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Transcendence of Rogers–Ramanujan continued fraction and reciprocal sums of Fibonacci numbers

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This is a report on the recent work of Duverney, Ke. Nishioka, Ku. Nishioka, and the author [11] concerning the title of this paper. Let $P(q)$, $Q(q)$, $R(q)$ be the Ramanujan's functions defined by

$$P(q)=1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},$$
$$Q(q)=1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n=1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},$$
$$R(q)=1-540\sum_{n=1}^{\infty}\sigma_5(n)q^n=1-540\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n},$$

which are the classical Eisenstein series $E_2(q)$, $E_4(q)$, $E_6(q)$ respectively, where $\sigma_i(n)=\sum_{d|n}d^i$. Mahler [17] proved the algebraically independency of the functions $P(q)$, $Q(q)$, $R(q)$ over $\mathbb{C}(q)$. Letting '$'$' denote the derivation $q\dfrac{d}{dq}$, we have

$$P'=\frac{1}{12}(P^2-Q),\quad Q'=\frac{1}{3}(PQ-R),\quad R'=\frac{1}{2}(PR-Q^2)$$

(cf. [15; Theorem 5.3]). We put

$$\Delta=\frac{1}{1728}(Q^3-R^2),\qquad J=\frac{Q^3}{\Delta}.$$

The modular function $j(\tau)$ is described as $j(\tau)=J(q)$, where $q=e^{2\pi i\tau}$, $\mathrm{Im}\tau>0$. Barré–Serieix, Diaz, Gramain, and Philibert [3] proved the transcendency of the value $J(\alpha)$ for any $\alpha\in\overline{\mathbb{Q}}$, $0<|\alpha|<1$. By the equalities

$$\frac{J'}{J}=-\frac{R}{Q},\quad \frac{J''}{J'}=\frac{1}{6}P-\frac{2}{3}\frac{R}{Q}-\frac{1}{2}\frac{Q^2}{R},$$

we have $Q \in \mathbb{Q}(J, J', J'')$, and hence

$$\mathbb{Q}(P, Q, R) = \mathbb{Q}(J, J', J'') = \boldsymbol{K}, \text{ say.}$$

We note that $\boldsymbol{K}$ is a differential field, i.e., closed under the derivation '$'$'. Now we state

Nesterenko's theorem ([19], [20]). If $\alpha \in \mathbb{C}$, $0 < |\alpha| < 1$, then

$$\text{trans.deg}_{\mathbb{Q}}\, \mathbb{Q}(\alpha, P(\alpha), Q(\alpha), R(\alpha)) \geq 3.$$

Corollary 1. If $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, then each of the following set 1) $P(\alpha)$, $Q(\alpha)$, $R(\alpha)$, 2) $J(\alpha)$, $J'(\alpha)$, $J''(\alpha)$ are algebraically independent.

Corollary 2. The numbers π, e^{π}, and $\Gamma(1/4)$ are algebraically independent.

Let $\eta(q)$ be the eta function defined by

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

which is known to satisfy

$$\eta(q)^{24} = \Delta(q).$$

Corollary 3. If $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, then

$$\text{trans.deg}_{\mathbb{Q}}\, \mathbb{Q}(\alpha, \eta(\alpha), \eta'(\alpha), \eta''(\alpha)) \geq 3.$$

In particular, the infinite product $\prod_{n=1}^{\infty} (1 - \alpha^n)$ is transcendental for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$.

Let ϑ_3, $\vartheta = \vartheta_4$, ϑ_2 be Jacobi's theta series defined by

$$\vartheta_3 = 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \quad \vartheta = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}, \quad \vartheta_2 = 2q^{1/4} \sum_{n=1}^{\infty} q^{n(n-1)}.$$

Corollary 4(Bertrand [5]). Let $y = y(q)$ be one of ϑ_3, ϑ, ϑ_2. If $\alpha \in \overline{\mathbb{C}}$, $0 < |\alpha| < 1$, then

$$\text{trans.deg}_{\mathbb{Q}}\, \mathbb{Q}(\alpha, y(\alpha), y'(\alpha), y''(\alpha)) \geq 3.$$

In particular, the number $\sum_{n=1}^{\infty} \alpha^{n^2}$ is transcendental for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$.

We note that Corollary 4 provides the best possible results as y is known to satisfy an algebraic differential equations of the third order defined over $\mathbb{Q}$ (cf. Jacobi [13]). A survey on Nesterenko's theorem can be found in Waldschmidt [23].

The following lemmas are useful to prove the transcendency of some numbers related to modular functions.

Lemma 1([10]). Let $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$. If a nonconstant function $f(q)$ is algebraic over $\boldsymbol{K}$ and defined at α, then $f(\alpha)$ is transcendental.

Lemma 2([10]). Let $y = y(q)$ be one of the functions η, ϑ_3, ϑ, ϑ_2. Then $y(q^k)$, $y'(q^k)$, $y''(q^k), \ldots$ are algebraic over $\boldsymbol{K}$ for any positive integer k.

The Rogers–Ramanujan continued fraction $RR(q)$ is defined by

$$RR(q) = 1 + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \frac{q^4}{1} + \cdots,$$

which is known to have the expressions

$$RR(q) = \frac{\displaystyle\sum_{k=0}^{\infty} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}}{\displaystyle\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)}} = \prod_{k=0}^{\infty} \frac{(1-q^{5k+2})(1-q^{5k+3})}{(1-q^{5k+1})(1-q^{5k+4})}$$

(cf. [4; Chap. 16, Entry 15, 38(iii)]). Irrationality measures for some values of this continued fraction were given by Osgood [21] and Shiokawa [22]. The latter proved that for any integer $d \geq 2$ there is a constant $C = C(d) > 0$ such that

$$\left| RR\left(\frac{1}{d}\right) - \frac{p}{q} \right| > C q^{-2-B/\sqrt{\log q}}$$

for all integers p, $q\,(\geq 2)$, where $B = \sqrt{\log d}$. Matala–Aho [18] obtained some higher degree irrationality results. For example, $RR((\sqrt{5}-1)/2) \notin \mathbb{Q}(\sqrt{5})$.

Theorem 1([11]). The Rogers–Ramanujan continued fraction $RR(\alpha)$ is transcendental for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$.

Proof. Let

$$F(q) = \frac{q^{1/5}}{1} \, \frac{q}{+\,1} \, \frac{q^2}{+\,1} \, \frac{q^3}{+\,1} \, \frac{q^4}{+\,1\,+} \cdots,$$

then

$$\frac{1}{F(q)} - F(q) - 1 = q^{-1/5} \frac{\prod_{n=1}^{\infty}(1-q^{n/5})}{\prod_{n=1}^{\infty}(1-q^{5n})} = \frac{\eta(q^{1/5})}{\eta(q^5)}$$

(see [4; p.85]). Applying Lemma 1 and 2 to the function $f(q) = \eta(q)/\eta(q^{25})$, we see that $f(\alpha)$ is transcendental for any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, and so is $F(\alpha)$ from the formula above.

We give here further examples of continued fractions whose transcendence can be easily deduced from Lemma 1 and 2. For any $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$, the following continued fractions (i), (ii), (iii) are transcendental:

(i) $\dfrac{1}{1} \, \dfrac{\alpha}{+\,1+\alpha} \, \dfrac{\alpha^2}{+\,1+\alpha^2} \, \dfrac{\alpha^3}{+\,1+\alpha^3\,+} \cdots$ (see [4; Chap.19, Entry 1 (i)]).

(ii) $\dfrac{1}{1+\alpha} \, \dfrac{\alpha^2}{+\,1+\alpha^3} \, \dfrac{\alpha^4}{+\,1+\alpha^5} \, \dfrac{\alpha^6}{+\,1+\alpha^7\,+} \cdots$ (see [4; Chap.19, Entry 1 (ii)]).

(iii) $\dfrac{1}{1\,+} \, \dfrac{\alpha+\alpha^2}{1\quad+} \, \dfrac{\alpha^2+\alpha^4}{1\quad+} \, \dfrac{\alpha^3+\alpha^6}{1\quad+} \cdots$ (see [4; Chap.20, Entry 1]).

Let α and β be algebraic numbers with $\alpha \neq \beta$ and $|\beta| < 1$. Put

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n.$$

Theorem 2([11]). If $\alpha\beta = \pm 1$, then the numbers

$$\sum_{n=1}^{\infty} \frac{1}{U_n^{2s}}, \qquad \sum_{n=1}^{\infty} \frac{1}{V_n^{2s}}$$

are transcendental for any positive integer s.

Theorem 3([11]). If $\alpha\beta = 1$, then the number

$$\sum_{n=1}^{\infty} \frac{1}{V_n^{s}}$$

is transcendental for any positive integer s.

Theorem 4([11]). If $\alpha\beta = -1$, then the number

$$\sum_{n=1}^{\infty} \frac{1}{U_{2n-1}^s}$$

is transcendental for any positive integer s.

In the special case of $s = 1$, these theorems are proved in [10] by direct calculation without using Lemma 3 below.

For the proof, we need another lemma. Let

$$k = \vartheta_2^2(q)/\vartheta_3^2(q)$$

be the modulus of the complete elliptic integrals

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad E = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt,$$

of the first and the second kind, respectively. Then we have

$$\frac{K}{\pi} = \frac{1}{2}\vartheta_3^2(q), \qquad \frac{E}{\pi} = \frac{K}{\pi} + \frac{\pi}{K}\frac{\vartheta'(q)}{\vartheta(q)},$$

where $\vartheta' = q\dfrac{d\vartheta}{dq}$ (cf. [6; (2.1.13), (2.3.17)]).

Lemma 3 ([11]). Let s be any positive integer and let

$$f_{2s}(q) = \sum_{n=1}^{\infty} \frac{1}{(q^{-n} - q^n)^{2s}}, \qquad g_s(q) = \sum_{n=1}^{\infty} \frac{1}{(q^{-n} + q^n)^s}.$$

Then $f_{2s}(q)$, $f_{2s}(q^2)$, $g_s(q)$, and $g_s(q^2)$ are algebraic over the field $\mathbb{Q}(P(q), Q(q), R(q))$.

Proof. Let s be a positive integer. We put

$$\mathrm{I}_{2s} = \sum_{n=1}^{\infty} \operatorname{cosech}^{2s}(n\pi c) = \sum_{n=1}^{\infty} \left(\frac{2}{q^{-n} - q^n}\right)^{2s}, \qquad q = e^{-\pi c},$$

$$\mathrm{II}_s = \sum_{n=1}^{\infty} \operatorname{sech}^s(n\pi c) = \sum_{n=1}^{\infty} \left(\frac{2}{q^{-n} + q^n}\right)^s,$$

so that

$$f_{2s}(q) = 2^{-2s}\mathrm{I}_{2s}, \qquad g_s(q) = 2^{-s}\mathrm{II}_s.$$

Then Zucker [26] obtained expansions of I_{2s}, II_s, and II_{2s+1} as polynomials of k, K/π, and E/π with rational coefficients, which can be found in Table 1(i), 1(ii), and 1(vi) in [26], respectively. Hence the lemma follows from Lemma 2.

Proof of Theorem 2. If $\alpha\beta = 1$, then we have

$$(\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_n^{2s}} = \sum_{n=1}^{\infty}\frac{1}{(\beta^{-n}-\beta^n)^{2s}} = f_{2s}(\beta),$$

$$\sum_{n=1}^{\infty}\frac{1}{V_n^{2s}} = \sum_{n=1}^{\infty}\frac{1}{(\beta^{-n}+\beta^n)^{2s}} = g_{2s}(\beta),$$

and the results follow from Lemma 3 and 1. If $\alpha\beta = -1$, then we have

$$\begin{aligned}(\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_n^{2s}} &= \sum_{n=1}^{\infty}\frac{1}{((-\beta)^{-n}-\beta^n)^{2s}}\\ &= \sum_{n=1}^{\infty}\frac{1}{((-\beta)^{-2n}-\beta^{2n})^{2s}} + \sum_{n=1}^{\infty}\frac{1}{((-\beta)^{-(2n-1)}-\beta^{2n-1})^{2s}}\\ &= f_{2s}(\beta^2)+g_{2s}(\beta)-g_{2s}(\beta^2),\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{V_n^{2s}} &= \sum_{n=1}^{\infty}\frac{1}{((-\beta)^{-n}+\beta^n)^{2s}}\\ &= \sum_{n=1}^{\infty}\frac{1}{(\beta^{-2n}+\beta^{2n})^{2s}} + \sum_{n=1}^{\infty}\frac{1}{(-\beta^{-(2n-1)}+\beta^{2n-1})^{2s}}\\ &= g_{2s}(\beta^2)+f_{2s}(\beta)-f_{2s}(\beta^2).\end{aligned}$$

Proof of Theorem 3.

$$\sum_{n=1}^{\infty}\frac{1}{V_n^{s}} = \sum_{n=1}^{\infty}\frac{1}{(\beta^{-n}+\beta^n)^{s}} = g_s(\beta).$$

Proof of Theorem 4.

$$(\alpha-\beta)^{-2s}\sum_{n=1}^{\infty}\frac{1}{U_{2n-1}^{2s}} = g_{2s}(\beta)-g_{2s}(\beta^2),$$

$$\begin{aligned}(\alpha-\beta)^{-(2s-1)}\sum_{n=1}^{\infty}\frac{1}{U_{2n-1}^{2s-1}} &= -\sum_{n=1}^{\infty}\frac{1}{(\beta^{-(2n-1)}+\beta^{2n-1})^{2s-1}}\\ &= -g_{2s-1}(\beta)+g_{2s-1}(\beta^2).\end{aligned}$$

Fibonacci numbers $\{F_n\}_{n\geq 1}$ and Lucas numbers $\{L_n\}_{n\geq 1}$ are defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0),$$
$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0),$$

and written as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n \qquad (n \geq 0),$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

Corollary([11]). The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^{s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^{s}}$$

are transcendental for any positive integer s.

André–Jeannin [1] proved the irrationality of the number

$$\sum_{n=1}^{\infty} \frac{1}{F_n}.$$

Duverney [8] gave another proof and Kato [14] showed by Duverney's method that the number

$$\sum_{n=1}^{\infty} \frac{1}{F_{an}}$$

is irrational for any positive integer a. It is not known whether these numbers are transcendental or not. Bundschuh and Väänänen [7] gave an irrationality measure for $\sum_{n=1}^{\infty} F_n^{-1}$; namely

$$\left| \sum_{n=1}^{\infty} \frac{1}{F_n} - \frac{p}{q} \right| > \frac{1}{q^{8.621}}$$

holds for all rationals p/q with sufficiently large q.

Finally, we state two problems which are interesting in comparison with the arithmetical properties of the values of the Riemann zeta function $\zeta(s)$ at $s = 2, 3, 4, \ldots$.

Problem 1. Is the number

$$\sum_{n=1}^{\infty} \frac{1}{F_n^3}$$

irrational ?

Problem 2. Are the numbers

$$\sum_{n=1}^{\infty}\frac{1}{F_n^2},\qquad \sum_{n=1}^{\infty}\frac{1}{F_n^4},\qquad \sum_{n=1}^{\infty}\frac{1}{F_n^6}$$

algebraically independent ?

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