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CUBIC MODULAR EQUATIONS AND NEW RAMANUJAN-TYPE SERIES FOR $1/\pi$ (TALK GIVEN AT THE CONFERENCE "TOPICS IN NUMBER THEORY AND ITS APPLICATIONS", RIMS, KYOTO)

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1. INTRODUCTION

In his famous paper "Modular equations and Approximations to π", Ramanujan offered 17 beautiful series for $1/\pi$. He then remarks that two of these series, namely,

$$\frac{27}{\pi}=\sum_{k=0}^{\infty}(2+15k)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{1}{3})_k}{(k!)^3}\left(\frac{2}{27}\right)^k \tag{1.1}$$

and

$$\frac{15\sqrt{3}}{2\pi}=\sum_{k=0}^{\infty}(4+33k)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(\frac{4}{125}\right)^k \tag{1.2}$$

where

$$(a)_0=1,(a)_k=(a)\cdot(a+1)\cdots(a+n-1),$$

"belong to the theory of q_2". Ramanujan did not elaborate on what he meant by "theory of q_2". Ramanujan's so-called "theory of q_2" has recently been developed by B. C. Berndt, S. Bhargava and F. G. Garvan (see TAMS, vol. 347, (1995), 4163–4244), after the discovery of the Borweins' cubic theta functions and is now known as "*Ramanujan's theory of elliptic function to alternative base 3*".

In this talk, we will see how one can derive new series for $1/\pi$ which belong to the aforementioned theory. Our fastest convergent new series takes the form

$$\frac{2153559\sqrt{3}}{\pi}=\sum_{k=0}^{\infty}(a+bk)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(\frac{73-40\sqrt{3}}{2^{1/3}\cdot 23^2(4+5\sqrt{3})}\right)^{3k}, \tag{1.3}$$

where

$$a=1028358\sqrt{3}-593849 \quad \text{and} \quad b=19101285\sqrt{3}-795.$$

For each term summed in this series, we get approximately 10 more decimal places of accuracies for π. As a corollary, we have

$$\pi \simeq \frac{1781547\sqrt{3} + 9255222}{3928247}.$$

2. THE BORWEINS' CUBIC SERIES

Let

$${}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}.$$

Further, let

$$K(x) := {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; 1; x), \dot{K}(x) := \frac{dK(x)}{dx},$$

and define the cubic singular modulus α_n as the unique number satisfying

$$\frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1 - \alpha_n)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \alpha_n)} = \sqrt{n}, \quad n \in \mathbb{Q}^+$$

The Borweins recorded in their book "Pi and the AGM" the following general series for $1/\pi$:

Theorem 2.1. *(The Borweins' general "cubic" series for* $1/\pi$*)*
Set

$$\epsilon(n) = \frac{3\sqrt{3}}{8\pi}(K(\alpha_n))^{-2} - \sqrt{n}\left(\frac{3}{2}\alpha_n(1 - \alpha_n)\frac{\dot{K}(\alpha_n)}{K(\alpha_n)} - \alpha_n\right),$$

$$a_n := \frac{8\sqrt{3}}{9}\left(\epsilon(n) - \sqrt{n}\alpha_n\right), \quad \textit{and} \quad b_n := \frac{2\sqrt{3n}}{3}\sqrt{1 - H_n},$$

where $H_n := 4\alpha_n(1 - \alpha_n)$*. Then*

$$\frac{1}{\pi} = \sum_{k=0}^{\infty}(a_n + b_n k)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}H_n^k.$$

Note that from this general series, we see that in order to construct series for $1/\pi$, it suffices to evaluate $\epsilon(n)$, a_n, b_n and H_n for various n. On the other hand, since b_n is dependent on H_n, and so does α_n, it suffices to compute $\epsilon(n)$ and H_n for various n. We have succeeded in using new modular equations and Kronecker's limit formula to compute H_n and $\epsilon(n)$ for $n = 7$, 10, 11, 14, 19, 19, 26, 31, 34 and 59. Our nine new series then follow from the table :

Our series (1.3) is the case $n = 59$. Before the discovery of these new series, there are only 5 known cubic series for $1/\pi$, namely $n = 2$, 3, 4, 5 and 6. Two of which are already in Ramanujan's paper while the other three were given by the Borweins in their book before the discovery of Ramanujan's alternative base theory. The Borweins discovered their series by solving a sixth degree polynomial expressing α_n in terms of Ramanujan-Weber class invariants.

n	a_n	b_n	H_n
7	$-\frac{10}{27}+\frac{7}{27}\sqrt{7}$	$\frac{13}{9}\sqrt{7}-\frac{7}{9}$	$-\frac{17}{27}+\frac{13}{54}\sqrt{7}$
10	$\frac{25}{243}\sqrt{15}-\frac{8}{243}\sqrt{6}$	$\frac{70}{81}\sqrt{15}+\frac{10}{81}\sqrt{6}$	$\frac{223}{1458}-\frac{35}{729}\sqrt{10}$
11	$\frac{6}{11}-\frac{13}{99}\sqrt{3}$	$\frac{45}{11}-\frac{5}{33}\sqrt{3}$	$-\frac{194}{1331}+\frac{225}{2662}\sqrt{3}$
14	$\frac{21}{125}\sqrt{7}-\frac{82}{1125}\sqrt{3}$	$\frac{198}{125}\sqrt{7}+\frac{28}{375}\sqrt{3}$	$\frac{1819}{31250}-\frac{198}{15625}\sqrt{21}$
19	$\frac{1654}{3375}-\frac{133}{3375}\sqrt{19}$	$\frac{5719}{1125}-\frac{13}{1125}\sqrt{19}$	$-\frac{8522}{421875}+\frac{3913}{843750}\sqrt{19}$
26	$-\frac{3967}{44217}\sqrt{3}+\frac{1118}{14739}\sqrt{39}$	$\frac{4620}{4913}\sqrt{39}+\frac{130}{14739}\sqrt{3}$	$\frac{249913}{48275138}-\frac{34650}{24137569}\sqrt{13}$
31	$-\frac{14662}{91125}+\frac{7843}{91125}\sqrt{31}$	$\frac{35113}{30375}\sqrt{31}-\frac{217}{30375}$	$-\frac{684197}{307546875}+\frac{245791}{615093750}\sqrt{31}$
34	$-\frac{7157}{323433}\sqrt{51}+\frac{62896}{323433}\sqrt{6}$	$\frac{296140}{107811}\sqrt{6}+\frac{70}{107811}\sqrt{51}$	$\frac{3555313}{2582935938}-\frac{304850}{1291467969}\sqrt{34}$
59	$\frac{342786}{717853}-\frac{593849}{6460677}\sqrt{3}$	$\frac{6367095}{717853}-\frac{265}{2153559}\sqrt{3}$	$-\frac{1461224894}{30403462846931}+\frac{1687280175}{60806925693862}\sqrt{3}$

TABLE 1. Class number = 4

Let us briefly describe the Borweins' method. The Borweins obtained their series $n = 2, 3$ and 6 by solving α_n from a sixth degree relations :

$$\frac{(9-8\alpha_n)^3}{64\alpha_n^3(1-\alpha_n)} = \frac{(4G_{3n}-1)^3}{27G_{3n}^{24}},$$

where the Ramanujan-Weber class invariant G_n is defined as

$$G_n = 2^{-1/4}e^{\pi\sqrt{n}/24}\prod_{k=1}^{\infty}(1-e^{-\pi\sqrt{n}(2k-1)}).$$

Examples :

$$G_{15}^{12} = 8\left(\frac{\sqrt{5}+1}{2}\right)^4 \quad \text{gives} \quad \alpha_5 = \frac{1}{2} - \frac{11\sqrt{5}}{50}.$$

However, the Borweins did not indicate how they obtain their $\epsilon(5)$'s. They leave the computations of $\epsilon(n)$ as exercises. Their method cannot be applied in our case since the class invariants G_{3n} are more complicated. So new methods have to be devised.

3. CUBIC MODULAR EQUATIONS

We say that β has degree n over α if

$$\frac{K(1-\beta)}{K(\beta)} = n\frac{K(1-\alpha)}{K(\alpha)}. \tag{3.1}$$

A relation between α and β induced by (3.1) is known as a *cubic modular equation.* The first few modular equations are given by Ramanujan.
For example, when β has degree 2 over α

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} = 1.$$

In general, we have

Theorem 3.1. *(Cubic Russell-type modular equations)*
Suppose $p > 3$ is an odd prime and $(p+1)/3=N/s$ in lowest terms. Suppose β has degree p over α. Then the relation between

$$u = (\alpha\beta)^{s/6} \quad \textit{and} \quad v = \{(1-\alpha)(1-\beta)\}^{s/6}$$

can be given in the form

$$B_0(v)u^N + B_1(v)u^{N-1} + \cdots + B_N(v) = 0,$$

where $B_0(v), ..., B_N(v)$ are polynomials of degrees at most N in v.

Next, define the *multiplier of degree n* to be

$$m(\alpha,\beta) = \frac{K(\alpha)}{K(\beta)}.$$

One can show that

$$m^2(\alpha,\beta) = n\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\frac{d\alpha}{d\beta}. \tag{3.2}$$

From (3.2), we see that m can be computed via differentiating a modular equation of degree n. This in turn allows us to conclude that

Lemma 3.1. $\frac{dm(\alpha,\beta)}{d\alpha}$ *can be expressed in terms of α and β.*

We are now ready to compute $\epsilon(n)$

Theorem 3.2. *(New formula for $\epsilon(n)$)*

$$\epsilon(n) = \sqrt{n}\alpha_n + \frac{3\alpha_n(1-\alpha_n)}{4}\frac{dm}{d\alpha}(1-\alpha_n, \alpha_n).$$

This formula has never appeared in print. It shows that $\epsilon(n)$ can be computed once we know α_n and at least a modular equation of degree n. This result guarantees us a modular equation of prime degree and so $\epsilon(p)$ can be computed from α_p. When $n = 2p$, as in our table, we can use modular equations of 2 and p to evaluate $\epsilon(n)$ but we will not go into the details. It remains to compute H_n from which α_n will follow.

4. COMPUTATIONS OF H_n

Theorem 4.1. *Suppose the class number of the imaginary quadratic field* $\mathbb{Q}(\sqrt{-3n})$ *is 4 and that each genus in the class group contains a single class. Then* $4H_n^{-1}$ *is of the form* $a+b\sqrt{d}$*, with* a *and* b *non-negative integers and* $d \in \{2,3,6,p,2p,3p,6p\}$.

This shows that $4H_n^{-1}$ can be determined in a finite number of steps. So, for example

$$4H_7^{-1} = 136.789534087679355... = 68 + 26\sqrt{7}.$$

α_n then follows from the H_n.

The proof of Theorem 4.1 follows from the fact that

$$4H_n^{-1} = 2 + u_n + u_n^{-1},$$

where

$$u_n = \frac{1}{27}\left(\frac{\eta(\sqrt{-n/3})}{\eta(\sqrt{-3n})}\right)^{12}.$$

Here,

$$\eta(\tau) := e^{\pi i\tau/12}\prod_{k=1}^{\infty}(1-e^{2\pi ik\tau}).$$

Then, the fact that u_n^2 is a product of two fundamental units follows from the following result which is a consequence of Kronecker's limit formula:

Theorem 4.2. *Let* χ *be a genus character arising from the decomposition* $D_K = d_1 d_2$*. Let* $h_{i,\chi}$ *be the class number of the field* $\mathbb{Q}(\sqrt{d_i})$*,* $\omega_{2,\chi}$ *be the number of roots of unity in* $\mathbb{Q}(\sqrt{d_2})$*, and* ϵ_χ *be the fundamental unit of* $\mathbb{Q}(\sqrt{d_1})$*. Suppose* $[\mathfrak{a}]$ *is an ideal class in* C_K*. Set*

$$F([\mathfrak{a}]) = \sqrt{N([1,\tau])}|\eta(\tau)|^2,$$

where $\eta(\tau)$ *denotes the Dedekind* η*-function defined by*

$$\eta(z) = e^{\pi iz/12}\prod_{k=1}^{\infty}(1-e^{2\pi ikz})$$

and

$$\tau = \frac{\tau_2}{\tau_1}, \quad Im\,\tau > 0, \quad \textit{where} \quad \mathfrak{a} = [\tau_1, \tau_2].$$

Then

$$\epsilon_\chi^{2h_{1,\chi}h_{2,\chi}/\omega_{2,\chi}} = \prod_{\mathfrak{a}\in C_K} F([\mathfrak{a}])^{-\chi([\mathfrak{a}])}.$$