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ON THE TRANSFORMATION FORMULA OF RIEMANN'S THETA SERIES

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§0 Introduction

0.1. Theta series is a double-faced monster like Janus[1]. One face is looking at geometry, the other face is looking at representation theory.

For example, let us consider Riemann's theta series;

$$\vartheta[\alpha](z,w)=\sum_{\ell\in\mathbb{Z}^n}\mathbf{e}\left(\frac{1}{2}\langle\ell+\alpha',(\ell+\alpha')z\rangle+\langle\ell+\alpha',w+\alpha''\rangle\right).$$

Here $\alpha=(\alpha',\alpha'')\in\mathbb{R}^{2n}$ with $\alpha',\alpha''\in\mathbb{R}^n$. z is an element of the Siegel upper half space $\mathfrak{H}_n$ of degree n, and $w\in\mathbb{C}^n$. $\mathbf{e}(t)=\exp 2\pi\sqrt{-1}t$ and $\langle x,y\rangle=x\cdot{}^ty$ for $x,y\in\mathbb{C}^n$.

In the geometry, Riemann's theta series plays an important role in the analytic theory of abelian varieties [Mum, Chap.I]. Fix a diagonal matrix

$$e=\begin{bmatrix}e_1 & & \\ & \ddots & \\ & & e_n\end{bmatrix},\quad 0<e_j\in\mathbb{Z}\quad\text{s.t. } e_j|e_{j+1}.$$

For each $z\in\mathfrak{H}_n$, an abelian variety $X_{z,e}=\mathbb{C}^n/(\mathbb{Z}^nz\oplus\mathbb{Z}^ne)$ is defined with a Riemann form $H(x,y)=\langle x\cdot\mathrm{Im}(z)^{-1},\overline{y}\rangle$. The polarization of $X_{z,e}$ is defined by a line bundle $\mathcal{L}_{z,\alpha}$ which is characterized by $z\in\mathfrak{H}_n$ and $\alpha\in\mathbb{R}^{2n}$. Let $\{\alpha_i'\}_{i=1,\cdots,N}$ be a complete set of the representatives of $\mathbb{Z}^ne^{-1}/\mathbb{Z}^n$. Put $\alpha_i=(\alpha_i',0)\in\mathbb{R}^{2n}$. Then $\{\vartheta[\alpha+\alpha_i](z,*)\}_{i=1,\cdots,N}$ is a $\mathbb{C}$-basis of the complex vector space $\Gamma(X_{z,e},\mathcal{L}_{z,\alpha})$ of the global sections of $\mathcal{L}_{z,\alpha}$. The basis is ortho-normal with respect to the canonical Hermitian inner product on $\Gamma(X_{z,e},\mathcal{L}_{z,\alpha})$. From this fact, we can deduce the transformation formula of Riemann's theta series with respect to the paramodular group $\Gamma(e)$ defined by

$$\Gamma(e)=\{\gamma\in Sp(n,\mathbb{R})\mid L\gamma=L\}$$

where $L=\mathbb{Z}^n\times\mathbb{Z}^ne$ is a $\mathbb{Z}$-lattice in $\mathbb{R}^{2n}$. For every $\gamma\in\Gamma(e)$, the abelian varieties $X_{z,e}$ and $X_{\gamma(z),e}$ are isomorphic over $\mathbb{C}$ via the isomorphism $w\mapsto wJ(\gamma,z)^{-1}$. Here

$$\gamma(z)=(az+b)(cz+d)^{-1},\quad\text{and}\quad J(\gamma,z)=cz+d$$

[1] January is the most suitable month to talk about Janus!

for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This isomorphism induces an isomorphism

$$\Gamma(X_{\gamma(z),e}, \mathcal{L}_{\gamma(z),\alpha}) \xrightarrow{\sim} \Gamma(X_{z,e}, \mathcal{L}_{z,(\alpha+\delta)\gamma})$$

which is unitary with respect to the canonical Hermitian inner product. Here we put

$$\delta = \frac{1}{2}((c^t d)_0 \cdot e, (a^t b)_0) \in \mathbb{R}^{2n}$$

and, for any symmetric matrix $S \in \mathrm{Sym}_n(\mathbb{R})$, $S_0 \in \mathbb{R}^n$ is the row vector whose i-th component is the i-th diagonal component of S. Then the representation of this unitary isomorphism with respect to the ortho-normal basis consisting of Riemann's theta series gives the following transformation formula of Riemann's theta series with respect to the paramodular group;

Theorem 0.1.1.

$$\begin{aligned}&[\vartheta[\alpha + \alpha_i](\gamma(z), w \cdot J(\gamma, z)^{-1})]_{i=1,\cdots,N} \\ &= \mathbf{e}\left(\frac{1}{2}\langle w, w \cdot J(\gamma, z)^{-1} c\rangle\right) \cdot \det J(\gamma, z)^{1/2} U \cdot [\vartheta[(\alpha + \delta)\gamma + \alpha_i](z, w)]_{i=1,\cdots,N}\end{aligned}$$

for any $\alpha \in \mathbb{R}^{2n}$. Here $U = U(\gamma; \alpha; \{\alpha_1, \cdots, \alpha_N\}) \in U(N, \mathbb{C})$ is a unitary matrix constant with respect to z and w.

0.2. The transformation formula of theta series comes also from representation theory, that is, Weil representation. Let

$$\varpi : \widetilde{Sp}(n, \mathbb{R}) \to Sp(n, \mathbb{R})$$

be the non-trivial two-fold covering of the symplectic group $Sp(n, \mathbb{R})$. Then $\widetilde{Sp}(n, \mathbb{R})$ has a unitary representation ω on $L^2(\mathbb{R}^n)$ called Weil representation (see §2 for the constructions of $\widetilde{Sp}(n, \mathbb{R})$ and ω). Let Γ_θ be the subgroup of $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(n, \mathbb{Z})$ such that the diagonal components of $a^t b$ and $c^t d$ are even. Γ_θ is called theta group. Put $\widetilde{\Gamma}_\theta = \varpi^{-1}(\Gamma_\theta) \subset \widetilde{Sp}(n, \mathbb{R})$. Then we have the following theorem of Weil [Wei];

Theorem 0.2.1. *For any $\widetilde{\gamma} \in \widetilde{\Gamma}_\theta$, we have*

$$\theta_\varphi(\widetilde{\gamma}\widetilde{\sigma}) = \rho(\widetilde{\gamma}) \cdot \cdot \theta_\varphi(\widetilde{\sigma}).$$

Here $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a Schwartz function on $\mathbb{R}^n$, and θ_φ is a function on $\widetilde{Sp}(n, \mathbb{R})$ defined by

$$\theta_\varphi(\widetilde{\sigma}) = \sum_{\ell \in \mathbb{Z}^n} (\omega(\widetilde{\sigma})\varphi)(\ell).$$

ρ is a group homomorphism of $\widetilde{\Gamma}_\theta$ to $\mathbb{C}_1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$.

In the special case of $\gamma = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$, Theorem 0.2.1 gives Poisson summation formula

$$\sum_{\ell \in \mathbb{Z}^n} \widehat{\varphi}(\ell) = \sum_{\ell \in \mathbb{Z}^n} \varphi(\ell), \quad \widehat{\varphi}(y) = \int_{\mathbb{R}^n} \varphi(x) \mathrm{e}(-\langle x, y \rangle) dx$$

for a Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^n)$. So Theorem 0.2.1 is called the generalized Poisson summation formula of Weil.

If we choose suitably a Schwartz function φ, Theorem 0.2.1 gives the transformation formula of Riemann's theta series (see [Mum2, Cor. 8.9]). However it gives the transformation formula only for the theta group Γ_θ in the case of principal polarization $e = 1_n$.

0.3. I have been wondering the reason why there exists such a gap between these two methods of proving the transformation formula of Riemann's theta series. One possible reason was that the special choice of a Schwartz function in Theorem 0.2.1 makes Riemann's theta series much more symmetric beyond the symmetry of theta group. In this lecture, I will show that this is not the case. That is, the generalized Poisson summation formula of Weil can be extended so that it gives the transformation formula of Riemann's theta series with respect to the paramodular group of any polarization.

§1 Weil representation

Let us recall the construction of Weil representation over the real number field.

1.1. Let V be a symplectic $\mathbb{R}$-space with a symplectic $\mathbb{R}$-form D. The group of the symplectic automorphisms of V is denoted by $Sp(V)$;

$$Sp(V) = \{\sigma \in GL_{\mathbb{R}}(V) \mid D(x\sigma, y\sigma) = D(x, y) \quad \forall x, y \in V\}$$

($GL_{\mathbb{R}}(V)$ acts on V from right). The Heisenberg group associated with V is denoted by $H[V]$, that is, $H[V] = V \times \mathbb{R}$ with multiplication law

$$(x, t) \cdot (y, u) = \left(x + y, t + u + \frac{1}{2} D(x, y) \right).$$

The center of $H[V]$ is $\{(0, t) \mid t \in \mathbb{R}\}$, and it is identified with the additive group $\mathbb{R}$. $\sigma \in Sp(V)$ acts on $h = (x, t) \in H[V]$ from right as an automorphism by $h^\sigma = (x\sigma, t)$. For any additive subgroup M of V, put $H[M] = M \times \mathbb{R}$ which is a normal subgroup of $H[V]$.

Let Λ_0 be a $\mathbb{Z}$-lattice in V which is self-dual with respect to D. Then there exists a polarization $V = W \oplus W'$ of V such that Λ_0 is a direct sum of lattices Λ_0' and Λ_0'' of W and W' respectively. A paring $W \times W' \to \mathbb{R}$ is defined by $\langle x, y \rangle = D(x, y)$ ($x \in W$, $y \in W'$). The Haar measure on W and W' is normalized so that $\mathrm{vol}(W/\Lambda_0') = 1$ and $\mathrm{vol}(W'/\Lambda_0'') = 1$ respectively.

1.2. $H[V]$ has an irreducible unitary representation π such that $\pi(t) = \mathbf{e}(t)$ for all $t \in Z(H[V]) = \mathbb{R}$. π is unique up to unitary isomorphism. The representation space of π is denote by E_π. There exist two realizations of π.

The first one is realized on $L^2(W)$ by

$$(\pi(h)\varphi)(w) = \mathbf{e}\left(t + \frac{1}{2}\langle x, y\rangle + \langle w, y\rangle\right) \cdot \varphi(w + x)$$

for $h = ((x, y), t) \in H[V]$ and $\varphi \in L^2(W)$. This realization is called Schrödinger model.

The second one is realized as an induced representation $\mathrm{Ind}(H[V], H[\Lambda_0]; \xi_0)$ with a character ξ_0 of $H[\Lambda_0]$ defined by

$$\xi_0(h) = \mathbf{e}\left(t + \frac{1}{2}\langle x, y\rangle\right) \quad \text{for } h = ((x, y), t) \in H[\Lambda_0].$$

(Let us recall the definition of an induced representation. Let G be a locally compact unimodular group, H a closed unimodular subgroup of G, and ρ a unitary representation of H with a representation space E_ρ. Then the induced representation $\pi = \mathrm{Ind}(G, H; \rho)$ consists of the E_ρ-valued function φ on G such that

(1) $\varphi(hg) = \rho(h)\varphi(g)$ for all $h \in H$,

(2) $|\varphi|^2 = \int_{H\backslash G} |\varphi(g)|^2 d\dot{g} < \infty$

with the action $(\pi(g)\varphi)(x) = \varphi(xg)$ for $g \in G$.) An isomorphism between these two realizations is given by

$$L^2(W) \ni \varphi \xrightarrow{\sim} \Theta_\varphi \in \mathrm{Ind}(H[V], H[\Lambda_0]; \xi_0)$$

where Θ_φ is defined by

$$\Theta_\varphi(h) = \sum_{\ell \in \Lambda_0} \varphi(x + \ell)\mathbf{e}\left(t + \frac{1}{2}\langle x, y\rangle + \langle \ell, y\rangle\right)$$

for $h = ((x, y), t) \in H[V]$ and a Schwartz function $\varphi \in \mathcal{S}(W)$ on W.

1.3. Let $U(E_\pi)$ be the group of the unitary automorphism of E_π as a complex Hilbert space. $U(E_\pi)$ is a Hausdorff topological group with respect to the weakest topology such that $U(E_\pi) \ni T \to Tv \in E_\pi$ is continuous for all $v \in E_\pi$. Let $Mp(V)$ be a subgroup of $Sp(V) \times U(E_\pi)$ consisting of (σ, T) such that $T^{-1} \circ \pi(h) \circ T = \pi(h^\sigma)$ for all $h \in H[V]$. Then $Mp(V)$ is a locally compact unimodular group [Igu].

1.4. Suppose $E_\pi = L^2(W)$. For any $c \in \mathrm{Hom}_{\mathbb{R}}(W', W)$, put $\det c = \det(\langle w_i c, w_j\rangle)_{i,j=1,\cdots,n}$ with a $\mathbb{R}$-basis $\{w_1, \cdots, w_n\}$ of W'. Now for any $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(V)$ such that $\det c \neq 0$, define $\mathbf{r}_0(\sigma) \in U(E_\pi)$ by

$$(\mathbf{r}_0(\sigma)\varphi)(w) = |\det c|^{1/2} \int_{W'} \varphi(wa + w'c)\mathbf{e}\left(\frac{1}{2}\langle wa + w'c, wb + w'd\rangle - \frac{1}{2}\langle w, w'\rangle\right) dw'$$

for all $\varphi \in L^2(W) \cap L^1(W)$. Then it is proved that $\mathbf{r}(\sigma) = (\sigma, \mathbf{r}_0(\sigma))$ is an element of $Mp(V)$.

There exists uniquely a continuous group homomorphism Φ of $Mp(V)$ to $\mathbb{C}_1$ such that

(1) $\Phi(1, \lambda) = \lambda^2$ for all $\lambda \in \mathbb{C}_1$,

(2) $\Phi(\mathbf{r}(\sigma)) = \mathbf{e}(\frac{1}{8}\dim V)\det c/|\det c|$ for all $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(V)$ such that $\det c \neq 0$.

Put $\widetilde{Sp}(V) = \mathrm{Ker}(\Phi)$. Then

$$\varpi : \widetilde{Sp}(V) \ni (\sigma, T) \mapsto \sigma \in Sp(V)$$

gives a non-trivial two-fold covering of $Sp(V)$ and

$$\omega : \widetilde{Sp}(V) \ni (\sigma, T) \mapsto T \in U(L^2(W))$$

defines a unitary representation of $\widetilde{Sp}(V)$ which is called Weil representation.

1.5. The two-fold covering group $\widetilde{Sp}(V)$ acts on $H[V]$ via the covering mapping. The action defines a semi-direct product $\widetilde{Sp}(V)_J = \widetilde{Sp}(V) \ltimes H[V]$ which is called Jacobi group. Put

$$\omega_J(g) = \omega(\widetilde{\sigma}) \circ \pi(h) \quad \text{for } g = (\widetilde{\sigma}, h) \in \widetilde{Sp}(V)_J.$$

Then ω_J is an irreducible unitary representation of Jacobi group on $L^2(W)$.

§2 An extension of generalized Poisson summation formula

Now we will construct an extension of generalized Poisson summation formula recalled in Introduction. Let L be a sub $\mathbb{Z}$-lattice of Λ_0. Then

$$\Lambda_0 \subset L^* = \{x \in V \mid D(L, x) \subset \mathbb{Z}\}.$$

Now put

$$Sp(L) = \{\gamma \in Sp(V) \mid L\gamma = L\} = \{\gamma \in Sp(V) \mid L^*\gamma = L^*\}$$

and call it the paramodular group of L.

2.1. Let us start with some preliminaries. Let $\mathfrak{X}_L$ be a set of pairs (Λ, ξ) such that

(1) Λ is a $\mathbb{Z}$-lattice in V, self-dual with respect to D,

(2) $L \subset \Lambda \subset L^*$,

(3) ξ is a continuous unitary character of $H[\Lambda]$ such that $\xi(t) = \mathbf{e}(t)$ for all $t \in Z(H[\Lambda]) = \mathbb{R}$.

For each $(\Lambda, \xi) \in \mathfrak{X}_L$, put $\chi_{\Lambda,\xi} = \mathrm{Ind}(H[L^*], H[\Lambda]; \xi)$ which is a finite dimensional irreducible unitary representation of $H[L^*]$. It is irreducible because

$$\mathrm{Ind}(H[V], H[L^*]; \chi_{\Lambda,\xi}) = \mathrm{Ind}(H[V], H[\Lambda]; \xi)$$

is an irreducible representation of $H[V]$. We have

$$\dim \chi_{\Lambda,\xi} = (H[L^*] : H[\Lambda]) = (L^* : L)^{1/2}.$$

The trace of $\chi_{\Lambda,\xi}$ is given by

$$\operatorname{tr} \chi_{\Lambda,\xi}(h) = \begin{cases} 0 & h \notin H[L] \\ (L^* : L)^{1/2}\xi(h) & h \in H[L] \end{cases} \tag{2.1.1}$$

for any $h \in H[L^*]$.

I am not sure if the representation $\chi_{\Lambda,\xi}$ is determined by its character. So let $\mathfrak{X}_L^0$ be a subset of $\mathfrak{X}_L$ consisting of the (Λ, ξ) such that the order of $\xi(x, 0)$ is bounded for all $x \in \Lambda$. Then, for any $(\Lambda, \xi) \in \mathfrak{X}_L^0$, the representation $\chi_{\Lambda,\xi}$ factors through a compact quotient group of $H[L^*]$, so the representation $\chi_{\Lambda,\xi}$ is determined by its character. Note that (Λ_0, ξ_0) is an element of $\mathfrak{X}_L^0$.

Let X_L be a set of continuous unitary character ζ of $H[L]$ such that $\zeta(t) = \mathbf{e}(t)$ for all $t \in Z(H[L]) = \mathbb{R}$. The paramodular group $Sp(L)$ acts from right on X_L by $(\gamma \cdot \zeta)(h) = \zeta(h^\gamma)$. We have a bijection $\delta \mapsto \zeta_\delta$ of V/L^* onto X_L defined by

$$\zeta_\delta(h) = \mathbf{e}\left(t + \frac{1}{2}\langle x', x''\rangle + D(x, \delta)\right)$$

for all $h = (x, t) \in H[L]$ such that $x = (x', x'') \in L$ with $x' \in W$ and $x'' \in W'$.

2.2. Take any element $\gamma \in Sp(L)$. For any vector $\varphi \in \operatorname{Ind}(H[V], H[\Lambda_0]; \xi_0)$, define a function φ^γ on $H[V]$ by $\varphi^\gamma(h) = \varphi(h^\gamma)$. Then φ^γ is an vector of $\operatorname{Ind}(H[V], H[\Lambda_0\gamma^{-1}]; \xi_0^\gamma)$ where $\xi_0^\gamma(h) = \xi_0(h^\gamma)$. So the mapping $\varphi \mapsto \varphi^\gamma$ gives a unitary isomorphism

$$[\gamma] : \operatorname{Ind}(H[V], H[\Lambda_0]; \xi_0) \xrightarrow{\sim} \operatorname{Ind}(H[V], H[\Lambda_0\gamma^{-1}]; \xi_0^\gamma)$$

as complex HIlbert spaces. By the formula of induction in the stages, we have

$$\operatorname{Ind}(H[V], H[\Lambda_0\gamma^{-1}]; \xi_0^\gamma) = \operatorname{Ind}(H[V], H[L^*]; \chi_{\Lambda_0\gamma^{-1},\xi_0^\gamma}).$$

Note that $(\Lambda_0\gamma^{-1}, \xi_0^\gamma)$ is an element of $\mathfrak{X}_L^0$. Choose a $\delta \in V$ such that $\gamma \cdot \zeta_0 = \zeta_\delta$ (see **2.1**). Now we will twist the representation $\operatorname{Ind}(H[V], H[L^*]; \chi_{\Lambda_0\gamma^{-1},\xi_0^\gamma})$ by an inner automorphism of $H[V]$ induced by $(\delta, 0) \in H[V]$. The twisted representation is $\operatorname{Ind}(H[V], H[L^*]; \chi_{\Lambda_0\gamma^{-1},\xi'})$ with an element $(\Lambda_0\gamma^{-1}, \xi')$ of $\mathfrak{X}_L^0$, because $H[L^*]$ and $H[\Lambda_0\gamma^{-1}]$ are normal subgroups of $H[V]$. The inner automorphism of $H[V]$ induces a unitary isomorphism

$$[(\delta, 0)] : \operatorname{Ind}(H[V], H[L^*]; \chi_{\Lambda_0\gamma^{-1},\xi_0^\gamma}) \xrightarrow{\sim} \operatorname{Ind}(H[V], H[L^*]; \chi_{\Lambda_0\gamma^{-1},\xi'}).$$

Because of the special choice of $\delta \in V$, we have $\xi'|_{H[L]} = \xi_0|_{H[L]}$, that is, $\operatorname{tr} \chi_{\Lambda_0\gamma^{-1},\xi'} = \operatorname{tr} \chi_{\Lambda_0,\xi_0}$ by the formula (2.1.1). Because $(\Lambda_0\gamma^{-1}, \xi')$ is an element of $\mathfrak{X}_L^0$, there exists a unitary isomorphism

$$U_{\gamma,\delta} : \chi_{\Lambda_0\gamma^{-1},\xi'} \xrightarrow{\sim} \chi_{\Lambda_0,\xi_0}$$

as representations of $H[L^*]$. Note that $U_{\gamma,\delta}$ is unique up to constant multiplication. Now $U_{\gamma,\delta}$ induces an unitary isomorphism

$$[U_{\gamma,\delta}] : \mathrm{Ind}(H[V], H[L^*]; \chi_{\Lambda_0\gamma^{-1},\xi'}) \tilde{\to} \mathrm{Ind}(H[V], H[L^*]; \chi_{\Lambda_0,\xi_0}) = \mathrm{Ind}(H[V], H[\Lambda_0]; \xi_0).$$

The composition of all the isomorphisms

$$\mathbf{r}(\gamma, \delta) = [U_{\gamma,\delta}] \circ [(\delta, 0)] \circ [\gamma]$$

is a unitary automorphism of $E_\pi = \mathrm{Ind}(H[V], H[\Lambda_0]; \xi_0)$. Now we can prove that $(\gamma, \mathbf{r}(\gamma, \delta)) \in Sp(V) \times U(E_\pi)$ is an element of $Mp(V)$.

2.3. Put $\widetilde{Sp}(L) = \varpi^{-1}(Sp(L)$ and take any $\widetilde{\gamma} = (\gamma, T) \in \widetilde{Sp}(L) \subset \widetilde{Sp}(V)$. Choose a $\delta \in V$ such that $\gamma \cdot \zeta_0 = \zeta_\delta$ (see **2.1**).

We have constructed an element $(\gamma, \mathbf{r}(\gamma, \delta)) \in Mp(V)$. Then T is equal to a constant multiple of $\mathbf{r}(\gamma, \delta)$. We can adjust the constant in the choice of $U_{\gamma,\delta}$ so that we have $T = \mathbf{r}(\gamma, \delta)$. This normalized $U_{\gamma,\delta}$ is determined uniquely by $\widetilde{\gamma}$. So we will denote it by $U_{\widetilde{\gamma},\delta}$.

Now look at the intermediate induced representation $\chi_{\Lambda_0,\xi_0} = \mathrm{Ind}(H[L^*], H[\Lambda_0]; \xi_0)$. The twist by γ induces an unitary isomorphism $\{\gamma\}$ of χ_{Λ_0,ξ_0} onto $\chi_{\Lambda_0\gamma^{-1},\xi_0^\gamma}$. The inner automorphism of $H[V]$ associated with $(\delta, 0) \in H[V]$ induces an unitary isomorphism $\{(\delta, 0)\}$ of $\chi_{\Lambda_0\gamma^{-1},\xi_0^\gamma}$ onto $\chi_{\Lambda_0\gamma^{-1},\xi'}$. Put

$$U(\widetilde{\gamma}, \delta) = U_{\widetilde{\gamma},\delta} \circ \{(\delta, 0)\} \circ \{\gamma\}$$

which is a unitary automorphism of $\chi_{\Lambda_0\xi_0}$ as a complex Hilbert space.

2.4. Now let us describe our main result. For any Schwartz function $\varphi \in \mathcal{S}(W)$ and $\alpha \in V$, define a function

$$\vartheta_\varphi[\alpha] : \widetilde{Sp}(V)_J \to \mathrm{Ind}(H[L^*], H[\Lambda_0]; \xi_0)$$

by

$$\vartheta_\varphi[\alpha](g) : H[L^*] \ni h \mapsto \Theta_{\omega_J(g)\varphi}(h(\alpha, 0)) \in \mathbb{C}$$

for all $g \in \widetilde{Sp}(V)_J$. Then our main theorem is

Theorem 2.4.1. *Take any $\widetilde{\gamma} \in \widetilde{Sp}(L)$ and put $\gamma \cdot \zeta_0 = \zeta_\delta$ with $\delta = (\delta', \delta'') \in V = W \oplus W'$. Then*

$$\vartheta_\varphi[\alpha](\widetilde{\gamma} g) = \varepsilon \cdot U(\widetilde{\gamma}, \delta) \vartheta_\varphi[(\alpha + \delta)\gamma](g)$$

with $\varepsilon = \mathbf{e}\left(\frac{1}{2}D(\delta, \alpha) - \frac{1}{2}\langle \delta', \delta'' \rangle\right)$ *for all* $\varphi \in \mathcal{S}(W)$ *and* $\alpha \in V$.

Let us reformulate Theorem 2.4.1. Let $\mathcal{G}_L$ be a subgroup of $\widetilde{Sp}(V)_J$ consisting of $h \cdot \widetilde{\gamma} \in \widetilde{Sp}(L) \ltimes H[V]$ with $h = (\delta, t) \in H[V]$ and $\widetilde{\gamma} \in \widetilde{Sp}(L)$ such that $\gamma \cdot \zeta_0 = \zeta_\delta$. For any $h \cdot \widetilde{\gamma} \in \mathcal{G}_L$ with $h = (\delta, t) \in H[V]$ and $\delta = (\delta', \delta'') \in V = W \oplus W''$, put

$$U^*(h \cdot \widetilde{\gamma}) = \mathbf{e}\left(t - \frac{1}{2}\langle \delta', \delta'' \rangle\right) \cdot U(\widetilde{\gamma}, \delta).$$

Then we have

Corollary 2.4.2. *For any $h \cdot \widetilde{\gamma} \in \mathcal{G}_L$ with $h = (\delta, t) \in H[V]$, we have*

$$\vartheta_\varphi[\alpha](h \cdot \widetilde{\gamma} g) = U^*(h \cdot \widetilde{\gamma})\vartheta_\varphi[(\alpha + 2\delta)\gamma](g)$$

for all $\varphi \in \mathcal{S}(W)$ and $\alpha \in V$.

Using Corollary 2.4.2, we can prove

Corollary 2.4.3.

(1) *U^* is a unitary representation of $\mathcal{G}_L$ on* $\mathrm{Ind}(H[L^*], H[\Lambda_0]; \xi_0)$,
(2) *$U^*(h) = \chi_{\Lambda_0,\xi_0}(-\ell, t)$ for $h = (\ell, t) \in H[L^*]$.*

Note that $H[L^*]$ is the kernel of a group homomorphism $h \cdot \widetilde{\gamma} \mapsto \widetilde{\gamma}$ of $\mathcal{G}_L$ onto $\widetilde{Sp}(L)$ and that $(x, t) \mapsto (-x, t)$ is an automorphism of $H[V]$. So the second statement tells how $U(\widetilde{\gamma}, \delta)$ behaves under various choice of $\delta \in V$ for each $\widetilde{\gamma}$.

§3 Automorphic factor of weight 1/2

Let us recall the construction of the automorphic factor of weight 1/2. For the sake of simplicity, we will assume that $V = \mathbb{R}^{2n}$ (row vectors) and $D(x, y) = x \cdot J_n \cdot {}^t y$ for $x, y \in V$ with $J_n = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$. See [Tak] for the details.

3.1. For any complex symmetric matrix $S \in \mathrm{Sym}_n(\mathbb{C})$ of size n, put

$$\det{}^{-1/2} S = \int_{\mathbb{R}^n} \exp(-\pi \langle xS, x \rangle) dx.$$

For any $z, z' \in \mathfrak{H}_n$ and $\sigma \in Sp(V)$, put

$$\varepsilon(\sigma; z', z) = \det{}^{-1/2}\left(\frac{\sigma(z') - \overline{\sigma(z)}}{2\sqrt{-1}}\right) \det{}^{1/2}\left(\frac{z' - \overline{z}}{2\sqrt{-1}}\right) \cdot |\det J(\sigma, z') J(\sigma, z)|^{-1/2}.$$

For any $z_0 \in \mathfrak{H}_n$ and $\sigma, \tau \in Sp(V)$, put

$$\beta_{z_0}(\sigma, \tau) = \varepsilon(\sigma; z_0, \tau(z_0))$$

Then β_{z_0} is a $\mathbb{C}_1$-valued real analytic two-cocycle on $Sp(V)$. The two-cocycle β_{z_0} is constructed by Satake [Sat]. The cohomology class of β_{z_0} is of order two;

$$\beta_{z_0}(\sigma, \tau)^2 = \alpha_{z_0}(\tau) \alpha_{z_0}(\sigma\tau)^{-1} \alpha_{z_0}(\sigma)$$

with $\alpha_{z_0}(\sigma) = \det J(\sigma, z_0)/|\det J(\sigma, z_0)|$. The two-fold covering group $\widetilde{Sp}(V)$ can be realized as a subgroup of the central extension of $Sp(V)$ associated with the two-cocycle β_{z_0}. Put

$$\widetilde{Sp}(V; z_0) = \left\{(\varepsilon, \sigma) \in \mathbb{C}_1 \times Sp(V) \mid \varepsilon^2 = \alpha_{z_0}(\sigma)^{-1}\right\}$$

with a multiplication law

$$(\varepsilon, \sigma) \cdot (\eta, \tau) = (\varepsilon\eta\beta_{z_0}(\sigma, \tau), \sigma\tau).$$

Then $\widetilde{Sp}(V; z_0)$ is a connected real Lie group and

$$\varpi : \widetilde{Sp}(V; z_0) \ni (\varepsilon, \sigma) \mapsto \sigma \in Sp(V)$$

gives a non-trivial two-fold covering of $Sp(V)$. We can give explicitly a topological group isomorphism of $\widetilde{Sp}(V; z_0)$ onto $\widetilde{Sp}(V)$ the two-fold covering of $Sp(V)$ constructed in **1.4** [Tak]. We will identify these two covering groups of $Sp(V)$.

3.2. For any $\widetilde{\sigma} = (\varepsilon, \sigma) \in \widetilde{Sp}(V; z_0) = \widetilde{Sp}(V)$ and $z \in \mathfrak{H}_n$, put

$$J_{\frac{1}{2}}(\widetilde{\sigma}, z) = \varepsilon^{-1} \cdot \varepsilon(\sigma; z, z_0) |\det J(\sigma, z)|^{1/2}.$$

Then $J_{\frac{1}{2}}$ is a non-zero-complex-valued function on $\widetilde{Sp}(V) \times \mathfrak{H}_n$ such that

(1) real analytic on $\widetilde{Sp}(V)$ and holomorphic on $\mathfrak{H}_n$,
(2) $J_{\frac{1}{2}}(\widetilde{\sigma}\widetilde{\tau}, z) = J_{\frac{1}{2}}(\widetilde{\sigma}, \tau(z)) J_{\frac{1}{2}}(\widetilde{\tau}, z)$ for all $\widetilde{\sigma}, \widetilde{\tau} \in \widetilde{Sp}(V)$,
(3) $J_{\frac{1}{2}}(\widetilde{\sigma}, z)^2 = \det J(\sigma, z)$.

That is $J_{\frac{1}{2}}$ is an automorphic factor of weight 1/2.

§4 Application to Riemann's theta series

Now we will apply our extension of generalized Poisson summation formula to prove the transformation formula of Riemann's theta series.

4.1. Let $V = \mathbb{R}^{2n}$ be a symplectic $\mathbb{R}$-space with a symplectic form $D(x, y) = x \cdot J_n \cdot {}^t y$ $\left(J_n = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}\right)$. A polarization $V = W \oplus W'$ is given by

$$W = \{(x, 0) \in V \mid x \in \mathbb{R}^n\}, \quad W' = \{(0, y) \in V \mid y \in \mathbb{R}^n\}$$

which are identified with $\mathbb{R}^n$. The semi-direct product $Sp(V)_J = Sp(V) \ltimes H[V]$ acts on $\mathfrak{H}_n \times \mathbb{C}^n$ by

$$g(z, w) = (\sigma(z), (w + xz + y) J(\sigma, z)^{-1})$$

for $g = (\sigma, h) \in Sp(V)_J$ with $h = ((x, y), t) \in H[V]$ and $(z, w) \in \mathfrak{H}_n \times \mathbb{C}^n$. Put

$$\eta(g; z, w) = \mathbf{e}\left(t + \frac{1}{2}\langle x, x\sigma(z) + y\rangle + \langle x, wJ(\sigma, z)^{-1}\rangle + \frac{1}{2}\langle w(-{}^t c), wJ(\sigma, z)^{-1}\rangle\right)$$

for $g = (\sigma, h^{\sigma}) \in Sp(V)_J$ with $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(V)$ and $H = ((x, y), t) \in H[V]$. Then $\eta(g; z, w)$ has a property of an automorphic factor;

$$\eta(gg'; Z) = \eta(g; g'(Z)) \cdot \eta(g'; Z)$$

for all $g, g' \in Sp(V)_J$ and $Z \in \mathfrak{H}_n \times \mathbb{C}^n$.

4.2. Put $\Lambda_0 = \mathbb{Z}^{2n}$ which is a self-dual $\mathbb{Z}$-lattice in V. Put $L = \mathbb{Z}^n \times \mathbb{Z}^n e$ with

$$e = \begin{bmatrix} e_1 & & \\ & \ddots & \\ & & e_n \end{bmatrix}, \quad 0 < e_j \in \mathbb{Z}, \quad e_j | e_{j+1}.$$

Let $\{\alpha'_1, \cdots, \alpha'_N\}$ be a complete set of representatives of $\mathbb{Z}^n e^{-1}/\mathbb{Z}^n$. Put $\alpha_j = (\alpha'_j, 0) \in V = W \oplus W'$. Then $\{(\alpha_j, 0)\}_{j=1,\cdots,N}$ is a complete set of representatives of the coset space $H[\Lambda_0] \backslash H[L^*]$. Define an element $\psi_j \in \mathrm{Ind}(H[L^*], H[\Lambda_0]; \xi_0)$ by

$$\psi_j(h) = \begin{cases} 0 & h \notin H[\Lambda_0](\alpha_j, 0) \\ \xi_0\left(h \cdot (\alpha_j, 0)^{-1}\right) & h \in H[\Lambda_0](\alpha_j, 0) \end{cases}$$

for $h \in H[L^*]$. Then $\{\psi_1, \cdots, \psi_N\}$ is a ortho-normal $\mathbb{C}$-basis of $\mathrm{Ind}(H[L^*], H[\Lambda_0]; \xi_0)$. Take any $\alpha \in V$. Consider the theta series $\vartheta_\varphi[\alpha](\widetilde{g})$ with a Schwartz function

$$\varphi(x) = \mathbf{e}\left(\frac{1}{2}\langle x, xz_0\rangle\right) \quad (x \in W)$$

with a fixed $z_0 \in \mathfrak{H}_n$. We will use the identification $\widetilde{Sp}(V) = \widetilde{Sp}(V; z_0)$ described in **3.1**. Now we can prove the formula

$$(\vartheta_\varphi[\alpha](\widetilde{g}), \psi_j) = J_{\frac{1}{2}}(\widetilde{\sigma}, z_0)^{-1}\eta(g; z_0, 0)\mathbf{e}\left(\frac{1}{2}D(\alpha_j, \alpha)\right) \cdot \vartheta^*[\alpha + \alpha_j](z, w)$$

for all $\widetilde{g} = (\widetilde{\sigma}, h) \in \widetilde{Sp}(V)_J$. Here we put $g = (\sigma,) \in Sp(V)_J$ and $(z, w) = g(z_0, 0) \in \mathfrak{H}_n \times \mathbb{C}^n$, and

$$\vartheta^*[\alpha](z, w) = \mathbf{e}\left(-\frac{1}{2}\langle \alpha', \alpha''\rangle\right) \cdot \vartheta[\alpha](z, w) \tag{4.2.1}$$

for all $\alpha = (\alpha', \alpha'') \in V$ with Riemann's theta series $\vartheta[\alpha](z, w)$.

4.3. Our extension of generalized Poisson summation formula (Theorem 2.4.1) and the formula (4.2.1) give a transformation formula of Riemann's theta series with respect to the paramodular group $Sp(L)$. Define a column vector of functions

$$\Theta[\alpha](z, w) = \left[\mathbf{e}\left(\frac{1}{2}D(\alpha_j, \alpha)\right)\vartheta^*[\alpha + \alpha_j](z, w)\right]_{j=1,\cdots,N}.$$

Take any $\widetilde{\gamma} \in \widetilde{Sp}(L)$ and put $\gamma \cdot \zeta_0 = \zeta_\delta$ with $\delta = (\delta', \delta'') \in V = W \oplus W'$. Then we have

Theorem 4.3.1.

$$\begin{aligned}\Theta[\alpha](\gamma(z), wJ(\gamma, z)^{-1}) =& \mathbf{e}\left(\frac{1}{2}D(\delta, \alpha) - \frac{1}{2}\langle \delta', \delta''\rangle\right) \\ & \times J_{\frac{1}{2}}(\widetilde{\gamma}, z)\eta(\gamma; z, w)^{-1}U(\widetilde{\gamma}, \delta)\Theta[(\alpha + \delta)\gamma](z, w).\end{aligned}$$

Here the representation matrix of $U(\widetilde{\gamma}, \delta)$ with respect to $\{\psi_1, \cdots, \psi_N\}$ is also denoted by $U(\widetilde{\gamma}, \delta)$.

4.4. Take any $\widetilde{\gamma} \in \widetilde{Sp}(L)$. Recall the identification $\widetilde{Sp}(V) = \widetilde{Sp}(V; z_0)$ and put $\widetilde{\gamma} = (\varepsilon, \gamma)$ with $\varepsilon \in \mathbb{C}_1$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(V)$. In the transformation formula of Theorem 0.1.1, replace the factor $\det J(\gamma, z)^{1/2}$ with $J_{\frac{1}{2}}(\widetilde{\gamma}, z)$. Comparing the formulae of Theorem 0.1.1 and Theorem 4.3.1, we can see that the unitary matrix $U(\gamma; \alpha; \{\alpha_1, \cdots, \alpha_N\})$ is equal to the representation matrix of $U(\widetilde{\gamma}, \delta)$ with $\delta = \frac{1}{2}((c\,{}^t\!d)_0 e, (a\,{}^t\!b)_0)$ up to trivial factors. We can calculate an explicit formula of $U(\gamma; \alpha; \{\alpha_1, \cdots, \alpha_N\})$ by a modification of the method of Siegel [Sie]. It gives an explicit formula of $U(\widetilde{\gamma}, \delta)$, or its matrix coefficients. Typical result is

Theorem 4.4.1. *Suppose* $\det c \neq 0$ *and put*

$$\delta = (\delta', \delta'') = \frac{1}{2}((c^t d)_0 e, (a^t b)_0).$$

Then

$$(U(\tilde{\gamma}, \delta)\psi_i, \psi_j) = \varepsilon \cdot G(\alpha'_i, \alpha'_j; c, d) \cdot \frac{\det^{1/2}((z_0 + c^{-1}d)/\sqrt{-1})}{|det(z_0 + c^{-1}d)|^{1/2}}$$
$$\times \mathbf{e}\left(\frac{1}{2}\langle(\alpha'_j - \delta', (\alpha'_j - \delta')ac^{-1}\rangle - \langle \alpha'_j - \delta', \delta''\rangle\right)$$

where $G(\alpha'_i, \alpha'_j; c, d)$ *is a Gauss sum defined by*

$$G(\alpha'_i, \alpha'_j; c, d) = |\det e|^{-1} |\det c|^{-1/2}$$
$$\times \sum_{\lambda \in \mathbb{Z}^n/\mathbb{Z}^n e} \mathbf{e}\left(\frac{1}{2}\langle \lambda + \alpha'_i, (\lambda + \alpha'_i)c^{-1}d\rangle - \langle \alpha'_j - \delta', (\lambda + \alpha'_i)c^{-1}\rangle\right).$$

§5 Application to reductive dual pair

Our extension of generalized Poisson summation formula (Theorem 2.4.1) can be applied to general reductive dual pairs. In this section, we will consider the simplest case of $(O(m), Sp(n, \mathbb{R}))$.

5.1. Fix a positive define $Q \in \mathrm{Sym}_m(\mathbb{Z})$. The orthogonal group of Q is denoted by $O(Q)$;

$$O(Q) = \{g \in GL(m, \mathbb{R}) \mid gQ^t g = Q\}.$$

Define a symplectic form on $V = M_{m,2n}(\mathbb{R})$ by $D(x, y) = \mathrm{tr}(x \cdot J_n \cdot {}^t y)$. V has a polarization $V = W \oplus W'$ with

$$W = \{(x, 0) \in V \mid x \in M_{m,n}(\mathbb{R})\}, \quad W' = \{(0, y) \in V \mid y \in M_{m,n}(\mathbb{R})\}$$

which are identified with $M_{m,n}(\mathbb{R})$. A pairing $\langle x, y\rangle \in \mathbb{R}$ between $x \in W = M_{m,n}(\mathbb{R})$ and $y \in W' = M_{m,n}(\mathbb{R})$ is defined by $\langle x, y\rangle = D(x, y) = \mathrm{tr}(x \cdot {}^t y)$. For any $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(n, \mathbb{R})$, put

$$1 \boxtimes_Q \sigma = \begin{bmatrix} 1 \boxtimes a & Q \boxtimes b \\ Q^{-1} \boxtimes c & 1 \boxtimes d \end{bmatrix} \in Sp(V).$$

Here $Q \boxtimes b$ is a $\mathbb{R}$-linear mapping of W to W' defined by $x \mapsto Qxb$. Other elements are defined similarly. For any $g \in O(Q)$, put

$$g \boxtimes_Q 1 = \begin{bmatrix} g \boxtimes 1 & 0 \\ 0 & {}^t g^{-1} \boxtimes 1 \end{bmatrix} \in Sp(V).$$

Then $\sigma \mapsto 1 \boxtimes_Q \sigma$ (resp. $g \mapsto g \boxtimes_Q 1$) is an injective group homomorphism of $Sp(n, \mathbb{R})$ (resp. $O(Q)$) into $Sp(V)$. Identifying $Sp(n, \mathbb{R})$ and $O(Q)$ with their image under the group homomorphisms, we have a reductive dual pair $(O(Q), Sp(n, \mathbb{R}))$ in $Sp(V)$.

Put $\Lambda_0 = M_{m,2n}(\mathbb{Z})$ and $L = M_{m,n}(\mathbb{Z}) \times Q \times M_{m,n}(\mathbb{Z}) \subset \Lambda_0$. Then the group homomorphism $\gamma \mapsto 1 \boxtimes_Q \gamma$ maps $Sp(n, \mathbb{Z})$ into $Sp(L)$. So our extension of generalized Poisoon summation formula, Theorem 2.4.1, gives the transformation formula of theta series with respect to the Siegel full modular group $Sp(n, \mathbb{Z})$.

5.2. Fix a $z_0 \in \mathfrak{H}_n$ and recall the identification $\widetilde{Sp}(n,\mathbb{R}) = \widetilde{Sp}(n,\mathbb{R};z_0)$ (see **3.1**). The polarization $V = W \oplus W'$ defines Siegel upper half space $\mathfrak{H}_V$ in $\mathrm{Hom}_{\mathbb{C}}(W'_{\mathbb{C}}, W_{\mathbb{C}})$ ($W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ etc.). Then $Q \boxtimes z_0$ is an element of $\mathfrak{H}_V$. Now identify $\widetilde{Sp}(V)$ with $\widetilde{Sp}(V; Q \boxtimes z_0)$. Then $\imath_Q(\widetilde{\sigma}) = (\varepsilon^m, 1 \boxtimes_Q \sigma)$ for $\widetilde{\sigma} = (\varepsilon, \sigma) \in \widetilde{Sp}(n,\mathbb{R})$ defines a continuous group homomorphism $\imath_Q$ of $\widetilde{Sp}(n,\mathbb{R})$ to $\widetilde{Sp}(V)$.

5.3. A $\mathbb{C}$-valued function f on $M_{m,n}(\mathbb{R})$ is called Q-harmonic if $\Delta_Q f = 0$ for

$$\Delta_Q = \mathrm{tr}\left({}^t\left(\frac{\partial}{\partial x}\right) Q^{-1} \left(\frac{\partial}{\partial x}\right)\right) \quad \text{with } \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_{ij}}\right)_{1 \le i \le m, 1 \le j \le n},$$

and f is called pluri-Q-harmonic if $(a \cdot f)(x) = f(xa)$ is Q-harmonic for all $a \in GL(n,\mathbb{R})$. We will denote by $\mathcal{H}_Q$ the space of the $\mathbb{C}$-valued pluri-Q-harmonic polynomial functions on $M_{m,n}(\mathbb{R})$. The orthogonal group $O(Q)$ acts on $\mathcal{H}_Q$ by $(g \cdot P)(x) = P({}^t g x)$ for $g \in O(Q)$ and $P \in \mathcal{H}_Q$. For any irreducible unitary representation λ of $O(Q)$, let us denote by $\mathcal{H}_Q(\lambda)$ the $\check{\lambda}$-isotypic component of $\mathcal{H}_Q$ with the contragredient representation $\check{\lambda}$ of λ. We will fix a λ such that $\mathcal{H}_Q(\lambda) \neq 0$. The highest weight of such a λ is determined by [K-V]. The representation τ_λ of $GL_n(\mathbb{C})$ on $\mathcal{H}_Q(\lambda)$ defined by $(\tau_\lambda(d)P)(x) = P(x{}^t d^{-1})$ is irreducible (see [K-V]). Put

$$J_\lambda(\sigma, z) = \tau_\lambda(J(\sigma, z)) \quad \text{for } \sigma \in Sp(n,\mathbb{R}),\ z \in \mathfrak{H}_n.$$

5.4. Let us denote by $\mathcal{H}_Q(\lambda)^*$ the complex dual space of $\mathcal{H}_Q(\lambda)$. The canonical pairing between $\mathcal{H}_Q(\lambda)^*$ and $\mathcal{H}_Q(\lambda)$ is denoted by $\langle , \rangle$. For any $\alpha = (\alpha', \alpha') \in V = W \oplus W'$, define a $\mathcal{H}_Q(\lambda)^*$-valued function $\vartheta_\lambda[\alpha]$ on $\mathfrak{H}_n$ by

$$\langle \vartheta_\lambda[\alpha](z), P \rangle = \sum_{\ell \in M_{m,n}(\mathbb{Z})} P(\ell + \alpha') \cdot \mathbf{e}\left(\frac{1}{2}\langle \ell + \alpha', Q(\ell + \alpha' z)\rangle + \langle \ell + \alpha', \alpha''\rangle\right)$$

for all $z \in \mathfrak{H}_n$ and $P \in \mathcal{H}_Q(\lambda)$. Put

$$\vartheta^*_\lambda[\alpha](z) = \mathbf{e}\left(-\frac{1}{2}\langle \alpha', \alpha''\rangle\right) \cdot \vartheta_\lambda[\alpha](z).$$

Let $\{\alpha'_1, \cdots, \alpha'_N\}$ be a complete set of representatives of $Q^{-1}M_{m,n}(\mathbb{Z})/M_{m,n}(\mathbb{Z})$, and put $\alpha_i = (\alpha'_i, 0) \in V = W \oplus W'$. Put

$$\Theta_\lambda[\alpha](z) = \left[\mathbf{e}\left(\frac{1}{2}D(\alpha_i, \alpha)\right) \cdot \vartheta^*_\lambda[\alpha + \alpha_i](z)\right]_{i=1,\cdots,N}.$$

For each $\widetilde{\gamma} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widetilde{Sp}(n,\mathbb{Z})$, choose any $\delta = (\delta', \delta'') \in V = W \oplus W'$ such that

$$\delta' \equiv \frac{1}{2}(Q^{-1} \boxtimes (c{}^t d))_0 \pmod{Q^{-1}M_{m,n}(\mathbb{Z})}, \quad \delta'' \equiv \frac{1}{2}(Q \boxtimes (a{}^t b))_0 \pmod{M_{m,n}(\mathbb{Z})}.$$

Here $(Q\boxtimes(c^td))_0 \in Q^{-1}M_{m,n}(\mathbb{Z})/2Q^{-1}M_{m,n}(\mathbb{Z})$ (resp. $(Q\boxtimes(a^tb))_0 \in M_{m,n}(\mathbb{Z})/2M_{m,n}(\mathbb{Z})$) is defined by the relation

$$\mathrm{tr}(y(Q \boxtimes (c^td))^ty) \equiv \mathrm{tr}((Q \boxtimes (c^td))_0{}^ty) \pmod{2\mathbb{Z}}$$

$$(\text{resp. } \mathrm{tr}(x(Q \boxtimes (a^tb))^tx) \equiv \mathrm{tr}((Q \boxtimes (a^tb))_0{}^tx) \pmod{2\mathbb{Z}}.)$$

Put $U_Q(\widetilde{\gamma},\delta) = U(\imath_Q(\widetilde{\gamma}),\delta)$ which is a unitary automorphism of $\mathrm{Ind}(H[L^*], H[\Lambda_0];\xi_0)$. The complete set of representatives $\{\alpha_1,\cdots,\alpha_N\}$ of L^*/Λ_0 defines an ortho-normal $\mathbb{C}$-basis of $\mathrm{Ind}(H[L^*], H[\Lambda_0];\xi_0)$. The representation matrix of $U_Q(\widetilde{\gamma},\delta)$ with respect to the ortho-normal $\mathbb{C}$-basis is also denoted by $U_Q(\widetilde{\gamma},\delta)$. Then we have the following transformation formula of $\Theta_\lambda[\alpha](z)$;

Theorem 5.4.1.

$$\begin{aligned}&\mathbf{e}\left(\frac{1}{2}\langle\delta',\delta''\rangle - \frac{1}{2}D(\delta,\alpha)\right)\cdot\Theta_\lambda[\alpha](\gamma(z))\\ &= U_Q(\widetilde{\gamma},\delta)\Theta_\lambda[(\alpha+\delta)(1\boxtimes_Q\gamma)](z)\circ J_\lambda(\gamma,z)^{-1}J_{\frac{1}{2}}(\widetilde{\gamma},z)^m.\end{aligned}$$

The explicit formula of the matrix coefficients of $U_Q(\widetilde{\gamma},\delta)$ is given by the explicit formula given by Theorem 4.4.1 and the group theoretic properties given by Corollary 2.4.3.

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