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The Levi Problem and the Structure Theorem for Non-Negatively Curved Complete Kähler Manifolds

Shigeharu TAKAYAMA : 高山 茂晴
Naruto Univ. Edu. / Osaka Univ.

§1. Introduction and Statement of Result.

Recent development of complex geometry enable us to get global sections of an adjoint bundle on a projective manifold, under a reasonable numerical condition [AS] [D2] [Tj]. The theory which is mainly developed by Demailly and Siu, is very concrete and constructive. Hence their method can be applied in various contexts [Ty2] [Ty4]. Here we apply their method to construct holomorphic functions on certain pseudoconvex manifolds.

Levi problem on a complex manifold.

Let us consider the following function theoritic properties of a complex manifold X:

(i) X is holomorphically convex;

(ii) there exists a proper holomorphic map $X \longrightarrow \mathbf{C}^N$;

(iii) X is weakly 1-complete, i.e., there exists a smooth function $\Phi : X \longrightarrow \mathbf{R}$ which is plurisubharmonic and exhaustive.

(iv) X is pseudoconvex, i.e., there exists a continuous plurisubharmonic exhaustion function.

It is well known that implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) hold. Then the Levi problem asks whether the implication (iv) $\Rightarrow$ (i) (or (iii) $\Rightarrow$ (i)) holds or not.

If X is a domain in $\mathbf{C}^n$, the answer is affirmative. However, as we will see below, there exists a quotient complex manifold of $\mathbf{C}^n$ such that it is weakly 1-complete but it is not holomorphically convex. Therefore we need some condition to get affirmative results in general. Here we impose a condition on the canonical bundle. The motive of this work in the following

Theorem 1.1 Ohsawa [O]. Let X be a 2-dimensional complex manifold with a negative canonical bundle K_X. Then X is holomorphically convex if and only if it is weakly 1-complete.

Here we generalize Ohsawa's theorem for higher dimensional cases as follows:

Main Theorem 1.2. Let X be a complex manifold with a negative canonical bundle K_X. Then X is holomorphically convex if and only if it is pseudoconvex.

The proof depends on effective construction of singular Hermitian metric and related vanishing theorem.

Structure theorem for non-negatively curved complete Kähler manifold. There are relations of metric properties of manifolds and the pseudoconvexity. We apply our Main Theorem to study such relations. Let us recall Riemannian case [CG].

Cheeger-Gromoll: Let (M, g) a complete Riemannian manifold with non-negative sectional curvature. Then there exists a totally geodesic compact submanifold S such that $M \approx N_{S/M}$, i.e., M is diffeomorphic to the total space of the normal bundle of S in M.

Their key result: basic construction [CG, §1] claim that, by using Busemann's function, a complete Riemannian manifold with non-negative sectional curvature has a continuous geodesically convex exhaustion function. Inspired by their works, Greene-Wu [GW] studied non-negatively curved Kähler manifolds.

Greene-Wu: Let (X, g) a complete Kähler manifold with positive (resp. non-negative) sectional curvature. Then there exists a continuous strictly plurisubharmonic (resp. plurisubharmonic) exhaustion function. Their main conclusion is as follows: Let (X, g) as above with positive sectional curvature, then X is Stein and diffeomorphic to $\mathbf{C}^n$. They also asked a kind of the Levi problem:

Conjecture 1.3 (Greene-Wu [GW]). Every complete Kähler manifold with non-negative sectional curvature and positive Ricci curvature is holomorphically convex.

Since a Kähler manifold (X, g) as in Conjecture 1.3 is pseudoconvex and the canonical bundle is negative, we can solve Conjecture 1.3 affirmatively by our Main Theorem. It means that there exists a proper holomorphic map: the Remmert reduction $R : X \longrightarrow Y$ to a Stein space. Moreover since X has semi-positive holomorphic tangent bundle, such a holomorphic reduction has very neat structure; for example, as in the structure theorem of Demailly-Peternell-Schneider [DPS]. Actually we can show that the Remmert reduction is a smooth holomorphic map over a Stein manifold, and that every fibre has a Kähler metric with non-negative sectional curvature with positive Ricci curvature. Then by the uniformization theorem of Mok [M] and the rigidity theorem of Bott [B], we can show

Theorem 1.4. Every complete Kähler manifold with non-negative sectional curvature and positive Ricci curvature, has a structure of holomorphic fiber bundle over a Stein mainfold whose typical fibre is biholomorphic to some compact Hermitian symmetric manifold.

The following example will explain on the positivity condition of Ricci curvature (or that of anti-canonical bundle), and differences from the Riemannian case. Let us consider a quotient complex Lie group $X_a := \mathbf{C}^2/\Gamma_a$ of $\mathbf{C}^2$ by a rank 3 discrete subgroup

$$\Gamma_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbf{Z} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbf{Z} + \begin{pmatrix} \sqrt{-1} \\ a \end{pmatrix}\mathbf{Z} \text{ with } a \in \mathbf{R}.$$

Then every X_a is a non-compact, weakly 1-complete Kähler manifold with a trivial (holomorphic) tangent bundle $T_{X_a} \cong \mathcal{O}_{X_a} \oplus \mathcal{O}_{X_a}$ (cf. [K1]). As real Lie groups, they have very simple strucure $X_a \cong T^3_{\mathbf{R}} \times \mathbf{R}$. On the other hand, their complex structures are very subtle [K2]. By an elementary argument we see the following four conditions are equivalent to each other:

(i) X_a has a non-constant holomorphic function;

(ii) X_a is a product of an elliptic curve and $\mathbf{C}^*$;

(iii) there exists a compact complex curve in X_a;

(iv) a is rational.

It may happen $H^1(X_a, \mathcal{O}_{X_a})$ is non-Hausdorff, if a belongs to a certain class of transcendental numbers. Although our approach can be applied to establish Lefschetz type theorems on certain kind of such non-compact quotients $\mathbf{C}^n/\Gamma$, so-called quasi-abelian varieties [Ty3].

§2. Vanishing Theorem.

Let us recall basic tools and notions [D1] [D2]. Let X a complex manifold, (L, h) a Hermitian holomorphic line bundle on X with positive curvature $\operatorname{curv} h := \sqrt{-1}\partial\bar{\partial}\log h > 0$. Let β a positive rational number, $s = \{s_j\}_{j\in J}$ a finite number of multivalued holomorphic sections of $L^{\otimes\beta}$ on X. The **multiplier ideal sheaf** $\mathcal{I}(s)$ of s is defined as follows: for every open set U,

$$\mathcal{I}(s)(U) := \left\{ f \in H^0(U, \mathcal{O});\ \int_U |f|^2(|s|^2)^{-1}dv < +\infty \right\}.$$

We see $\mathcal{I}(s)$ is a coherent ideal sheaf of $\mathcal{O}_X$. We denote $V\mathcal{I}(s) := \mathcal{O}_X/\mathcal{I}(s)$ the closed complex subspace of X definied by $\mathcal{I}(s)$. For every integer $m > \beta$, we define a **singular Hermitian metric** of $L^{\otimes m}$ by

$$H_m := \frac{h^m}{|s|^2} = h^{m-\beta}\frac{h^\beta}{|s|^2}.$$

We see, by the local computation, the curvature current satisfies

$$\operatorname{curv} H_m \geq (m-\beta)\operatorname{curv} h > 0.$$

We also define $\mathcal{I}(H_m) := \mathcal{I}(s)$ the multiplier ideal sheaf of H_m. Now we recall

Demailly's Nadel vanishing theorem 2.1. Assume that X is pseudoconvex. Then $H^1(X, K_X \otimes L^{\otimes m} \otimes \mathcal{I}(H_m)) = 0$. As a consequence, the restriction map

$$H^0(X, K_X \otimes L^{\otimes m}) \longrightarrow H^0(V\mathcal{I}(s), K_X \otimes L^{\otimes m} \otimes \mathcal{O}_X/\mathcal{I}(s))$$

is surjective. $\square$

If we have sections on $V\mathcal{I}(s)$, then we have global sections of $K_X \otimes L^{\otimes m}$. This is rather abstruct existence theorem. We have to controle the tensor power m and the locus $V\mathcal{I}(s)$. A zero-dimensional subspace is a candidate of a good locus, but then it is hard to controle the tensor power m. In the next section, we do effective construction of singular Hermitian metrics for another candidate of a locus which has a nice property.

§3. Effective Vanishing Theorem.

We let X a non-compact pseudoconvex manifold with a continuous plurisubharmonic exhaustion function $\Phi : X \longrightarrow \mathbf{R}$, and let (L, h) a positive line bundle on it. We consider the following equivalent relation on X: Let $x, y \in X$. Then "$x \sim y$" iff x and y are joined by a finite number of irreducible **compact** complex subspaces of X. We take the quotient $R : X \longrightarrow RX := X/\sim$, as sets, which we call the formal Remmert reduction of X.

Remark 3.1.

(1) If X is holomorphically convex, then $R : X \longrightarrow RX$ is the Remmert reduction.

(2) the plurisubharmonic function Φ is constant along each fibre of R.

(3) RX is not a point.

A fibre is a candidate of a good locus. We would like to separate two distinct fibres of R by holomorphic functions.

Let us take two distinct points $x'_i \in RX$ $(i = 1, 2)$, and set $V_i := R^{-1}(x'_i)$. We note that every V_i is a relatively compact set of X and that $V_1 \cap V_2 = \emptyset$. Take a sublevel set $X_c := \{x \in X;\ \Phi(x) < c\}$ of (X, Φ) containing V_1 and V_2. Our main technical result is as follows.

Theorem 3.2. Let β be a rational number with $0 < \beta < 1$. Then there exist a finite number of multivalued holomorphic sections $s = \{s_j\}_{j \in J}$ of $L^{\otimes \beta}$ on X with the following three properties:

(1) $V\mathcal{I}(s) \cap V_i \neq \emptyset$ $(i = 1, 2)$;
(2) $V\mathcal{I}(s)$ has only one irreducible component Z which intersects V_1;
(3) The irreducible component Z in (2) is compact. □

We can decompose $V\mathcal{I}(s) = Z \amalg V$ with Z is compact and $Z \subset V_1$; $V \cap V_2 \neq \emptyset$ but $V \cap V_1 = \emptyset$. Then applying Demailly's Nadel vanishing theorem 2.1, we have

Corollary 3.3. The restriction map

$$H^0(X_c, K_X \otimes L) \longrightarrow H^0(Z, K_X \otimes L \otimes \mathcal{O}_Z) \oplus H^0(V, K_X \otimes L \otimes \mathcal{O}_V)$$

is surjective. □

We do not know existence of sections. However in a special case, we do have. For example the formal Remmert reduction is bijective, another example is negative canonical bundle case as follows.

§4. Levi Problem.

Let us go back to the original problem. We let (X, Φ) be a non-compact pseudoconvex manifold with negative canonical bundle K_X. We take $L = K_X^{\otimes(-1)}$ in §3. We use the same notation as in §3. Then by Corollary 3.3 we have a surjection

$$H^0(X_c, \mathcal{O}) \longrightarrow H^0(Z, \mathcal{O}_Z) \oplus H^0(V, \mathcal{O}_V).$$

We extend a holomorphic function $(1, 0) \in H^0(Z, \mathcal{O}_Z) \oplus H^0(V, \mathcal{O}_V)$ on X_c. Since every holomorphic function is constant along the fibre of $R : X \longrightarrow RX$, there exists a holomorphic function $f \in H^0(X_c, \mathcal{O})$ such that $f|_{V_1} \equiv 1$ and $f|_{V_2} \equiv 0$. We can separate every pair of two distinct fibres of R by a holomorphic function on X_c. Then after some arguments we see every X_c is holomorphically convex, and then by Narasimhan's approximation theorem [Nr], X is holomorphically convex.

§5. Example of Construction.

We consider 2-dimensional case in §3. We set $n = 2$. Let us take a rational number ε_0 with $0 < \varepsilon_0 < \beta$, and positive integers p and q

such that $np/q < \varepsilon_0$. By the so-called line bundle convexity property of the sublevel set X_c with respect to the positive line bundle L, we see $\dim H^0(X_c, L^{\otimes m}) = +\infty$ for every large m. We take a point $x_i \in V_i$ $(i = 1, 2)$. Then we can take a non-zero section

$$\sigma \in H^0(X_c, L^{\otimes pm} \otimes \mathcal{M}_{x_1}^{mq} \mathcal{M}_{x_2}^{mq}),$$

here $\mathcal{M}_x$ is the maximal ideal sheaf of x in X. We consider (log-canonical thresholds)

$$\begin{aligned} \alpha_i &:= \sup\{t \geq 0;\ V\mathcal{I}(\sigma^{tn/(mq)}) \cap V_i = \emptyset\}; \\ \alpha &:= \max\{\alpha_1,\ \alpha_2\}. \end{aligned}$$

We see every α_i is a rational number with $0 < \alpha_i \leq 1$. The section $\sigma_1 := \sigma^{\alpha n/(mq)}$ may be desired one. If it is not, we continue the above procedure on the locus $V\mathcal{I}(\sigma_1)$. Let $V\mathcal{I}(\sigma_1) = \bigcup Y_i$ be the irreducible decomposition. For example the following cases may happen:

(1) $\alpha_1 = \alpha_2$, $x_1, x_2 \in Y_1$ but $(V_1 \cup V_2) \cap Y_i = \emptyset$ for any $i \neq 1$;

(2) $\alpha_1 > \alpha_2$, $x_1 \in Y_1 \not\subset V_1$ but $V_1 \cap Y_i = \emptyset$ for any $i \neq 1$.

We should note that these Y_1 are non-compact.

We consider the first case. After replacing X_c by a smaller sublevel set of (X, Φ), we can take a multivalued holomorphic section τ of $L^{\otimes \beta - \varepsilon_0}$ on X_c such that the restriction $\tau|_{Y_1}$ is not identically zero, and vanishes at x_1 and x_2 with high multiplicities in Y_1. This follows from the line bundle convexity property of Y_1 with respect to L. Then, for a sufficiently small positive rational number δ, we have a multivalued holomorphic section $s := \sigma_1^{1-\delta} \times \tau$ of $L^{\otimes \beta'}$ on X_c for some $0 < \beta' < \beta$. Then we will have

$$x_1, x_2 \in V\mathcal{I}(s) \subset V\mathcal{I}(\sigma_1) \cap (\tau)_0,$$

which means $V\mathcal{I}(s)$ is zero-dimensional around V_1 and V_2. Thus we obtained a desired section s. Here we need the semicontinuity argument of Angehrn-Siu [AS].

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Shigeharu TAKAYAMA

Department of Mathematics
Naruto University of Education
Takashima, Naruto-cho, Naruto-shi, 772 Japan.
e-mail address: takayama@naruto-u.ac.jp

Current address:
Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka, 560 JAPAN