde{c}_{k_1+1,l_1;k_2,l_2-1}, \end{aligned}$$

$$(23) \qquad \begin{aligned} &k_1\big\{(p^2 - z_+(t)p+1)(\partial_p + l_2 - s) \\ &+ (s-l_1-l_2)(2p - z_+(t))p\big\}\tilde{c}_{k_1,l_1;k_2,l_2} \\ &= (r-k_1+1)(l_1+1)p\tilde{c}_{k_1-1,l_1+1;k_2,l_2} \\ &+ (r-k_1+1)(l_2+1)\tilde{c}_{k_1-1,l_1;k_2,l_2+1}. \end{aligned}$$

As we know, the equations (21), (22) and (23) can be obtained by flipping indices 1 and 2:

*Remark* 4.5. We have similar equations when $s \leq l_1 + l_2 \leq 2s$.

## 5. Solution for the holonomic system: the main theorem

5.1. **Separation of variables.** We treat the case when $l_1 + l_2 \leq s$.

**Proposition 5.1.** *Write $M = (k_1, l_1; k_2, l_2)$. Then $\tilde{c}_M$ can be written in the form of "separation of variables":*

$$\tilde{c}_M(a) = \sum_{i=0}^{l_1+l_2} (-1)^{r-k_1-l_1} \binom{r}{k_1}\binom{r}{k_2} p^{l_1+l_2-i} s_{M,i}(t).$$

We can prove it by induction on $l_1 + l_2$. Assume that $l_1 = l_2 = 0$. By (18), we have,

$$\partial_p \tilde{c}_{k_1,0;k_2,0} = 0,$$

so that actually we can put $s_{(k_1,0;k_2,0),0}(t) := \tilde{c}_{k_1,0;k_2,0}(a)$. Next assume that $l_1 + l_2 > 0$. If $l_1 \neq l_2$ and $k_1 < r$, $k_2 < r$, then $(18)/(r-k_2) - (21)/(r-k_1)$ shows the assertion. Otherwise, we can assume $k_2 \neq r$. Consulting (18), we readily prove the formula.

According to Proposition 5.1, we can rewrite the difference equations of Lemma 4.4 in terms of $p$ and $t$. Comparing the coefficients as a polynomial of $p$, we have the following.

**Lemma 5.2.** 1. *If $0 \leq k_2 < r$, then,*

$$(24)\quad (l_2 - i) s_{(k_1,l_1;k_2,l_2),i} = (s - l_2 + 1) s_{(k_1,l_1;k_2+1,l_2-1),i} - (s - l_1 + 1) s_{(k_1,l_1-1;k_2+1,l_2),i-1},$$

$$(25)\quad \left(2\partial_t + \frac{u}{2} + (k_2+1)\frac{t+1}{t-1}\right) s_{(k_1,l_1;k_2,l_2),i} - \frac{2k_1}{z_-(t)} s_{(k_1-1,l_1;k_2+1,l_2),i}$$
$$= (s - l_2 + 1)\left(\frac{2}{z_-(t)} s_{(k_1,l_1;k_2+1,l_2-1),i-1} - \frac{t+1}{t-1} s_{(k_1,l_1;k_2+1,l_2-1),i}\right)$$
$$- (s - l_1 + 1)\left(\frac{t+1}{t-1} s_{(k_1,l_1-1;k_2+1,l_2),i-1} - \frac{2}{z_-(t)} s_{(k_1,l_1-1;k_2+1,l_2),i}\right),$$

$$(26)\quad l_2 s_{(k_1,l_1;k_2,l_2),i+1} - (l_1+1) s_{(k_1,l_1+1;k_2,l_2-1),i}$$
$$= (s - l_2 - i) s_{(k_1,l_1;k_2+1,l_2-1),i+1} - (l_1 - i) z_+(t) s_{(k_1,l_1;k_2+1,l_2-1),i}$$
$$+ (2l_1 + l_2 - s - i) s_{(k_1,l_1;k_2+1,l_2-1),i-1}.$$

2. *If $0 \leq k_1 < r$, then,*

$$(27)\quad (l_1 - i) s_{(k_1,l_1;k_2,l_2),i} = (s - l_1 + 1) s_{(k_1+1,l_1-1;k_2,l_2),i} - (s - l_2 + 1) s_{(k_1+1,l_1;k_2,l_2-1),i-1},$$

$$(28)\quad \left(2\partial_t - \frac{u}{2} + (k_1+1)\frac{t+1}{t-1}\right) s_{(k_1,l_1;k_2,l_2),i} - \frac{2k_2}{z_-(t)} s_{(k_1+1,l_1;k_2-1,l_2),i}$$
$$= (s - l_1 + 1)\left(\frac{2}{z_-(t)} s_{(k_1+1,l_1-1;k_2,l_2),i-1} - \frac{t+1}{t-1} s_{(k_1+1,l_1-1;k_2,l_2),i}\right)$$
$$- (s - l_2 + 1)\left(\frac{t+1}{t-1} s_{(k_1+1,l_1;k_2,l_2-1),i-1} - \frac{2}{z_-(t)} s_{(k_1+1,l_1;k_2,l_2-1),i}\right),$$

$$(29)\quad l_1 s_{(k_1,l_1;k_2,l_2),i+1} - (l_2+1) s_{(k_1,l_1-1;k_2,l_2+1),i}$$
$$= (s - l_1 - i) s_{(k_1+1,l_1-1;k_2,l_2),i+1} - (l_2 - i) z_+(t) s_{(k_1+1,l_1-1;k_2,l_2),i}$$
$$+ (2l_2 + l_1 - s - i) s_{(k_1+1,l_1-1;k_2,l_2),i-1}.$$

5.2. **Expression of peripheral entries using Gaussian hypergeometric functions.** First assume that $l_1 = l_2 = 0$. We simply write $s_{k_1,k_2} = s_{(k_1,0;k_2,0),0}$. By (19) and (22), we have

$$(r-k_2)\left(\partial_t + \frac{u}{4} + \frac{k_2+1}{2}\frac{t+1}{t-1}\right) s_{k_1;k_2} + \frac{(k_2+1)(r-k_1+1)}{z_-(t)} s_{k_1-1;k_2+1} = 0,$$
$$(r-k_1+1)\left(\partial_t - \frac{u}{4} + \frac{k_1}{2}\frac{t+1}{t-1}\right) s_{k_1-1;k_2+1} + \frac{k_1(r-k_2)}{z_-(t)} s_{k_1;k_2} = 0.$$

Eliminating $s_{k_1-1;k_2+1}$, we have

$$\left\{\left(\partial_t - \frac{u}{4} + \frac{k_1+1}{2}\frac{t+1}{t-1}\right)\left(\partial_t + \frac{u}{4} + \frac{k_2+1}{2}\frac{t+1}{t-1}\right) - k_1(k_2+1)z_-(t)^{-2}\right\} s_{k_1;k_2} = 0.$$

Considering $r+s = k_1+k_2$, we have

$$\left(\partial_t^2 + \frac{r+s+2}{2}\frac{t+1}{t-1}\partial_t + \frac{u(k_1-k_2)}{8}\frac{t+1}{t-1} + \frac{(r+s+2)^2-(k_1-k_2)^2-u^2}{16}\right) s_{k_1;k_2} = 0$$

and its Riemann's $P$ scheme is:

$$P\begin{bmatrix} 0 & 1 & \infty \\ \frac{r+s+2}{4} - \frac{k_1-k_2+u}{4} & 0 & \frac{r+s+2}{4} + \frac{k_1-k_2-u}{4} \\ \frac{r+s+2}{4} + \frac{k_1-k_2+u}{4} & -(r+s+1) & \frac{r+s+2}{4} - \frac{k_1-k_2-u}{4} \end{bmatrix}.$$

In general, let $\Phi(m_1,m_2) = \Phi(m_1,m_2;u;t)$ be a regular function around 1 having the $P$-scheme

$$P\begin{bmatrix} 0 & 1 & \infty \\ \frac{m_1+m_2+2}{4} - \frac{m_1-m_2+u}{4} & 0 & \frac{m_1+m_2+2}{4} - \frac{m_1-m_2-u}{4} \\ \frac{m_1+m_2+2}{4} + \frac{m_1-m_2+u}{4} & -(m_1+m_2+1) & \frac{m_1+m_2+2}{4} + \frac{m_1-m_2-u}{4} \end{bmatrix}$$

with condition $\Phi(m_1,m_2;u;1) = \binom{r+s}{m_1}^{-1}$. We also write $\Phi(m) = \Phi(m, r+s-m)$ for simplicity. Then it follows $s_{(k_1,0;k_2,0),0} = c_0\Phi(k_1,k_2)$.

5.3. **Reduction of general coefficients $s_{M,i}$.** To describe general solutions, we introduce the notion of height and bias. Write $M = (k_1,l_1;k_2,l_2)$ as before. Define $h = h(M,i) = \min(i, l_1, l_2, l_1+l_2-i)$ and

$$b = b(M,i) = \begin{cases} 0 & i \le \min(l_1,l_2), \\ \operatorname{sgn}(l_2-l_1)(i-\min(l_1,l_2)) & \min(l_1,l_2) \le i \le \max(l_1,l_2), \\ l_2 - l_1 & \max(l_1,l_2) \le i. \end{cases}$$

Then we have,

**Proposition 5.3.**

$$s_{M,i} = \sum_{j=b-h}^{b+h} Q_j(-z_+)\Phi(k_1+l_1+j) \tag{30}$$

*for a polynomial $Q_j(t) = Q_j(M,i;t)$ which is actually independent of the choice of $r$, $k_1$ and $k_2$. The degree of $Q_j$ is equal to $h - |j-b|$ and it follows*

$$Q_j(-z_+) = (-1)^{\deg Q_j} Q_j(z_+).$$

5.4. **Polynomials $Q_j(z_+)$.** The remaining paper deals with the determination of the polynomial $Q_j$. We can deduce the difference equations of $Q_j$ equivalent to (26). Proposition 5.3 says that $Q_j$ is in the form

$$Q_j(z_+) = \sum_{m \geq 0} \tilde{\beta}_m(M,i,j) z_+^{h-|j|-2m}.$$

For simplicity, we put $\tilde{\beta}_m(M,i,j) = \binom{s}{l_1}\binom{s}{l_2}\beta_m(M,i,j)$. Comparing the coefficient of $z_+^{h-|j|-2m}$, we see that our difference equations become as follows: If $j \geq 0$, then,

$$\begin{aligned}(31)\qquad (s-i+1)i\beta_m(M,i,j) &= (s-l_1)l_2\beta_m(M+(0,1;0,-1),i-1,j-1)\\ &+ l_1(s-l_2-i+1)\beta_{m-1}(M+(0,-1;1,0),i-1,j+1)\\ &+ (l_1-i+1)l_2\beta_m(M+(0,0;1,-1),i-1,j)\\ &+ (2l_1+l_2-s-i+1)l_2\beta_{m-1}(M+(0,0;1,-1),i-2,j).\end{aligned}$$

If $j < 0$, then,

$$\begin{aligned}(32)\qquad (s-i+1)i\beta_m(M,i,j) &= (s-l_1)l_2\beta_{m-1}(M+(0,1;0,-1),i-1,j-1)\\ &+ l_1(s-l_2-i+1)\beta_m(M+(0,-1;1,0),i-1,j+1)\\ &+ (l_1-i+1)l_2\beta_m(M+(0,0;1,-1),i-1,j)\\ &+ (2l_1+l_2-s-i+1)l_2\beta_{m-1}(M+(0,0;1,-1),i-2,j).\end{aligned}$$

The solution can be expressed as follows:

**Proposition 5.4.** *Assume that* $0 \leq l_1 + l_2 \leq s$. *Then,*

$$\beta_m = \alpha(m;i,|j|)\binom{l_1}{i-j_+-m}\binom{l_2}{i+j_--m}\sum_{n=0}^{m}\binom{s-l_1}{j+m-n}\binom{s-l_2}{m-n}\binom{s-i+n}{n}$$

*for*

$$\alpha(m;i,j) = \binom{i-j-m}{m}\binom{i}{m}^{-1}\binom{s}{i}^{-1}, \quad j_+ = \begin{cases} j & (j \geq 0),\\ 0 & (j < 0)\end{cases}$$

*and* $j_- = (-j)_+$.

We can check that each $\beta_m(M,i,j)$ fits the definition of $h$ as $\beta_m(M,i,j)$ is nonzero if and only if $i - |j| - 2m \geq 0$, $l_1 - i + j_+ + m \geq 0$ and $l_2 - i - j_- + m \geq 0$.

**Main Theorem 5.5.** *Let $\pi_\Lambda$ be a middle discrete series representation with $\Lambda = [r-1, s+1; u] \in \Xi_{\mathrm{III}}$, and $\tau_d$ the minimal $K$-type of $\pi_\Lambda$ with $d = [r,s;u]$. For a $(\tau_d, \tau_d^*)$-matrix coefficient $\Phi_{\pi,\tau}$, put*

$$\Phi_{\pi,\tau}(a) = \sum_{k_1,l_1;k_2,l_2} c_{k_1,l_1;k_2,l_2}(a) f_{k_1,l_1;k_2,l_2}.$$

*Then, $c_M(a)$ $(M = (k_1,l_1;\,k_2,l_2))$ is given by the following:*

1. *Suppose that $l_1 + l_2 \leq s$. The matrix coefficients $c_{k_1,l_1;k_2,l_2}(a_1, a_2)$ can be expressed as follows*:

$$
\begin{aligned}
c_M(a_1,a_2) &= c_0(-1)^{r-k_1-l_1}(\mathrm{sh}(a_1)\,\mathrm{sh}(a_2))^{s-l_1-l_2} \\
&\times \sum_{i=0}^{l_1+l_2} (\mathrm{ch}(a_1)\,\mathrm{ch}(a_2))^{-(r+s+2)/2+l_1+l_2-i} \binom{r}{k_1}\binom{r}{k_2}\binom{s}{l_1}\binom{s}{l_2} \\
&\times \sum_{j=b-h}^{b+h} (-1)^{h-|j-b|} \sum_{\mu=0}^{[\frac{i-|j-b|}{2}]} \beta_\mu(M,i,j) \left(\frac{\mathrm{ch}(a_1)}{\mathrm{ch}(a_2)} + \frac{\mathrm{ch}(a_2)}{\mathrm{ch}(a_1)}\right)^{h-|j-b|-2\mu} \\
&\times \Phi\left(k_1+l_1+j, k_2+l_2-j; u; \left(\frac{\mathrm{ch}(a_1)}{\mathrm{ch}(a_2)}\right)^2\right).
\end{aligned}
$$

2. *Suppose that $s < l_1+l_2 \leq 2s$. Define $M^\wedge = (r-k_1, s-l_1; r-k_2, s-l_2)$, $b^\wedge = b(M^\wedge, i)$ and $h^\wedge = h(M^\wedge, i)$. Then,*

$$
\begin{aligned}
c_M(a_1,a_2) &= c_0(-1)^{r-k_1-l_1}(\mathrm{sh}(a_1)\,\mathrm{sh}(a_2))^{-s+l_1+l_2} \\
&\times \sum_{i=0}^{l_1+l_2} (\mathrm{ch}(a_1)\,\mathrm{ch}(a_2))^{-(r+s+2)/2+2s-l_1-l_2-i} \binom{r}{k_1}\binom{r}{k_2}\binom{s}{l_1}\binom{s}{l_2} \\
&\times \sum_{j=b^\wedge-h^\wedge}^{b^\wedge+h^\wedge} (-1)^{h^\wedge-|j-b^\wedge|} \sum_{m=0}^{[\frac{i-|j-b^\wedge|}{2}]} \beta_m(M^\wedge,i,j) \left(\frac{\mathrm{ch}(a_1)}{\mathrm{ch}(a_2)} + \frac{\mathrm{ch}(a_2)}{\mathrm{ch}(a_1)}\right)^{h^\wedge-|j-b^\wedge|-2m} \\
&\times \Phi\left(k_2+l_2+j, k_1+l_1-j; -u; \left(\frac{\mathrm{ch}(a_1)}{\mathrm{ch}(a_2)}\right)^2\right).
\end{aligned}
$$

*Remark* 5.6. We can determine the unique unknown constant $c_0$ by using the normalization condition, *i.e.*, by specification of the value of $\Phi$ at the identity of $G$.

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES,
THE UNIVERSITY OF TOKYO,
KOMABA, TOKYO 153, JAPAN
DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION,
MIE UNIVERSITY,
TSU, MIE 514, JAPAN
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES,
THE UNIVERSITY OF TOKYO,
KOMABA, TOKYO 153, JAPAN

## REFERENCES


[1] Y. Gon, *The generalized Whittaker functions on* $SU(2,2)$ *with respect to the Siegel parabolic subgroup*, Dr. Thesis.

[2] T. Hayata, *Differential equations of principal series Whittaker functions on* $SU(2,2)$, Indag. Math. **8** (1997), No. 4, 493–528.

[3] ———, *Whittaker functions of generalized principal series on* $SU(2,2)$, to appear in J. Math. Kyoto Univ.

[4] T. Hayata and T. Oda, *An explicit integral representation of Whittaker functions for the representations of the discrete series — The case of* $SU(2,2)$ —, to appear in J. Math. Kyoto Univ.

[5] H. Koseki and T. Oda, *Matrix coefficients of* $P_J$*-principal series of* $SU(2,2)$, preprint, 1998.

[6] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th ed., Cambridge University Press, London, 1927.

[7] H. Yamashita, *Embeddings of discrete series into induced representations of semisimple Lie groups I, – general theory and the case of* $SU(2,2)$ –, Japan. J. Math. (N.S.) **16** (1990), No. 1, 31–95.

[8] ———, *Embeddings of discrete series into induced representations of semisimple Lie groups II, – generalized Whittaker models for* $SU(2,2)$ –, J. Math. Kyoto Univ. **31-2** (1991), 543–571.