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ON THE RESULTS OF SARNAK-RUDNICK-KATZ-IWANIEC-LUO ON ZEROS OF ZETA FUNCTIONS

慶大理工 小山信也 (Shin-ya Koyama)

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0. Introduction

In this article we will survey the results of Sarnak and surrounding mathematicians of him on the distribution of zeros of zeta functions. The chief concern is the spacing distribution of zeros. We begin with explaining what spacing distribution is.

Spacing Distribution.

Let $\lambda_0 \leq \lambda_1 \leq \cdots$ be a sequence of real numbers. Put $N(T) = \#\{j \mid \lambda_j \leq T\}$. For considering the spacing distribution, we can normalize λ_j such that $N(T) \sim T \quad (T \to \infty)$

Define the measure by

$$\mu(N) = \mu(\lambda_0, \cdots, \lambda_N)[a,b] = \frac{1}{N}\#\{0 \leq j \leq N-1 \mid \Delta_j \in [a,b]\}$$

$$\left(\Delta_j = \lambda_{j+1} - \lambda_j\right)$$

which we call the spacing distribution.

Examples.

(1) Prime Numbers

Let p_j be the j-th prime. By the prime number theorem, its normalization is given by $\lambda_j = \dfrac{p_j}{\log p_j}$. Numerical experiments [KS2, Figure 1] suggest that $\mu(N) \to e^{-x}dx$. But we have no way to prove it up to the present.

(2) Zeros of the Riemann zeta function

For numerical example we can assume the Riemann Hypothesis. Let $\rho_j = \dfrac{1}{2} + \gamma_j\sqrt{-1}$ be non-trivial zeros with $0 < \gamma_1 \le \gamma_2 \le \cdots$. The normalization is given by $\tilde{\gamma}_j = \dfrac{\gamma_j \log \gamma_j}{2\pi}$ by Riemann. The well-known numerical experiments by Odlyzko [O] suggest that the spacing distribution $\mu(N)$ tends to that of GUE. We call this phenomenon the Montgomery-Odlyzko Law.

1. The Riemann Zeta Function

In this section we explain the result of Rudnick and Sarnak [RS] on the spacing distribution of zeros of the Riemann zeta function. Although their main theorem does not assume the Riemann Hypothesis, here we assume it for simplicity. As was given in the previous section, we have the normalized sequence $\tilde{\gamma}_j = \frac{\gamma_j \log \gamma_j}{2\pi}$. Let $B_N = \{\tilde{\gamma}_1, ..., \tilde{\gamma}_N\}$ be the set of first N elements. What we want to know is the number

$$N(a,b) = \#\{k \mid \tilde{\gamma}_{k+1} - \tilde{\gamma}_k \in [a,b]\}$$

for any interval $[a,b]$. But this quantity is hard to treat directly, since it is hard to tell if two elements are consecutive or not unless we know all members in the sequence. So we put

$$N_2 = N_2(a,b) = \#\{(\tilde{\gamma}, \tilde{\gamma}') \in B_N^2 \mid \tilde{\gamma} < \tilde{\gamma}', \ \ \tilde{\gamma}' - \tilde{\gamma} \in [a,b]\}$$

and

$$N_3 = N_3(a,b) = \#\{(\tilde{\gamma}, \tilde{\gamma}', \tilde{\gamma}'') \in B_N^3 \mid \tilde{\gamma} < \tilde{\gamma}' < \tilde{\gamma}'', \ \ \tilde{\gamma}'' - \tilde{\gamma} \in [a,b]\}$$

and so on. To be precise, for a positive integer n, the integer N_n is the number of n-tuples whose difference between the biggest and the smallest elements belong to $[a,b]$. Then we have inductively

$$N(a,b) = N_2 - N_3 + N_4 - \cdots$$

which is a finite alternating sum. Therefore it suffices to obtain the numbers N_n for $n = 2, 3, 4....$ Rudnick and Sarnak generalized N_n as follows:

Definition. *(n-level correlation) Let f be a function of n-variables. The n-level correlation is*

$$R^{(n)}(f, B_N) = \frac{1}{N} \sum_{\substack{(j_1,\dots,j_n) \\ \text{distinct}}} f(\tilde{\gamma}_{j_1}, \dots, \tilde{\gamma}_{j_n})$$

The special case when

$$f(\cdots) = \begin{cases} 1 & \max\limits_k \tilde{\gamma}_{j_k} - \min\limits_k \tilde{\gamma}_{j_k} \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

gives

$$R^{(n)}(f, B_N) = \frac{n!}{N} N_n.$$

So the n-level correlation is a generalization of N_n. In what follows we will study $R^{(n)}(f, B_N)$.

For our purpose, it suffices to consider functions f satisfying the following three conditions:

(1) $f(x_1, \dots, x_n)$ is symmetric.
(2) $f(x_1 + t, \dots, x_n + t) = f(x_1, \dots, x_n)$ for $t \in \mathbf{R}$.
(3) $f(x) \to 0$ rapidly as $|x| \to \infty$ in the hyperplane $\sum\limits_j x_j = 0$.

Theorem 1 (Rudnick-Sarnak)[RS]. *Assume that f satisfies the above three conditions and that its Fourier transform $\hat{f}$ satisfies*

$$\operatorname{supp}(\hat{f}(\xi)) \subset \left\{ \sum_j |\xi_j| < 2 \right\},$$

then

$$R_n(f, B_N) \to \int_{\mathbf{R}^n} f(x) W_n(x) \delta\left(\tfrac{x_1 + \cdots + x_n}{n}\right) dx_1 \dots dx_n$$

as $N \to \infty$, where $\delta(x)$ is the Dirac mass at 0 and $W_n(x) = \det\left(\frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)}\right)$.

Idea of Proof:. By using the explicit formula, we can transform the sum in the definition of the n-level correlation to the sum over prime numbers. It is the formula of the type $\sum\limits_{\substack{(j_1,\dots,j_n) \\ \text{distinct}}} = \sum\limits_{\substack{(p_1,\dots,p_n) \\ \text{primes}}}$. Then we calculate the latter sum by a very complicated combinatorial technique. □

2. Congruence Zeta Functions

In this section we introduce the results of Katz and Sarnak [KS] [KS2]. Let $C/\mathbf{F}_q$ be a curve with function field k. The congruence zeta function is defined as

$$\zeta(C,T) = \prod_{v:\text{place of } k} \left(1 - T^{\deg(v)}\right)^{-1}.$$

It is well-known that it has the following expression.

$$\zeta(C,T) = \frac{P(C,T)}{(1-T)(1-qT)},$$

where P is a polynomial of degree $2g$ with g being the genus. The Riemann Hypothesis, which was proved by Deligne, asserts all zeros of P lie on $|T| = q^{-1/2}$. We put zeros as $e^{i\theta_j} q^{-1/2}$ $(j = 1, 2, ..., 2g)$ Out interest is the spacing distribution of $\{\theta_j\}$. Although there are only finite number of elements, an interesting phenomenon is observed if the curve C varies. We will consider a family of congruence zeta functions and will take a certain limit in the family.

By the process described before we have the measure μ_C which we are interested in. Their main theorem asserts the measure μ_C tends to a universal one deriving from general classical compact groups.

For a unitary matrix A, we have a finite sequence of eigenvalues of it. We define the measure μ_A by the same procedure from the sequence. The following lemma assures the existence of the universal measure.

Lemma (Katz-Sarnak). *There exists a measure $\mu_{\text{universal}}$ such that*

$$\lim_{N\to\infty} \int_{G(N)} \mu_A dA = \mu_{\text{universal}}$$

with $G = U, SU, O, SO, USp$.

When two measures μ and ν are given, we put $D(\mu,\nu) = \sup_{x\in\mathbf{R}} |\mu((-\infty, x] - \nu(-\infty, x])|$ which is called the discrepancy of the measures. If $D(\mu,\nu)$ is zero, the two measures are essentially equal. The following theorem says our measure μ_C "converges" to the universal measure.

Theorem 2. (Katz-Sarnak). *Let $M_g(\mathbf{F}_q)$ be the set of isomorphism classes of curves with genus g. Then*

$$\lim_{\substack{g\to\infty \\ q\to\infty}} \frac{\sum_{C\in M_g(\mathbf{F}_q)} D(\mu_C, \mu_{universal})}{\# M_g(\mathbf{F}_q)} = 0$$

Sketch of the Proof. There are following three keys in the proof.

(1) The monodromy group = full $Sp(2g)$.

(2) Equidistribution theorem of Deligne

(3) Law of large numbers

We deduce the map

$$M_g(\mathbf{F}_q) \ni C \to \text{Frobenius } \theta(C) \in USp(2g)$$

is surjective. $\square$

3. Examples of "Global Monodromy"

In the title, Global Monodromy has to be in double quotation marks, because it cannot be defined precisely. What we learned from the function field case is that interesting phenomena arise when we consider a family of zeta functions. The monodromy group plays the role of gluing those zeta functions. In the proof of the Riemann Hypothesis for function field cases, Deligne proved it not for a single zeta function, but for a family of zeta functions altogether. So if the original Riemann Hypothesis should be solved, we expect that we will be able to discover a family of zeta functions together with its monodromy group, which dominates the zeros. Although the definition of family and monodromy is not discovered at the present, we will introduce some results which will make us believe the existence of monodromy groups for various families of zeta functions.

Philosophy. *Let f be a source of L-function such as an automorphic form. We want to consider a family $\mathcal{F}$ of f. Although we have no precise rule on what family we can treat, we can give some examples later. For an element $f \in \mathcal{F}$, we assume the conductor of f is defined, which is a positive number and is denoted by c_f. Put $\mathcal{F}_X = \{f \in \mathcal{F} \mid c_f \leq X\}$. Let $\frac{1}{2} + i\gamma_f^{(j)}$ $(j = 1, 2, \ldots)$ be nontrivial zeros of $L(s, f)$. The Generalized Riemann Hypothesis asserts $\gamma_f^{(j)} \in \mathbf{R}$. For zeros with $\gamma_f^{(j)} \in \mathbf{R}$, we assume $\gamma_f^{(i)} \leq \gamma_f^{(j)}$ if $i \leq j$. We put the sequence $\hat{\gamma}_f^{(j)} = \frac{\gamma_f^{(j)} \log c_f}{2\pi}$ The distribution of the j-th lowest zero is defined by*

$$\mu_j(X, \mathcal{F})[a, b] = \frac{1}{|\mathcal{F}_X|} \#\{f \in \mathcal{F} \mid c_f \leq X, \hat{\gamma}_f^{(j)} \in [a, b]\}$$

The density of low-lying zeros in $O(\frac{1}{\log c_f})$ is defined by

$$W(X, \mathcal{F}, \phi) = \frac{1}{|\mathcal{F}_X|} \sum_{c_f \leq X} D(f, \phi),$$

where

$$D(f,\phi) = \sum_j \phi\left(\hat{\gamma}_f^{(j)}\right)$$

and ϕ is a rapidly decreasing function defined on $\mathbf{R}$. *(If $\gamma_f^{(j)}$ is not real, the value of ϕ should be considered as zero.)*

We hope that

$$\mu_j(X,\mathcal{F}) \to \mu_j(\mathcal{F})$$

for some $\mu_j(\mathcal{F})$ and that

$$W(X,\mathcal{F},\phi) \to \int_{-\infty}^{\infty} \phi(x)W(\mathcal{F})dx$$

for some $W(\mathcal{F})dx$ as $X \to \infty$.

In the function field case we observed that $\mu_j(X,\mathcal{F})$ and $W(X,\mathcal{F},\phi)$ were determined by the limit of the monodromy group. We expect such phenomena for global cases as well. Although we can't define the monodromy groups, we have some examples as below. In what follows the conjectural monodromy group will be denoted by "$G(\mathcal{F})$".

Example 1.

Let $\mathcal{F} = \{\chi \mid \text{primitive character mod } q,\ \chi^2 = 1\}$. Then $L(s,f) = L(s,\chi)$ which is the Dirichlet L-function, and the conductor $c_\chi = q$ is in the usual sense.

Prediction. "$G(\mathcal{F})$" $= Sp(\infty)$

We have some evidences for this prediction. The first is the following theorem.

Theorem 3.1 (Katz-Sarnak). *If* $\mathrm{supp}(\hat{\phi}) \subset (-2,2)$,

$$W(X,\mathcal{F},\phi) \to \int_{-\infty}^{\infty} \phi(x)\omega(Sp,x)dx$$

where $\omega(Sp,x) = 1 - \frac{\sin 2\pi x}{2\pi x}$

The second evidence is Rubinstein's experiment [R]. He investigated numerically the distributions of $\nu_j(\mathcal{F},X)$ $(j=1,2)$ and $W(X,\mathcal{F})$ for $X \approx 10^{12}$. He finds an excellent fit with the $Sp(\infty)$ predictions.

As the third evidence, we have the Hazelgrove phenomenon as follows. Hazelgrove numerically computed zeros in this family for moderate sized q. He found that the zeros repel the point $s = \frac{1}{2}$. The density of $\nu_1(Sp)$ vanishes to second order at 0 and this is unique to the Sp symmetry. So this phenomenon is a manifestation of the symplectic symmetry.

Example 2.

Let Δ be the cusp form for $SL_2(\mathbf{Z})$ of weight 12. Let $\mathcal{F} = \{\Delta \otimes \chi \mid \chi \bmod q\}$. We consider the family of L-functions

$$L(s, \Delta \otimes \chi) = \sum_{n=1}^{\infty} \frac{\tau(n)\chi(n)}{n^{\frac{11}{2}+s}}.$$

We have two subfamilies $\mathcal{F}^+$ and $\mathcal{F}^-$ according to the signature of the functional equation of $L(s, \Delta \otimes \chi)$. Putting the conductor $c_{\Delta \otimes \chi} = q^2$, the global monodromy is predicted as follows:

Prediction.

$$\text{“}G(\mathcal{F}^+)\text{”} = \lim_{n \to \infty} SO(2n)$$

$$\text{“}G(\mathcal{F}^-)\text{”} = \lim_{n \to \infty} SO(2n-1)$$

The first evidence is the following theorem.

Theorem 3.2 (Katz-Sarnak). *If* $\mathrm{supp}(\hat{\phi}) \subset (-1, 1)$, *as* $X \to \infty$

$$W(X, \mathcal{F}^+, \phi) \to \int_{-\infty}^{\infty} \phi(x)\omega(SO(even), x)dx.$$

$$W(X, \mathcal{F}^-, \phi) \to \int_{-\infty}^{\infty} \phi(x)\omega(SO(odd), x)dx.$$

We also have Rubinstein's numerical experiments [R] with $\nu_j(X, \mathcal{F}^{\pm})$ $(j = 1, 2)$ and $W(X, \mathcal{F}^{\pm})$ with $X \approx 10^6$, which agree with the $O(\infty)$-predictions.

Example 3.

Let $\mathcal{F} = \{f \mid \text{holo Hecke eigen cusp form for } PSL(2, \mathbf{Z}) \text{of wt } k\}$. We consider the family of automorphic L-functions. We again have two subfamilies $\mathcal{F}^+, \mathcal{F}^-$ owing to the functional equation. In fact the sign is $+1$ if $k \equiv 0(4)$ and -1 if $k \equiv 2(4)$. By putting $c_f = k^2$, we predict as follows.

Prediction.

$$\text{“}G(\mathcal{F}^+)\text{”} = \lim_{n \to \infty} SO(2n)$$

$$\text{“}G(\mathcal{F}^-)\text{”} = \lim_{n \to \infty} SO(2n-1)$$

Theorem 3.3 (Iwaniec-Luo-Sarnak). *If* $\mathrm{supp}(\hat{\phi}) \subset (-1,1)$, *as* $X \to \infty$

$$W(X, \mathcal{F}^+, \phi) \to \int_{-\infty}^{\infty} \phi(x)\omega(SO(even), x)dx$$

$$W(X, \mathcal{F}^-, \phi) \to \int_{-\infty}^{\infty} \phi(x)\omega(SO(odd), x)dx$$

If we assume the Generalized Riemann Hypothesis, we get it for $\mathrm{supp}(\hat{\phi}) \subset (-2,2)$.

Example 4.

Let $\mathcal{F} = \{f \mid \text{holo cusp form, wt } k \text{ for } \Gamma_0(N)\}$. Here we are interested in the subfamily of newforms $H_k(N) = \{f \mid \text{new form}\} \subset \mathcal{F}$. We assume that the central character of f is trivial and for simplicity we also assume N is prime. Let $H_k^+(N), H_K^-(N)$ be the subfamilies defined from the functional equation as the preceding examples. Putting $c_f = N$, we predict as follows.

Prediction.

$$\text{“}G(\mathcal{F}^+)\text{”} = \text{“}G(\mathcal{H}_k^+(N))\text{”} = \lim_{n\to\infty} SO(2n).$$

$$\text{“}G(\mathcal{F}^-)\text{”} = \text{“}G(\mathcal{H}_k^-(N))\text{”} = \lim_{n\to\infty} SO(2n-1).$$

We have the following theorem as an evidence.

Theorem 3.4 (Iwaniec-Luo-Sarnak). *If* $\mathrm{supp}(\hat{\phi}) \subset (-1,1)$, *as* $X \to \infty$

$$W(X, H_k^+(N), \phi) \to \int_{-\infty}^{\infty} \phi(x)\omega(SO(even), x)dx$$

$$W(X, H_k^-(N), \phi) \to \int_{-\infty}^{\infty} \phi(x)\omega(SO(odd), x)dx$$

If we assume the Generalized Riemann Hypothesis, we get it for $\mathrm{supp}(\hat{\phi}) \subset (-2,2)$.

Example 5.

Let $\mathcal{F} = \{\vee^2 f \mid f \in \text{Example 3}\}$. We consider the symmetric squared L-functions $L(s, \vee^2 f)$. It is equal to $L(s, \bar{f})$ for $\bar{f}$ a self-dual cusp form on GL_3. It has an Euler product of degree 3. By putting $c_{\vee^2 f} = k^2$, we have

Prediction. “$G(\mathcal{F})$” $= Sp(\infty)$

The following theorem is proved as an evidence.

Theorem 3.5 (Iwaniec-Luo-Sarnak). *If* $\mathrm{supp}(\hat{\phi}) \subset (-1,1)$*, as* $X \to \infty$

$$W(X, \mathcal{F}, \phi) \to \int_{-\infty}^{\infty} \phi(x)\omega(Sp, x)dx$$

If we assume the Generalized Riemann Hypothesis, we get it for $\mathrm{supp}(\hat{\phi}) \subset (-\frac{4}{3}, \frac{4}{3})$.

From the examples given above, we have the following general conjecture.

Density Conjecture.

$W(X, \mathcal{F}, \phi)$ converges to the claimed density without any restriction on $\hat{\phi}$.

4. Applications

In the settings given in Example 4, we have some remarkable applications. The Density Conjecture implies that

$$\frac{\#\{f \in \mathcal{F} \mid c_f \leq X,\ \epsilon_f = 1,\ L(\frac{1}{2}, f) \neq 0\}}{\#\{f \in \mathcal{F} \mid c_f \leq X,\ \epsilon_f = 1\}} \to 1$$

as $X \to \infty$. Towards the Density Conjecture, we have some partial results as follows:

Theorem 4.1 (Iwaniec-Sarnak-Luo). *Assume the Generalized Riemann Hypothesis, then*

$$\frac{\#\{f \in H_2^+(N) \mid L(\frac{1}{2}, f) \neq 0\}}{\#\{f \in H_2^+(N)\}} > \frac{9}{16}$$

Theorem 4.2 (Iwaniec-Sarnak).

$$\lim_{N \to \infty} \frac{\#\{f \in H_2^+(N) | L(\frac{1}{2}, f) \geq \frac{1}{(\log N)^2}\}}{\#\{f \in H_2^+(N)\}} \geq \frac{1}{2}$$

Theorem 4.3 (Iwaniec-Sarnak).

If Theorem 4.2 holds with any $C > \frac{1}{2}$ *in place of* $\frac{1}{2}$*, there are no Siegel zeros!*

References

[ILS] H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of L-functions of modular forms* (preprint).

[IS] H. Iwaniec and P. Sarnak, *The non-vanishing of central values of automorphic L-functions and Siegel zeros* (preprint).

[KS1] N. Katz and P. Sarnak, *Random matrices, Frobenius eigenvalues and monodromy* (1998), AMS (to appear).

[KS2] N. Katz and P. Sarnak, *Zeros of zeta functions, their spacings and their spectral nature* (preprint).

[O] A. M. Odlyzko,, *On the distribution of spacings between zeros of the zeta function.*, Math. of Computation **48** (1987), 273-308.

[R] M. Rubinstein, *Evidence for a spectral interpretation of the zeros of L-functions.*

[RS] Z. Rudnick and P. Sarnak, *The n-level correlations of zeros of the zeta function,*, C. R. Acad. Sci. Paris **319** (1994), 1027-1032.

Dept of Math., Keio University, 223-8522 Japan