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Substitution in two letters and transcendence

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1 Introduction

Let $A=\{a_1,\ldots,a_n\}$ be a finite nonempty set of letters and let A^* and A^ω denote the sets of all finite words over A and all sequences $x_0x_1\cdots x_k\cdots$ $(x_k\in A)$, respectively. Let λ be the empty word. A *substitution* (over A) is a map $\sigma : A\to A^*\setminus\{\lambda\}$, which has a natural extension to $\Omega=A^*\cup A^\omega$ by concatenation: $\sigma(x_0x_1\cdots)=\sigma(x_0)\sigma(x_1)\cdots$. If a_i is a prefix of $\sigma(a_i)$ and the length of $\sigma(a_i)$ is greater than 1, then there is a unique $w\in\Omega$ having a prefix a_i and being a fixed point of σ, which means that $\sigma(w)=w$. Any real algebraic irrational θ can be uniquely expressed as

$$\theta=\sum_{k=-m}^{\infty}\varepsilon_k 2^{-k}, \tag{1}$$

where m is a nonnegative integer depending on θ and $\varepsilon_k=0$ or 1. The problem we are interested in is whether the sequence $\varepsilon_0\varepsilon_1\cdots\in\{0,1\}^\omega$ is a fixed point of any substitution over $\{0,1\}$ or not.

Generally, for a fixed point $w=x_0x_1\cdots$ of the given substitution σ, we define the generating function of w for a_i by

$$f_i(z)=\sum_{k=0}^{\infty}\chi_k(w;a_i)z^k, \tag{2}$$

where $\chi_k(w;a_i)=1$ if $x_k=a_i$, and otherwise $\chi_k(w;a_i)=0$, so that

$$\sum_{i=1}^{n}f_i(z)=\sum_{k=0}^{\infty}z^k=\frac{1}{1-z}.$$

It is known that $f_i(z)$ $(1\le i\le n)$ satisfy a Mahler type functional equation if σ is of constant length, which means that each $\sigma(a_i)$ $(1\le i\le n)$ has the same length ≥ 2, and it is also known that if σ is of nonconstant length, i.e., the lengths of $\sigma(a_i)$ $(1 \le i \le n)$ are not equal, then we can construct $g_1(\boldsymbol{z}), \ldots, g_n(\boldsymbol{z}) \in \boldsymbol{Q}[[z_1, \ldots, z_n]]$ satisfying a Mahler type functional equation and $g_i(z, \ldots, z) = f_i(z)$ $(1 \le i \le n)$. We shall give here a detailed explanation of these facts, following Loxton [3].

First we consider the case where the substitution σ is of constant length. Suppose that each $\sigma(a_i)$ $(1 \le i \le n)$ has the same length $d \ge 2$. Since $\sigma(w) = w$, we observe that for any k, the string $x_{dk}x_{dk+1}\cdots x_{dk+d-1}$ coincides with $\sigma(a_j)$ if $x_k = a_j$. If we set

$$\psi_{ijl} = \begin{cases} 1 & \text{if } a_i \text{ is the } (l+1)\text{-st letter of } \sigma(a_j) \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\chi_{dk+l}(w; a_i) = \sum_{j=1}^{n} \psi_{ijl}\chi_k(w; a_j).$$

We can now obtain a system of functional equations for the functions $f_i(z)$ $(1 \le i \le n)$, since

$$\sum_{h=0}^{\infty} \chi_h(w; a_i) z^h = \sum_{k=0}^{\infty}\sum_{l=0}^{d-1} \chi_{dk+l}(w; a_i) z^{dk+l} = \sum_{j=1}^{n} \left(\sum_{l=0}^{d-1} \psi_{ijl} z^l\right) \left(\sum_{k=0}^{\infty} \chi_k(w; a_j) z^{dk}\right),$$

that is

$$f_i(z) = \sum_{j=1}^{n} p_{ij}(z) f_j(z^d) \quad (1 \le i \le n), \tag{3}$$

where $p_{ij}(z) = \sum_{l=0}^{d-1} \psi_{ijl} z^l$ are polynomials.

Next we consider the case where the substitution σ is not necessarily of constant length. We adopt the usual vector notations: if $\mu = (\mu_1, \ldots, \mu_n) \in \boldsymbol{N}_0{}^n$ with $\boldsymbol{N}_0$ the set of nonnegative integers, we write $\boldsymbol{z}^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}$ and $|\mu| = \mu_1 + \cdots + \mu_n$. Define the functions $g_1(\boldsymbol{z}), \ldots, g_n(\boldsymbol{z}) \in \boldsymbol{Q}[[z_1, \ldots, z_n]]$ by

$$g_i(\boldsymbol{z}) = \sum_{\mu} \phi_{i\mu} \boldsymbol{z}^\mu \quad (1 \le i \le n),$$

where the sum is taken over all n-tuples $\mu = (\mu_1, \ldots, \mu_n) \in \boldsymbol{N}_0{}^n$, $\phi_{i\mu} = 1$ whenever $x_{|\mu|} = a_i$ and for each k there are exactly μ_k occurrences of a_k in the string $x_0x_1\cdots x_{|\mu|-1}$, and $\phi_{i\mu} = 0$ otherwise. Then $g_i(z, \ldots, z) = f_i(z)$ $(1 \le i \le n)$.

In what follows, $|u|_{a_i}$ denotes the number of occurrences of the letter a_i in the word $u \in A^*$. Suppose that the term $\boldsymbol{z}^\mu$ occurs in the series $g_j(\boldsymbol{z})$. Imagine the fixed point $w = x_0x_1\cdots$ being constructed by applying the substitution σ successively to $x_0, x_1, \ldots$. When we reach $x_{|\mu|}$, we must have examined the letter a_i exactly μ_i times and so we must have written out the word $\sigma(a_i)$ exactly μ_i times. Let $t_{ik} = |\sigma(a_i)|_{a_k}$. Then the part of the sequence constructed by the time the substitution σ reaches $x_{|\mu|}$ contains the letter a_k exactly $\sum_{i=1}^{n} \mu_i t_{ik}$ times and altogether $\sum_{i=1}^{n}\sum_{k=1}^{n} \mu_i t_{ik}$

letters have been written down. The next letter to be written will be the first letter, say a_l, of $\sigma(a_j)$, so that $g_l(\boldsymbol{z})$ must contain the term $\boldsymbol{z}^\nu$ with $\nu_k = \sum_{i=1}^n \mu_i t_{ik}$. If a_m, say, is the second letter of $\sigma(a_j)$, then $g_m(\boldsymbol{z})$ contains the term z^λ with $\lambda_k = \nu_k$ $(k \neq l)$, $\lambda_l = \nu_l + 1$, and so on. We introduce the $n \times n$ matrix $T = (t_{ik})$. If $\boldsymbol{z} = (z_1, \ldots, z_n)$ is a point of $\boldsymbol{C}^n$ with $\boldsymbol{C}$ the set of complex numbers, we define a transformation $T : \boldsymbol{C}^n \to \boldsymbol{C}^n$ by

$$T\boldsymbol{z} = \left(\prod_{k=1}^n {z_k}^{t_{1k}}, \ldots, \prod_{k=1}^n {z_k}^{t_{nk}}\right). \tag{4}$$

Noting that

$$(T\boldsymbol{z})^\mu = \boldsymbol{z}^{\mu T},$$

where the exponent μT on the right-hand side is the usual product of the row vector μ and the matrix T, and so $\boldsymbol{z}^\nu = (T\boldsymbol{z})^\mu$, we can expect that each $g_i(\boldsymbol{z})$ will be expressible by means of $g_j(T\boldsymbol{z})$ $(1 \leq j \leq n)$. This works as in the preceding case. Set $\psi_{ij\kappa} = 1$ if a_i is the $(|\kappa| + 1)$-st letter of $\sigma(a_j)$ and is preceded by exactly κ_k occurrences of the letter a_k for each k, and set $\psi_{ij\kappa} = 0$ otherwise. Let the length of each $\sigma(a_j)$ $(1 \leq j \leq n)$ be not greater than s. Then

$$\begin{aligned}
\sum_\nu \phi_{i\nu} \boldsymbol{z}^\nu &= \sum_\mu \sum_{j=1}^n \sum_{|\kappa|<s} \psi_{ij\kappa} \phi_{j\mu} \boldsymbol{z}^{\mu T + \kappa} \\
&= \sum_{j=1}^n \sum_\mu \Big(\sum_{|\kappa|<s} \psi_{ij\kappa} \boldsymbol{z}^\kappa\Big) \phi_{j\mu} \boldsymbol{z}^{\mu T} \\
&= \sum_{j=1}^n \Big(\sum_{|\kappa|<s} \psi_{ij\kappa} \boldsymbol{z}^\kappa\Big)\Big(\sum_\mu \phi_{j\mu} \boldsymbol{z}^{\mu T}\Big),
\end{aligned}$$

that is

$$g_i(\boldsymbol{z}) = \sum_{j=1}^n p_{ij}(\boldsymbol{z}) g_j(T\boldsymbol{z}) \quad (1 \leq i \leq n), \tag{5}$$

where $p_{ij}(\boldsymbol{z})$ are certain polynomials whose coefficients are 0 and 1. The functional equations such as (3) and (5) are called Mahler type functional equations.

In this paper we study substitutions in two letters in connection with the dyadic expansion of real algebraic irrationals. Hence, in what follows, we consider the case of $n = 2$ and write $a_1 = a$ and $a_2 = b$ for abbreviation, so that in this case $A = \{a, b\}$. The generating functions defined by (2) are denoted by $f_1(z) = f_a(z)$, $f_2(z) = f_b(z)$. Similarly we denote $g_1(\boldsymbol{z}) = g_a(\boldsymbol{z})$, $g_2(\boldsymbol{z}) = g_b(\boldsymbol{z})$, which satisfy $g_a(z, z) = f_a(z)$, $g_b(z, z) = f_b(z)$, and

$$\begin{pmatrix} g_a(\boldsymbol{z}) \\ g_b(\boldsymbol{z}) \end{pmatrix} = M(\boldsymbol{z}) \begin{pmatrix} g_a(T\boldsymbol{z}) \\ g_b(T\boldsymbol{z}) \end{pmatrix},$$

where

$$M(\boldsymbol{z}) = \begin{pmatrix} p(\boldsymbol{z}) & q(\boldsymbol{z}) \\ r(\boldsymbol{z}) & s(\boldsymbol{z}) \end{pmatrix}, \quad p(\boldsymbol{z}), q(\boldsymbol{z}), r(\boldsymbol{z}), s(\boldsymbol{z}) \in \boldsymbol{Z}[z_1, z_2].$$

Further

$$T = \begin{pmatrix} t_{aa} & t_{ab} \\ t_{ba} & t_{bb} \end{pmatrix}, \tag{6}$$

where $t_{\alpha\beta} = |\sigma(\alpha)|_\beta$ $(\alpha, \beta \in A)$, and the characteristic polynomial of the matrix T is defined by

$$\Phi(X) = X^2 - (t_{aa} + t_{bb})X + (t_{aa}t_{bb} - t_{ab}t_{ba}).$$

If we proved that the value $f_a(2^{-1})$ or $f_b(2^{-1})$ of the generating function of a nonperiodic fixed point w of a substitution σ in two letters is transcendental, we could conclude that the sequence $\varepsilon_0\varepsilon_1\cdots$ appearing in the dyadic expansion (1) of any real algebraic irrational is not a fixed point of any substitution over $\{0,1\}$. This has not been proved so far. In the present paper, we prove it in the case of constant length (see Theorem 2 and Corollary below) and also in the case of nonconstant length, however, with some exceptional cases.

THEOREM 1. *Let w be any fixed point of a substitution σ in two letters and let $f_a(z)$ and $f_b(z)$ be the generating functions of w for a and for b, respectively. If $t_{ab}t_{ba}\Phi(1)\Phi(0)\Phi(-1) \neq 0$, then the numbers $f_a(l^{-1})$ and $f_b(l^{-1})$ are transcendental for any integer $l \geq 2$.*

EXAMPLE (cf. Wen and Wen [8]). We consider the substitution $(\sigma(a), \sigma(b)) = (ab, a)$, which is called Fibonacci substitution and has a fixed point

$$w = abaababaabaababaababa\cdots.$$

Let $f_a(z)$ and $f_b(z)$ be the generating functions of w for a and for b, respectively. Then the numbers $f_a(l^{-1})$ and $f_b(l^{-1})$ are transcendental for any integer $l \geq 2$.

THEOREM 2. *Let w be any nonperiodic fixed point of a substitution σ in two letters which is of constant length and let $f_a(z)$ and $f_b(z)$ be the generating functions of w for a and for b, respectively. Then the numbers $f_a(l^{-1})$ and $f_b(l^{-1})$ are transcendental for any integer $l \geq 2$.*

COROLLARY. *The dyadic expansion of any real algebraic irrational is not a fixed point of any substitution over $\{0,1\}$ which is of constant length.*

Therefore the problem which remains unsolved is to remove the condition $t_{ab}t_{ba}\Phi(1)\Phi(0)\Phi(-1) \neq 0$ in Theorem 1, in the case of substitutions in two letters of nonconstant length.

2 Lemmas

Let $T=(t_{ij})$ be an $n\times n$ matrix with nonnegative integer entries. Then the maximum ρ of the absolute values of the eigenvalues of T is itself an eigenvalue (cf. Gantmacher [2]). We suppose that the matrix T and an algebraic point $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)$, where α_i are nonzero algebraic numbers, have the following four properties:

(I) T is non-singular and none of its eigenvalues is a root of unity, so that in particular $\rho>1$.

(II) Every entry of the matrix T^k is $O(\rho^k)$ as k tends to infinity.

(III) If we put $T^k\boldsymbol{\alpha}=(\alpha_1^{(k)},\dots,\alpha_n^{(k)})$, then

$$\log|\alpha_i^{(k)}|\le -c\rho^k \quad (1\le i\le n)$$

for all sufficiently large k, where c is a positive constant.

(IV) For any nonzero power series $f(\boldsymbol{z})$ in n variables with complex coefficients which converges in some neighborhood of the origin, there are infinitely many positive integers k such that $f(T^k\boldsymbol{\alpha})\neq 0$.

Let K be an algebraic number field and I_K the integer ring of K. We denote by $K[[z_1,\dots,z_n]]$ the ring of formal power series in variables $z_1,\dots,z_n$ with coefficients in K. Suppose that $f(\boldsymbol{z})\in K[[z_1,\dots,z_n]]$ converges in an n-polydisc U around the origin and satisfies the functional equation

$$f(T\boldsymbol{z}) = \frac{\sum_{i=0}^{m} a_i(\boldsymbol{z})f(\boldsymbol{z})^i}{\sum_{i=0}^{m} b_i(\boldsymbol{z})f(\boldsymbol{z})^i}, \tag{7}$$

where $1\le m<\rho$ and $a_i(\boldsymbol{z})$, $b_i(\boldsymbol{z})$ are polynomials in $z_1,\dots,z_n$ with coefficients in I_K. We denote by $\Delta(\boldsymbol{z})$ the resultant of polynomials $\sum_{i=0}^m a_i(\boldsymbol{z})u^i$ and $\sum_{i=0}^m b_i(\boldsymbol{z})u^i$ in u. If one of them is a constant $c(\boldsymbol{z})$ in u, we set $\Delta(\boldsymbol{z})=c(\boldsymbol{z})$. Then Mahler proved the following:

LEMMA 1 (Mahler [4], cf. Nishioka [6]). *Assume that T and $\boldsymbol{\alpha}$ have the properties* (I)–(IV) *and $f(\boldsymbol{z})$ satisfying* (7) *is transcendental over the rational function field $K(z_1,\dots,z_n)$. If $T^k\boldsymbol{\alpha}\in U$ and $\Delta(T^k\boldsymbol{\alpha})\neq 0$ for any $k\ge 0$, then $f(\boldsymbol{\alpha})$ is transcendental.*

The following lemma will be used in the proof of Lemma 3 below.

LEMMA 2 (Masser [5]). *Let T be an $n\times n$ matrix with nonnegative integer entries for which the property* (I) *holds. Let $\boldsymbol{\alpha}$ be an n-dimensional vector whose components $\alpha_1,\ldots,\alpha_n$ are nonzero algebraic numbers such that $T^k\boldsymbol{\alpha} \to (0,\ldots,0)$ as k tends to infinity. Then the negation of the property* (IV) *is equivalent to the following:*

There exist integers $i_1,\ldots,i_n$, not all zero, and positive integers a,b such that

$$(\alpha_1^{(k)})^{i_1}\cdots(\alpha_n^{(k)})^{i_n} = 1$$

for all $k=a+lb$ $(l=0,1,2,\ldots)$.

In what follows, let $\mathbf{1}=(1,1)$ and $x\mathbf{1}=(x,x)$.

LEMMA 3. *Suppose that $t_{aa}+t_{bb}>0$, $t_{ab}t_{ba}\Phi(1)\Phi(0)\Phi(-1)\neq 0$, and $t_{aa}+t_{ab}\neq t_{ba}+t_{bb}$. Then the matrix T defined by* (6) *and $l^{-1}\mathbf{1}$, where l is an integer greater than* 1, *have the properties* (I)–(IV).

REMARK. If a substitution σ in two letters has a fixed point, then $t_{aa}+t_{bb}>0$.

Proof of Lemma 3. We denote

$$T=\begin{pmatrix} t_{aa} & t_{ab} \\ t_{ba} & t_{bb}\end{pmatrix} =: \begin{pmatrix} p & q \\ r & s\end{pmatrix}$$

for abbreviation. The eigenvalues of T are

$$\Lambda=(p+s+\sqrt{D})/2, \quad \lambda=(p+s-\sqrt{D})/2,$$

where $D=(p-s)^2+4qr>0$. Hence the property (II) is satisfied, since $p+s>0$ and so $\Lambda>|\lambda|$, and the property (I) is also satisfied, since the characteristic polynomial of the matrix T is $\Phi(X)$ and so $\Lambda,\lambda\neq 0,\pm 1$.

Letting

$$T^k\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}x_k\\y_k\end{pmatrix} \quad (k\geq 0), \tag{8}$$

we see that

$$T^k l^{-1}\mathbf{1}=(l^{-x_k},l^{-y_k})$$

by (4) and that $x_k,y_k>0$ for any $k\geq 0$. We can write

$$x_k=\xi_1\Lambda^k+\xi_2\lambda^k, \quad y_k=\eta_1\Lambda^k+\eta_2\lambda^k, \tag{9}$$

where $\xi_1,\xi_2,\eta_1,\eta_2\in\boldsymbol{Q}(\sqrt{D})$, and $\xi_1,\eta_1\geq 0$ since $\Lambda>|\lambda|$. We assert that $\xi_1,\eta_1>0$, which implies that the property (III) is satisfied. Since

$$\begin{pmatrix}x_{k+1}\\y_{k+1}\end{pmatrix}=T\begin{pmatrix}x_k\\y_k\end{pmatrix}=\begin{pmatrix}px_k+qy_k\\rx_k+sy_k\end{pmatrix}$$

with (9), we have

$$\xi_1\Lambda\Lambda^k + \xi_2\lambda\lambda^k = (p\xi_1 + q\eta_1)\Lambda^k + (p\xi_2 + q\eta_2)\lambda^k,$$
$$\eta_1\Lambda\Lambda^k + \eta_2\lambda\lambda^k = (r\xi_1 + s\eta_1)\Lambda^k + (r\xi_2 + s\eta_2)\lambda^k$$

for any $k \geq 0$. Hence, by the assumption that $q, r > 0$, we see that $\xi_1 = (p\xi_1 + q\eta_1)/\Lambda > 0$ if $\eta_1 > 0$, and that $\eta_1 = (r\xi_1 + s\eta_1)/\Lambda > 0$ if $\xi_1 > 0$. Therefore, noting that ξ_1 and η_1 are not both zero, we can conclude that $\xi_1, \eta_1 > 0$.

Finally, using Lemma 2, we prove that the property (IV) is satisfied. Assume that there exist integers t, u, not both zero, and positive integers m, n such that

$$(l^{-x_k})^t (l^{-y_k})^u = l^{-(tx_k + uy_k)} = 1$$

for all $k \in \mathcal{A} := \{\, m + ln \mid l \in \boldsymbol{N}_0 \,\}$. Then $w_k := tx_k + uy_k = 0$ $(k \in \mathcal{A})$. Since we can write $w_k = \zeta_1\Lambda^k + \zeta_2\lambda^k$, where $\zeta_1, \zeta_2 \in \boldsymbol{Q}(\sqrt{D})$,

$$\zeta_1 = -\zeta_2(\lambda/\Lambda)^k \quad (k \in \mathcal{A}).$$

Then the right-hand side converges to 0 as $k \in \mathcal{A}$ tends to infinity, but the left-hand side is a constant. Therefore $\zeta_1 = 0$ and so $\zeta_2 = 0$. Hence $w_k = 0$ for all $k \geq 0$. By the equations $w_0 = t + u = 0$ and $w_1 = t(p+q) + u(r+s) = 0$, we have $p + q = r + s$, which contradicts the assumption in the lemma. Therefore the property (IV) is satisfied, and the proof of the lemma is completed.

LEMMA 4. *Let w be any fixed point of a substitution σ in two letters. If $t_{ab}t_{ba}\Phi(1)\Phi(0)\Phi(-1) \neq 0$, then w is nonperiodic.*

Proof. We may assume that a is a prefix of w without loss of generality. Suppose that w is periodic. Let Λ, λ $(|\Lambda| \geq |\lambda|)$ be the eigenvalues of T. By the same reason as in the proof of Lemma 3, we see that $\Lambda > |\lambda|$. Define the frequency of $\alpha \in A = \{a, b\}$ occurring in $w = x_0x_1 \cdots x_n \cdots$ by

$$d_\alpha = \lim_{n\to\infty} (|x_0x_1 \cdots x_n|_\alpha / n),$$

so that $d_a + d_b = 1$. Then

$$(d_a, d_b)T = \Lambda(d_a, d_b), \tag{10}$$

since $t_{ab}t_{ba} \neq 0$ and $\Lambda > |\lambda|$ (cf. Queffélec [7]). By (10) and $t_{aa} + t_{bb} = \Lambda + \lambda$, we have

$$T\begin{pmatrix} d_b \\ -d_a \end{pmatrix} = \lambda \begin{pmatrix} d_b \\ -d_a \end{pmatrix}. \tag{11}$$

We can verify by induction that

$$T^n = \begin{pmatrix} |\sigma^n(a)|_a & |\sigma^n(a)|_b \\ |\sigma^n(b)|_a & |\sigma^n(b)|_b \end{pmatrix} \quad (n \geq 0), \tag{12}$$

where $\sigma^n(\alpha)$ $(\alpha \in A)$ denotes the n-fold iteration of σ. Then by (11) and (12),

$$\lambda^n \begin{pmatrix} d_b \\ -d_a \end{pmatrix} = T^n \begin{pmatrix} d_b \\ -d_a \end{pmatrix} = \begin{pmatrix} |\sigma^n(a)|_a & |\sigma^n(a)|_b \\ |\sigma^n(b)|_a & |\sigma^n(b)|_b \end{pmatrix} \begin{pmatrix} d_b \\ -d_a \end{pmatrix}$$

and so

$$\lambda^n d_b = |\sigma^n(a)|_a d_b - |\sigma^n(a)|_b d_a \quad (n \geq 0), \tag{13}$$

where $d_b \neq 0$ by (10) and $t_{ab} \neq 0$. Since w is periodic, we can write $w = luu\cdots$ with $l, u \in A^*$; thereby

$$|u|_a d_b - |u|_b d_a = 0. \tag{14}$$

Noting that $\sigma^n(a)$ is a prefix of w for any $n \geq 0$, we can write

$$\sigma^n(a) = l \underbrace{u \cdots u}_{k(n)} r_n \quad (n \geq 0),$$

where $k(n)$ is an integer depending on n and r_n is a word over A whose length is less than that of u. Therefore

$$|\sigma^n(a)|_\alpha = |l|_\alpha + k(n)|u|_\alpha + |r_n|_\alpha \quad (n \geq 0) \tag{15}$$

for $\alpha \in A$. By (13), (14), and (15), we have

$$\lambda^n d_b = (|l|_a + |r_n|_a) d_b - (|l|_b + |r_n|_b) d_a \quad (n \geq 0),$$

where the right-hand side is bounded since the length of r_n is less than that of u. Hence $|\lambda| \leq 1$. By (11) and (14), λ is a rational number. Since λ is an algebraic integer, it is a rational integer. Hence λ is $1, 0$, or -1, and the proof of the lemma is completed.

3 Proof of Theorems

Proof of Theorem 1. First we consider the case where the substitution σ is of nonconstant length, i.e., $t_{aa} + t_{ab} \neq t_{ba} + t_{bb}$. As mentioned in Section 1, we can construct $g_a(\boldsymbol{z}), g_b(\boldsymbol{z}) \in \boldsymbol{Q}[[\boldsymbol{z}]] = \boldsymbol{Q}[[z_1, z_2]]$ satisfying $g_a(z, z) = f_a(z)$, $g_b(z, z) = f_b(z)$, and

$$\begin{pmatrix} g_a(\boldsymbol{z}) \\ g_b(\boldsymbol{z}) \end{pmatrix} = M(\boldsymbol{z}) \begin{pmatrix} g_a(T\boldsymbol{z}) \\ g_b(T\boldsymbol{z}) \end{pmatrix}, \tag{16}$$

where

$$M(\boldsymbol{z}) = \begin{pmatrix} p(\boldsymbol{z}) & q(\boldsymbol{z}) \\ r(\boldsymbol{z}) & s(\boldsymbol{z}) \end{pmatrix}, \quad p(\boldsymbol{z}), q(\boldsymbol{z}), r(\boldsymbol{z}), s(\boldsymbol{z}) \in \boldsymbol{Z}[z_1, z_2].$$

Letting $h(\boldsymbol{z}) = g_a(\boldsymbol{z})/g_b(\boldsymbol{z})$, we get

$$h(\boldsymbol{z}) = \frac{p(\boldsymbol{z})h(T\boldsymbol{z}) + q(\boldsymbol{z})}{r(\boldsymbol{z})h(T\boldsymbol{z}) + s(\boldsymbol{z})}$$

by (16), so that

$$h(T\boldsymbol{z}) = \frac{-s(\boldsymbol{z})h(\boldsymbol{z}) + q(\boldsymbol{z})}{r(\boldsymbol{z})h(\boldsymbol{z}) - p(\boldsymbol{z})},$$

which is a functional equation of the form (7).

We shall apply Lemma 1. The properties (I)–(IV) are satisfied by Lemma 3. We have to check the remaining conditions in Lemma 1. We firstly verify that the function $h(\boldsymbol{z})$ is transcendental over the field $\boldsymbol{C}(z_1, z_2)$. For this, we show that $g_b(z, z) = f_b(z)$ is transcendental over the field $\boldsymbol{C}(z)$. Noting that the coefficients of the power series $f_b(z)$ are 0 and 1, we see by the theorem of Carlson [1] that if $f_b(z)$ is algebraic over $\boldsymbol{C}(z)$, then $f_b(z) \in \boldsymbol{C}(z)$; thereby the sequence of its coefficients is a linear recurrence, so that it is periodic, which contradicts Lemma 4. Therefore $g_b(z, z)$ is transcendental over $\boldsymbol{C}(z)$. Since $g_a(z, z) + g_b(z, z) = 1/(1 - z)$ and so $h(z, z) + 1 = 1/((1 - z)g_b(z, z))$, $h(z, z)$ is transcendental over $\boldsymbol{C}(z)$. Hence $h(\boldsymbol{z})$ is transcendental over $\boldsymbol{C}(z_1, z_2)$. Secondly we verify that $h(\boldsymbol{z})$ converges at all the $T^k l^{-1}\mathbf{1}$ $(k \geq 0)$. We have $T^k l^{-1}\mathbf{1} = (l^{-x_k}, l^{-y_k})$, where x_k and y_k are defined by (8). Since $x_k, y_k > 0$, $g_a(\boldsymbol{z})$ and $g_b(\boldsymbol{z})$ converge at $T^k l^{-1}\mathbf{1}$ for any $k \geq 0$. Hence $h(\boldsymbol{z})$ converges at all the $T^k l^{-1}\mathbf{1}$ $(k \geq 0)$, since $g_b(T^k l^{-1}\mathbf{1}) > 0$. Finally we assert that the resultant $\Delta(\boldsymbol{z})$ of polynomials $-s(\boldsymbol{z})u + q(\boldsymbol{z})$ and $r(\boldsymbol{z})u - p(\boldsymbol{z})$ in u satisfies $\Delta(T^k l^{-1}\mathbf{1}) \neq 0$ for any $k \geq 0$. Noting that $\Delta(\boldsymbol{z})$ divides $\det M(\boldsymbol{z}) = p(\boldsymbol{z})s(\boldsymbol{z}) - q(\boldsymbol{z})r(\boldsymbol{z})$ and letting $M^{(n)}(\boldsymbol{z}) = M(\boldsymbol{z})M(T\boldsymbol{z})\cdots M(T^{n-1}\boldsymbol{z})$, we see that if $\prod_{k=0}^{n-1} \Delta(T^k l^{-1}\mathbf{1}) = 0$, then $\det M^{(n)}(l^{-1}\mathbf{1}) = \prod_{k=0}^{n-1} \det M(T^k l^{-1}\mathbf{1}) = 0$. Hence it suffices to prove that $\det M^{(n)}(l^{-1}\mathbf{1}) \neq 0$ for any $n \geq 1$. To the contrary we assume that $\det M^{(n)}(l^{-1}\mathbf{1}) = 0$ for some n. Since the entries of $M^{(n)}(\boldsymbol{z})$ are elements of $\boldsymbol{Z}[z_1, z_2]$, those of $M^{(n)}(l^{-1}\mathbf{1})$ are rational numbers. Hence there exist integers t and u, not both zero, such that $(t, u)M^{(n)}(l^{-1}\mathbf{1}) = (0, 0)$. Noting that

$$\begin{pmatrix} g_a(\boldsymbol{z}) \\ g_b(\boldsymbol{z}) \end{pmatrix} = M^{(n)}(\boldsymbol{z}) \begin{pmatrix} g_a(T^n\boldsymbol{z}) \\ g_b(T^n\boldsymbol{z}) \end{pmatrix},$$

we have $tg_a(l^{-1}\mathbf{1}) + ug_b(l^{-1}\mathbf{1}) = 0$, so that $t\,h(l^{-1}\mathbf{1}) + u = 0$. Hence $t \neq 0$ and so $h(l^{-1}\mathbf{1}) = -u/t$. Since $h(z, z) + 1 = 1/((1 - z)g_b(z, z))$, $g_b(l^{-1}\mathbf{1})$ is a rational number. Therefore the l-adic decimal expansion of $g_b(l^{-1}\mathbf{1})$, which is given by

$$g_b(l^{-1}\mathbf{1}) = f_b(l^{-1}) = \sum_{k \geq 0} \chi_k(w; b) l^{-k},$$

is periodic, which contradicts Lemma 4, and the assertion is proved. Therefore it follows from Lemma 1 that $h(l^{-1}\mathbf{1})$ is transcendental. Hence $f_b(l^{-1}) = g_b(l^{-1}\mathbf{1}) = l/((l - 1)(h(l^{-1}\mathbf{1}) + 1))$ and $f_a(l^{-1}) = l/(l - 1) - f_b(l^{-1})$ are transcendental.

Next we consider the case where the substitution σ is of constant length, i.e., $t_{aa}+t_{ab}=t_{ba}+t_{bb}=d\geq 2$. As mentioned in Section 1, $f_a(z), f_b(z)$ satisfy

$$\begin{pmatrix} f_a(z) \\ f_b(z) \end{pmatrix} = M(z) \begin{pmatrix} f_a(z^d) \\ f_b(z^d) \end{pmatrix},$$

where

$$M(z) = \begin{pmatrix} p(z) & q(z) \\ r(z) & s(z) \end{pmatrix}, \quad p(z), q(z), r(z), s(z) \in \boldsymbol{Z}[z].$$

In this case, a matrix $T=(d)$ and a point l^{-1} obviously have the properties (I)–(IV) and the rest of the proof is similar to that of the preceding case.

We omit the proof of Theorem 2, since it is the same as the latter case in the proof of Theorem 1.

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