数理解析研究所講究録
1056 巻 1998 年 19-28

CRITICAL NONLINEAR WAVE EQUATIONS IN FRACTIONAL ORDER SOBOLEV SPACES

T. OZAWA (小澤 徹)

Department of Mathematics, Hokkaido University

In this note I describe some recent work on nonlinear wave equations, done jointly with M. Nakamura [20, 21]. We consider the nonlinear wave equations of the form

$$\partial_t^2 u - \Delta u = f(u) \tag{1}$$

where u is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\mathbb{R}^n$, and f is a complex-valued function, a typical form of which is the single power interaction

$$f(u) = \lambda |u|^{p-1} u \tag{2}$$

with $\lambda \in \mathbb{R}$ and $1 < p < \infty$.

There is a large literature on the Cauchy problem for the equation (1) and on the asymptotic behavior in time of the global solutions [3, 4, 7-12, 14, 15, 24, 25, 28 and references therein]. The Cauchy problem for (1) has been studied mainly in the space of classical solutions and in the energy space, while there arises a new interest in the treatment of the Cauchy problem in the Sobolev spaces $H^s = (1-\Delta)^{-s/2} L^2(\mathbb{R}^n)$ of fractional order s with $0 \le s < n/2$. In connection with the H^s theory for (1) with (2), a homogeneity argument indicates that the power p in (2) is critical [resp. subcritical] at the level of H^s if and only if $p = 1 + 4/(n-2s)$ [resp. $p < 1 + 4/(n-2s)$]. Though the critical power $p = 1 + 4/(n-2s)$ at the level of H^s is the same at that of nonlinear Schrödinger equations [2, 6, 13, 17, 18, 26], it would be natural to regard the power as $p = 1 + 4/((n-1) - 2(s-1/2))$ by the following reasons.

(i) In view of the sharp decay estimates for the free wave and Schrödinger equations, there is a natural shift in the corresponding space dimensions with difference by one. This implies that results in the nonlinear wave equations should be often compatible with the corresponding results in the nonlinear Schrödinger equations by reducing the space dimension by one. The origin of the discrepancy may be traced back to the rank of the Hessian of phase functions in the oscillatory integrals for fundamental solutions.

(ii) In view of the Strichartz estimates in the diagonal case [28], there is a natural shift in the corresponding regularity requirements on the data with difference by one half.

(iii) In view of the symmetry groups acting on the associated Lagrangeans, the conformal powers of the nonlinear wave and Schrödinger equations are given respectively by $p = 1 + 4/(n-1)$ and $p = 1 + 4/n$, while the corresponding space of data are given respectively by $H^{1/2}$ and L^2.

By the arguments above, we could expect the H^s theory for (1) at the level compatible with that of the nonlinear Schrödinger equations in the critical case where $p = 1 + 4/(n - 2s)$ with $1/2 \leq s < n/2$. This in turn implies that $n \geq 2$ and $1 + 4/(n-1) \leq p < \infty$ and that the critical power $p = 1 + 4/(n-2s)$ loses its meaning at the level of $H^{n/2}$.

The purpose of this note is twofold. The first is to make the H^s theory for (1) complete with the whole admissible range $1/2 \leq s < n/2$. This means that we intend to extend the results of Lindblad and Sogge [15] to the spaces with higher regularity with the notion of criticality preserved. The second is to examine the critical phenomenon as the index s grows to $n/2$ and to construct the $H^{n/2}$ theory for (1) with critical nonlinearity of specific growth at infinity.

As regards the H^s theory with $0 \leq s < n/2$, the power behavior of the nonlinearity determines the order of the Sobolev space where smallness of the data is imposed to ensure the existence and uniqueness of global H^s solutions. This is the right phenomenon, as is usual with other nonlinear evolution equations with dilation structure, such as the nonlinear heat and Schrödinger equations with single power interaction and the Navier-Stokes equations.

In contrast, when $s > n/2$, no specific behavior of nonlinearity is required of the H^s theory for (1) at least locally in time. In fact, when $s > n/2$, for the existence and uniqueness of local H^s solutions one has only to assume that $f \in C^k(\mathbb{C};\mathbb{C})$ with $f(0) = 0$, where differentiability refers to the real sense and k is the largest integer less than or equal to s [19]. The proof depends on the usual Sobolev embedding $H^s \hookrightarrow L^\infty$ for $s > n/2$ in an essential way.

The case $s = n/2$ may therefore be regarded as the borderline in two aspects: (1) No power behavior of interaction amounts to the critical nonlinearity at the level of $H^{n/2}$. (2) Poitwise control of solutions falls beyond the scope of the $H^{n/2}$ theory, so that any argument similar to that of the H^s theory with $s > n/2$ breaks down even for local theory without specific behavior of interaction.

In addition to the critical phenomena described above, $H^{n/2}$ solutions deserve attention as finite energy solutions for $n = 2$ and as strong solutions for $n = 4$.

We prove the existence and uniqueness of global $H^{n/2}$ solutions to (1) with small Cauchy data under the nonlinearity of exponential type. This is reminiscent of Trudinger's inequality which replaces the Sobolev embedding in the limiting case on the basis of the exponential estimates in terms of functions in the critical order Sobolev space $H^{n/2}$ [16, 22, 23, 27, 29].

To state the results precisely, we use the following notation. For any r with $1 \le r \le \infty$, $L^r = L^r(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and any r with $1 < r < \infty$, $H^s_r = (1 - \Delta)^{-s/2} L^r$ denotes the Sobolev space defined in terms of Bessel potentials. For any $s \in \mathbb{R}$ and any r, m with $1 \le r, m \le \infty$, $B^s_{r,m}$ denotes the Besov space defined as the space of distributions u such that $\{2^{sj} \| \phi_j * u; L^r \|\}_{j=0}^\infty \in \ell^m$, where $\{\phi_j\}$ is a dyadic decomposition on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and any r with $1 < r < \infty$, $\dot{H}^s_r$ denotes the homogeneous Sobolev space defined as the space of classes of distributions u modulo polynomials such that $(-\Delta)^{-s/2} u \in L^r$. For any $s \in \mathbb{R}$ and any r, m with $1 \le r, m \le \infty$, $\dot{B}^s_{r,m}$ denotes the homogeneous Besov space defined as the space of classes of distributions u modulo polynomials such that $\{2^{sj} \| \psi_j * u; L^r \|\}_{j=-\infty}^\infty \in \ell^m$, where $\{\psi_j\}$ is a dyadic decomposition on $\mathbb{R}^n \backslash \{0\}$. We refer to [1, 8, 30] for general information on Besov and Triebel-Lizorkin spaces and their homogeneous counterparts. For simplicity, we put $H^s = H^s_2, \dot{H}^s = \dot{H}^s_2$, $B^s_r = B^s_{r,2}, \dot{B}^s_r = \dot{B}^s_{r,2}$. For any interval $I \subset \mathbb{R}$ and any Banach space X we denote by $C(I; X)$ the space of strongly continuous functions from I to X and by $L^q(I; X)$ the space of strongly measurable functions u from I to X such that $\|u(\cdot); X\| \in L^q(I)$. The Cauchy problem for the equation (1) with data $(u(0), \partial_t u(0)) = (\phi, \psi)$ will be treated in the form of the integral equation

$$u(t) = K(t)\psi + \dot{K}(t)\phi + \int_0^t K(t-t')f(u(t'))dt', \tag{3}$$

where $K(t) = \omega^{-1}\sin t\omega$, $\dot{K}(t) = \cos t\omega$, and $\omega = (-\Delta)^{1/2}$. To treat the Cauchy problem both at finite and infinite times on the basis of the free unitary group in the generalized energy space, we formally differentiate (3) in time and introduce the following system of equations

$$\begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} = U(t)\begin{pmatrix} \phi \\ \psi \end{pmatrix} + \int_{t_0}^t U(t-t')\begin{pmatrix} 0 \\ f(u(t')) \end{pmatrix} dt', \tag{4}$$

where $\binom{u(t_0)}{\partial_t u(t_0)} = U(t_0)\binom{\phi}{\psi}$ is the prescribed Cauchy data at time t_0 and

$$U(t) = \begin{pmatrix} \dot{K}(t) & K(t) \\ -\omega^2 K(t) & \dot{K}(t) \end{pmatrix} = \exp t \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$$

is a unitary group in the Hilbert spaces $E^s \equiv \dot{H}^s \oplus \dot{H}^{s-1}$ if $1/2 \le s < n/2$ and $E^{n/2} = (\dot{H}^{n/2} \cap \dot{H}^{1/2}) \oplus (\dot{H}^{n/2-1} \cap \dot{H}^{-1/2})$. The equation (4) will be studied in the space X^s with $1/2 \le s \le n/2$ defined as

$$X^s = C(\mathbb{R}; E^s) \cap \bigcap_{(1/q, 1/r, \rho) \in \Lambda^s} L^q(\mathbb{R}; \dot{B}^{\rho}_{r} \oplus \dot{B}^{\rho-1}_{r})$$

if $1/2 \le s < n/2$, and

$$X^{n/2} = (C \cap L^\infty)(\mathbb{R}; E^s) \cap L^{q_0}(\mathbb{R}; (B^{(n-1)/2}_{q_0} \cap \dot{B}^0_{q_0}) \oplus (B^{(n-3)/2}_{q_0} \cap \dot{B}^{-1}_{q_0}))$$

where $q_0 = 2(n+1)/(n-1)$,

$$\begin{aligned} \Lambda^s &= \{(1/q, 1/r, \rho);\ (1/q, 1/r) \in \Lambda_0,\ 0 \le \rho \le s,\ 0 \le 1/q \le n/2 - s\}, \\ \Lambda_0 &= \{(1/q, 1/r);\ 0 \le 1/q, 1/r \le 1/2,\ (1/q, 1/r) \ne (1/2, 1/2 - 1/(n-1)), \\ &\qquad 1/r + 2/((n-1)q) \le 1/2\}. \end{aligned}$$

For the nonlinear interaction behaving as a power p at zero, we introduce the following assumption.

(A)$_k$ $f \in C^k(\mathbb{C};\mathbb{C})$ and $f^{(j)}(0) = 0$ for any j with $0 \le j \le k$. There exists a constant C such that for all $z_1, z_2 \in \mathbb{C}$

$$|f^{(k)}(z_1) - f^{(k)}(z_2)| \le \begin{cases} C(|z_1|^{p-k-1} + |z_2|^{p-k-1})|z_1 - z_2| & \text{if } p \ge k+1, \\ C|z_1 - z_2|^{p-k} & \text{if } p < k+1. \end{cases}$$

Here $f^{(j)}$ denotes any of the j-th order derivatives of f with respect to z and $\bar{z}$ and $|f^{(j)}|$ denotes the maximum of the moduli of those derivatives. Single power interaction (2) satisfies (A)$_k$ with $0 \le k < p$.

For the nonlinear interaction having an exponential growth at infinity, we introduce the following assumption.

(B) $f \in C^{[n/2]}(\mathbb{C};\mathbb{C}), f(0) = 0$. There exist two positive constants λ and C such that for all $z \in \mathbb{C}$

$$|f'(z)| \le C|z|^{4/(n-1)} \exp(\lambda|z|^2).$$

Moreover,

$$|f''(z)| \le C|z|^{1/3} \exp(\lambda|z|^2) \quad \text{if } n = 4,$$
$$\mathrm{Max}_{2\le k\le[n/2]}|f^{(k)}(z)| \le C \exp(\lambda|z|^2) \quad \text{if } n \ge 5.$$

With the notation above we now state the main results in this paper. For any s with $1/2 \le s \le n/2$ and $\varepsilon > 0$, we denote by E^s_ε the closed ball in E^s with the center at zero and radius ε.

Theorem I (Critical case at the level of H^s with $1/2 \le s < n/2$).

Let $n \ge 2$. Let s and p satisfy

$$1/2 \le s < n/2,$$
$$[s] < p = 1 + 4/(n - 2s).$$

Let f satisfy $(A)_{[s]}$. Then there exists $\varepsilon > 0$ with the following property.
(1) For any $(\phi, \psi) \in E^s_\varepsilon$ at $t_0 = 0$ the equation (4) has a unique solution $(u, \partial_t u) \in X^s$. Moreover, there exist two pairs of asymptotic states $(\phi_\pm, \psi_\pm) \in E^s$ such that

$$\left\|\begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - U(t)\begin{pmatrix} \phi_\pm \\ \psi_\pm \end{pmatrix}; E^s\right\| \to 0 \qquad (5)_\pm$$

as $t \to \pm\infty$.

(2) For any $(\phi_+, \psi_+) \in E^s_\varepsilon$ at $t_0 = +\infty$ [resp. $(\phi_-, \psi_-) \in E^s_\varepsilon$ at $t_0 = -\infty$] the equation (4) has a unique solution $(u, \partial_t u) \in X^s$ satisfying $(5)_+$ [resp. $(5)_-$].

Theorem II (Critical case at the level of $H^{n/2}$).

Let $n \geq 2$. Let f satisfy (B). Then there exists $\varepsilon > 0$ with the following property.
(1) For any $(\phi, \psi) \in E^{n/2}_\varepsilon$ at $t_0 = 0$ the equation (4) has a unique solution $(u, \partial_t u) \in X^{n/2}$. Moreover, there exist two pairs of asymptotic states $(\phi_\pm, \psi_\pm) \in E^{n/2}$ such that

$$||\begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - U(t)\begin{pmatrix} \phi_\pm \\ \psi_\pm \end{pmatrix}; E^{n/2}|| \to 0 \tag{6$_\pm$}$$

as $t \to \pm\infty$.

(2) For any $(\phi_\pm, \psi_\pm) \in E^{n/2}_\varepsilon$ at $t_0 = +\infty$ [resp. $(\phi_-, \psi_-) \in E^{n/2}_\varepsilon$ satisfying $(6)_+$ [resp. $(6)_-$].

Remark 1. The theorems above shows the existence and asymptotic completeness of the wave operators $W_\pm : (\phi_\pm, \psi_\pm) \mapsto (u(0), \partial_t u(0)) = (\phi, \psi)$ defined on small asymptotic states in E^s with $1/2 \leq s \leq n/2$. The scattering operator S is then defined as $S = W_+^{-1} \circ W_+$.

Remark 2. A part of Theorem I is proved by Pecher [24] and Lindblad-Sogge [15] in the cases where $s = 1$ with $3 \leq n \leq 5$ and $1/2 \leq s \leq 3/2$ with $n \geq 2$, respectively. There are several results on the ill-posedness for (1) with $s < 1/2$ [14, 15].

Remark 3. The power $p = 1 + 4/(n - 2s)$ comes out as a critical one in $\dot{H}^s \oplus \dot{H}^{s-1}$ in the sense that $||(u, \partial_t u); \dot{H}^s \oplus \dot{H}^{s-1}||$ is invariant under the dilation $u \mapsto u_\lambda$ if and only if $s = n/2 - 2/(p-1)$, where $u_\lambda(t, x) \equiv \lambda^{-2/(p-1)} u(\lambda^{-1} t, \lambda^{-1} x), \lambda > 0$. We note here that $u \mapsto u_\lambda$ leaves (1) with (2) invariant in the sense that u solves (1) with (2) if and only if u_λ does.

Remark 4. The argument in Remark 3 makes sense only when $s < n/2$ and loses its meaning when $s = n/2$ for instance. In view of Trudinger's inequality the growth rate as $\exp(\lambda |z|^2)$ at infinity seems to be optimal at the level of $\dot{H}^{n/2}$. We note that the L^∞ norm is out of control of the $H^{n/2}$ norm even when the latter is infinitesimally small.

We now give a sketch of the proofs. As usual the method depends on a partial contraction argument on (4) in the space X^s where the Strichartz type estimates for the free propagatpr fit naturally. To be more specific we use the inequality

$$||U(\cdot)\begin{pmatrix}\phi\\ \psi\end{pmatrix}; L^q(\mathbb{R}; \dot{B}^{\rho}_r \oplus \dot{B}^{\rho-1}_r)|| \le C||\begin{pmatrix}\phi\\ \psi\end{pmatrix}; E^s||,$$

where $(1/q, 1/r) \in \Lambda_0$ and $s = \rho + n(1/2 - 1/r) - 1/q$, and its inhomogeneous version [10, 15, 28]. We combine those Strichartz estimates with the inequality in the following

Proposition 1 [17]. *Let p and s satisfy $1 \le p < \infty$ and $0 \le s < p$. Let ℓ, r, m satisfy $1 < \ell \le r < \infty, 2 \le r, m < \infty, 1/\ell = 1/r + (p-1)/m$. Let f satisfy $f \in C^{[s]}(\mathbb{C}; \mathbb{C})$. Then*

$$||f(u); \dot{B}^s_\ell|| \le C||u; \dot{B}^0_m||^{p-1}||u; \dot{B}^s_r||.$$

The basic estimates that the required iteraction scheme goes though are completed by embedding theorems and convexity inequalities for the homogeneous Besov spaces. For the proof of Theorem II we expand the exponential nonlinearity, estimate individual L^p noems, and consider the convergence of the resulting series of norms. For that purpose we need information on the growth rate in p of the L^p estimate in terms of the $H^{n/2}$ norm. To be more specific we use the inequalities in the following

Proposition 2 [18]. *Let $1 < r < \infty$. Then there exists a constant C_0 such that for any q with $r \le q < \infty$ the following estimates hold.*

$$||u; L^q|| \le C_0 q^{1/2+(r-2)/(2q)}||u; \dot{H}^{n/2}||^{1-r/q}||u; L^r||^{r/q},$$
$$||u; \dot{B}^0_q|| \le C_0 q^{1/2+(r-2)/(2q)}||u; \dot{H}^{n/2}||^{1-r/q}||u; \dot{B}^0_r||^{r/q}.$$

The proof of Proposition 1 follows closely that of [6, Lemma 3.4] in the sense that it depends on an equivalent norm on the homogeneous Beson spaces in terms of modulus of continuity with differences of second order, though actual proof is rather involved because of higher derivatives of functions coming from derivatives of the composite function $f \circ u$. Proposition 2 follows from a sharp form of the Hardy-Littlewood-Sobolev inequality [22; Inequality (2.6)] and convexity inequalities between homogeneous Besov and Sobolev spaces. See [18] for details. The corresponding results have been obtained in [17, 18] for the nonlinear Schrödinger equations in the fractional order Sobolev spaces.

References

[1] J. Bergh and J. Löfström, "Interpolation Spaces," Springer-Verlag, Berlin-Heidelberg-New York, 1976

[2] T. Cazenave, F. B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in* H^s, Nonlinear Analysis, TMA, 14(1990), 807-836.

[3] V. Georgiev, P. P. Schirmer, *Global existence of low regularity solutions of nonlinear wave equations*, Math. Z., **219**(1995), 1-19.

[4] J. Ginibre, *Scattering theory in the energy space for a class of nonlinear wave equation*, Advanced Studies in Pure Mathematics, **23**(1994), 83-103.

[5] J. Ginibre, *An Introduction to Nonlinear Schrödinger Equations*, in *"Nonlinear Waves,"* GAKUTO International Series, Mathematical Sciences and Applications, **10**(1997), 85-133.

[6] J. Ginibre, T. Ozawa, G. Velo, *On the existence of the wave operators for a class of nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, Physique théorique, **60**(1994), 211-239.

[7] J.Ginibre, A. Soffer, G.Velo, *The global Cauchy problem for the critical nonlinear wave equation*, J. Funct. Anal., **110**(1992), 96-130.

[8] J. Ginibre, G. Velo, *The global Cauchy problem for the nonlinear Klein-Gordon equation*, Math. Z., **189**(1985), 487-505.

[9] J. Ginibre, G. Velo, *Regularity of solutions of critical and subcritical nonlinear wave equations*, Nonlinear Analysis, Theory, Methods & Applications, **22**(1994), No.1, 1-19.

[10] J. Ginibre, G. Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal., **133**(1995), 50-68.

[11] L. V. Kapitanski, *Weak and yet weaker solutions of semilinear wave equations*, Commun. PDE **19**(1994), 1629-1676.

[12] L. V. Kapitanski, *Global and unique weak solutions of nonlinear wave equations*, Mathematical Research Letters, 1(1994), 211-223.

[13] T.Kato, *On nonlinear Schrödinger equations II. H^s-solutions and unconditional well-posedness*, J.d'Anal.Math., **67**(1995), 281-306.

[14] H. Lindblad, *A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations*, Duke Math. J., **72**(1993), 503-539.

[15] H. Lindblad, D. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal., **130**(1995), 357-426.

[16] J.Moser, *A sharp form of an inequality by N.Trudinger*, Indiana Univ.Math.J. **20** (1971) 1077-1092.

[17] M. Nakamura, T. Ozawa, *Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces*, Rev. Math. Phys., **9**(1997), 397-410.

[18] M. Nakamura, T. Ozawa, *Nonlinear Schrödinger equations in the Sobolev space of critical order*, J. Funct. Anal., (in press).

[19] M. Nakamura, T. Ozawa, *The Cauchy problem for nonlinear wave equations in the Sobolev space of critical order*, Discrete and Continuous Dynamical Systems (in press).

[20] M. Nakamura, T. Ozawa, *The Cauchy problem for nonlinear wave equations in the homogeneous Sobolev space*, Ann. Inst. Henri Poincaré, Physique théorique (in press).

[21] M. Nakamura, T. Ozawa, *Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth* (preprint).

[22] T. Ozawa, *On critical cases of Sobolev's inequalities*, J. Funct. Anal., **127**(1995), 259-269.

[23] T. Ozawa, *Characterization of Trudinger's inequality*, J. Inequal. Appl., 1(1997), 369-374.

[24] H. Pecher, *Nonlinear small data scattering for the wave and Klein-Gordon equation*, Math. Z., **185**(1984), 261-270.

[25] H. Pecher, *Local solutions of semilinear wave equations in H^{s+1}*, Math. Methods Appl. Sci., **19**(1996), 145-170.

[26] H. Pecher, *Solutions of semilinear Schrödinger equations in H^s*, Ann. Inst. Henri Poincaré, Physique théorique, **67** (1997), 259-296.

[27] R. S. Strichartz, *A note on Trudinger's extension of Sobolev's inequalities*, Indiana Univ. Math. J. **21**(1972), 841-842.

[28] R. S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and*

decay of solutions of wave equations, Duke Math. J., **44** (1977), 705-714.

[29] M. Struwe, *Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces*, Ann. Inst. Henri Pointcaré, Analyse nonlinéaire **5**(1988), 425-464.

[30] H. Triebel, "Theory of Function Spaces," Birkhäuser, 1983.

[31] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17**(1967), 473-483.