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An Approximation Scheme for Gauss Curvature Flow

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§1 Introduction. We propose here an approximation scheme for Gauss curvature flow of a convex hypersurface in $\mathbf{R}^{n+1}$ and explain how to prove the convergence of the scheme to the Gauss curvature flow.

The Gauss curvature flow of a convex hypersurface in $\mathbf{R}^{n+1}$ is described as follows. Let Γ_0 be a convex hypersurface of $\mathbf{R}^{n+1}$ and $F_0 : S^n \to \mathbf{R}^{n+1}$ be a parametric representation of Γ_0. The Gauss curvature flow of this hypersurface Γ_0 is a collection $F(\cdot, t) : S^n \to \mathbf{R}^{n+1}$ of closed hypersurfaces with parameter $t \in [0, T)$ which is a solution of the initial value problem

$$
(1.1) \qquad \begin{cases} \dfrac{\partial F}{\partial t}(s,t) = -K^{\beta}(s,t)n(s,t) & (0 < t < T,\ s \in S^n) \\ F(s,0) = F_0(s) & (s \in S^n), \end{cases}
$$

where $\beta > 0$ is a constant and $K(s,t)$ and $n(s,t)$ denote the Gauss curvature and the outward unit normal vector, respectively, at $F(s,t)$ of the hypersurface $F(\cdot, t)$.

W. J. Firey [F] proposed problem (1.1) with $\beta = 1$ as a mathematical model of the wearing process of stones on beach by waves and studied some basic properties of the solution F of this problem. Afterwards, K. Tso [T] and then B. Chow [C] studied problem (1.1) and established the following existence theorem.

Theorem 1. *If F_0 represents a smooth, strictly convex hypersurface, then there exist a positive $T > 0$ and a unique smooth solution $F : S^n \times [0,T) \to \mathbf{R}^{n+1}$ of* (1.1) *such that $F(\cdot,t)$ represents a strictly convex set $\Gamma_t \equiv F(S^n,t) \subset \mathbf{R}^{n+1}$ for all $0 < t < T$ and Γ_t converge to a point as $t \nearrow T$.*

Another way of describing Gauss curvature flow is the so-called level-set approach, in which the evolving convex hypersurfaces Γ_t are regarded as the 0-level set of a function u defined on $\mathbf{R}^{n+1} \times [0,\infty)$. More precisely, the idea is explained as follows. Given a compact convex hypersurface Γ_0, we choose a function $g \in BUC(\mathbf{R}^{n+1})$ so that

$$\text{(1.2)} \qquad \Gamma_0 = \{z \in \mathbf{R}^{n+1} \mid g(z) = 0\} \quad \text{and} \quad \{z \in \mathbf{R}^{n+1} \mid g(z) \leq 0\} \ \text{ is convex,}$$

and consider the initial value problem

$$\text{(1.3)} \qquad \begin{cases} \text{(i)} & u_t + G(Du, D^2u) = 0 \quad \text{in } \mathbf{R}^{n+1} \times (0,\ \infty), \\ \text{(ii)} & u(z,0) = g(z) \qquad (z \in \mathbf{R}^{n+1}). \end{cases}$$

Here the function $G : \left(\mathbf{R}^{n+1} \setminus \{0\}\right) \times \mathcal{S}(n+1) \to \mathbf{R}$ is defined by

$$G(p,X) = -|p| \left\{ \det{}_+ \left(\frac{1}{|p|}(I - \overline{p} \otimes \overline{p})X(I - \overline{p} \otimes \overline{p}) + \overline{p} \otimes \overline{p} \right) \right\}^{\beta}$$

where $\overline{p} = p/|p|$ and

$$\det{}_+ A = \prod_{i=1}^{n+1} \max\{\lambda_i, 0\} \quad \text{for } A \in \mathcal{S}(n+1),$$

with λ_i $(i = 1, ..., n+1)$ denoting the eigenvalues of $A \in \mathcal{S}(n+1)$. Now, the (generalized) *Gauss curvature flow* of Γ_0 is defined as the collection $\{\Gamma_t\}_{t \geq 0}$ of the closed subsets

$$\text{(1.4)} \qquad \Gamma_t = \{z \in \mathbf{R}^{n+1} \mid u(z,t) = 0\} \subset \mathbf{R}^{n+1}.$$

One of main results in [IS] (see [IS, Theorems 1.8 and 1.9]) states that if $g \in BUC(\mathbf{R}^{n+1})$, then there is a unique viscosity solution $u \in BUC(\mathbf{R}^{n+1} \times [0,\infty))$ of (1.3) and that the collection $\{\Gamma_t\}_{t \geq 0}$ defined by (1.4) is independent of the choice of g. See [IS] as well for the correct definition of viscosity solution for (1.3), (i). This assertion is, of course, a generalization of a well-known, similar observation due to Chen-Giga-Goto [CGG] and Evans-Spruck [ES] for mean curvature flow and alike.

An argument in [CEI] guarantees that under assumption (1.2), if we set

$$V_t = \{z \in \mathbf{R}^{n+1} \mid u(z,t) \le 0\}, \tag{1.5}$$

then V_t is a convex set and $\Gamma_t = \partial V_t$ for all $t \ge 0$. We shall also call the collection $\{V_t\}_{t\ge 0}$ the generalized *Gauss curvature flow* of the convex body V_0.

See [CEI] for a discussion on the consistency of this level-set approach and the parametric representaion approach based on (1.1).

In what follows we discuss only on the generalized Gauss curvature flow defined via the level-set approach as above and hence suppress the word "generalized" in the argument below.

§2 An approximation scheme and the main result.

Now, we introduce an approximation scheme for Gauss curvature flow. We need notation. We denote by $\mathcal{C}(m)$ the collection of all closed subsets of $\mathbf{R}^m$. Let $A \in \mathcal{C}(n+1)$ and $p \in S^n$. Define

$$\ell_0(A,p) = \sup\{\langle z,p\rangle \mid z \in A\}. \tag{2.1}$$

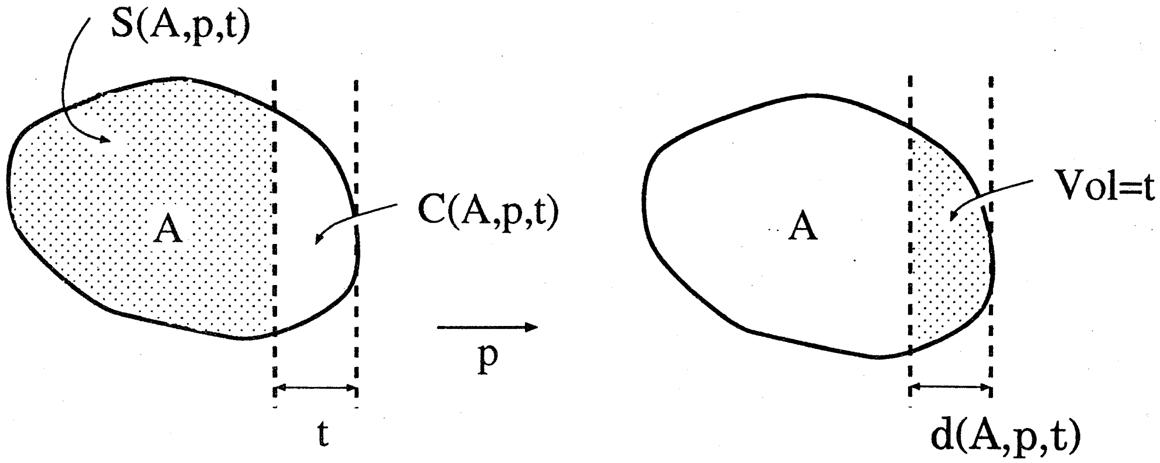
Of course, $\ell_0(\emptyset, p) = -\infty$ and, if the set $\{\langle z,p\rangle \mid z \in A\}$ is not bounded above, $\ell_0(A,p) = \infty$.

For $t > 0$ define $S(A,p,t)$ and $C(A,p,t)$, subsets of $\mathbf{R}^{n+1}$, by

$$S(A,p,t) = \{z \in A \mid \langle z,p\rangle \le \ell_0(A,p) - t\} \tag{2.2}$$

and

$$C(A,p,t) = \{z \in A \mid \langle z,p\rangle > \ell_0(A,p) - t\} \ \big(= A \setminus S(A,p,t)\big). \tag{2.3}$$



Moreover, we define $d(A,p,t) \in [0,\infty]$ by

$$d(A,p,t) = \inf\{s > 0 \mid \mathcal{L}^{n+1}(C(A,p,s)) \geq t\}, \tag{2.4}$$

where $\mathcal{L}^{n+1}(B)$ denotes the $(n+1)$-dimensional Lebesgue measure of the set B, and for $\mu > 0$ set

$$d_\mu(A,p,t) = \min\{d(A,p,t), \mu\} \tag{2.5}$$

Finally, for any $A \in \mathcal{C}(n+1)$, $h > 0$, and $\mu > 0$ we define

$$T_h^\mu(A) = \bigcap_{p \in S^n} S\left(A, p, d_\mu\left(A, p, \alpha_n h^{\frac{1}{2\beta}}\right)^{\beta(n+2)}\right), \tag{2.6}$$

where

$$\alpha_n = \frac{2^{\frac{n+2}{2}}\omega_n}{n+2}, \quad \text{with } \omega_n = \text{ the volume of the unit ball } \subset \mathbf{R}^n.$$

It is clear that for all $A \in \mathcal{C}(n+1)$, $T_h^\mu(A) \subset A$ and $T_h^\mu(A) \in \mathcal{C}(n+1)$ and if A is convex then so is $T_h^\mu(A)$.

Fix a compact convex set $V_0 \subset \mathbf{R}^{n+1}$. Fix $h > 0$ and $\mu > 0$. Define the sequence $\{C_i\}_{i\in\mathbf{N}}$ of subsets of $\mathbf{R}^{n+1}$ by the recursion formula

$$C_1 = V_0 \quad \text{and} \quad C_{i+1} = T_h^\mu(C_i) \quad \text{for } i \in \mathbf{N}$$

and the collection $\{V_t^{\mu,h}\}_{t\geq 0}$ of subsets of $\mathbf{R}^{n+1}$ by

$$V_t^{\mu,h} = C_i \quad \text{if } (i-1)h \leq t < ih \text{ and } i \in \mathbf{N}. \tag{2.7}$$

This collection $\{V_t^{\mu,h}\}_{t\geq 0}$, with $\mu > 0$ and $h > 0$, is our approximation scheme for the Gauss curvature flow $\{V_t\}_{t\geq 0}$ defined by (1.5).

The main result in this paper is the following

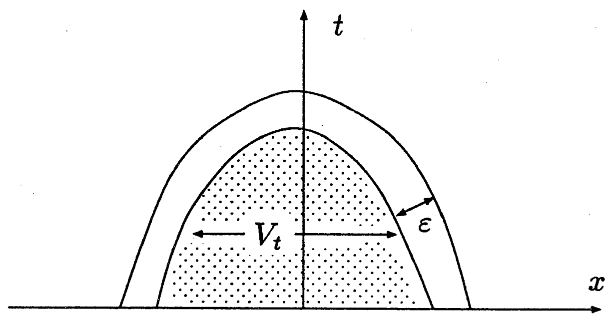
Theorem 2. *Assume that V_0 is compact and convex, $\beta \geq \frac{1}{n+2}$, and $\mu \in \left(0, \frac{1}{6}\right)$. For each $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < h < \delta$, then*

$$\bigcup_{t\geq 0} V_t \times \{t\} \subset \bigcup_{t\geq 0} V_t^{\mu,h} \times \{t\} + B^{n+2}(0,\varepsilon) \tag{2.8}$$

and

$$\bigcup_{t\geq 0} V_t^{\mu,h} \times \{t\} \subset \bigcup_{t\geq 0} V_t \times \{t\} + B^{n+2}(0,\varepsilon), \tag{2.9}$$

where $B^{n+2}(0,\varepsilon)$ denotes the closed ball of radius ε and center at the origin in $\mathbf{R}^{n+2}$. That is, as $h \searrow 0$, the sets $\bigcup_{t\geq 0} V_t^{\mu,h} \times \{t\}$ converge to the set $\bigcup_{t\geq 0} V_t \times \{t\}$ in the Hausdorff distance.



The underlying idea of our definition of T_h^μ (or our approximation scheme) can be explained as follows. Let $A \in \mathcal{C}(n+1)$ be a smooth, strictly convex domain and $B_0 \in \partial A$.

Now, we assume that $B_0 = 0$ and $(0,-1) \in \mathbf{R}^n \times \mathbf{R}$ is the outward unit normal vector of A at 0. Writing $z = (x,y) \in \mathbf{R}^n \times \mathbf{R}$ for generic points in $\mathbf{R}^{n+1}$, we may claim that in a neighborhood of 0, the set A is almost identical to the paraboloid

$$P = \left\{(x,y) \mid y \geq \frac{1}{2}\left(\kappa_1 x_1^2 + \cdots + \kappa_n x_n^2\right)\right\},$$

where κ_i denotes the principal curvatures of the surface ∂A at 0.

For $d > 0$ we denote

$$P(d) = P \cap \{(x,y) \mid y \leq d\},$$

and compute the volume of the set $P(d)$, to find

$$\mathcal{L}^{n+1}(P(d)) = \alpha_n \frac{d^{\frac{n+2}{2}}}{\sqrt{K}}, \tag{2.10}$$

K denoting the Gauss curvature of ∂A at 0, i.e., $K = \prod_{i=1}^{n} \kappa_i$.

If a convex body, starting with A at time 0, is shrinking with velocity

$$v = K^{\beta},$$

in the directions of inward normal vectors of A, then the boundary point of the convex body at time $h > 0$ with $(0, -1)$ as its outward normal direction must be somewhere near the point B_1 with coordinates

$$(0, K^{\beta} h).$$

If we set

$$d = \left(K^{\beta} h\right)^{\frac{1}{\beta(n+2)}},$$

and plug this into (2.10), then we find the formula

$$\mathcal{L}^{n+1}(P(d)) = \alpha_n h^{\frac{1}{2\beta}}. \tag{2.11}$$

A nice feature in formula (2.11) is that it does not involve the Gauss curvautre K explicitly any more. Moreover, the formula determines d uniquely as a function of $h > 0$. Thus, reversing the above process, i.e., fixing first $d > 0$ by formula (2.11) and then setting $a = d^{\beta(n+2)}$, we can identify the point B_1 as the point with coordinates $(0, a)$ without knowing the Gauss curvature K.

Roughly speaking, the set $T_h^{\mu}(A)$ is defined as the convex hull of all the points B_1 obatined from $B_0 \in \partial A$ by the process described above.

§3 Some properties of T_h^μ. We begin with the following proposition, which says that the mapping T_h^μ is invariant under translation and orthogonal transformation. Henceforth h, β, and μ denote fixed positive constants.

Propostion 1. *For any $A \in \mathcal{C}(n+1)$, $U \in O(n+1)$, and $z \in \mathbf{R}^{n+1}$ we have*

$$T_h^\mu(U(A)) = U(T_h^\mu(A)), \tag{3.1}$$

$$T_h^\mu(z + A) = z + T_h^\mu(A). \tag{3.2}$$

Here and later $O(n+1)$ denotes the set of orthogonal matrices of order $n+1$.

The proof is straightforward and left to the reader.

The next proposition asserts the monotonicity of T_h^μ.

Proposition 2. *Assume that $\beta \geq \frac{1}{n+2}$ and $0 < \mu < \frac{1}{6}$. Then, for any $A, B \in \mathcal{C}(n+1)$, if $A \subset B$ we have*

$$T_h^\mu(A) \subset T_h^\mu(B). \tag{3.3}$$

Remark. The restriction that $\beta \geq \frac{1}{n+2}$ and $\mu < \frac{1}{6}$ in Theorem 2 is due to the above proposition. The condition, $\mu < \frac{1}{6}$, is not optimal in this respect and we will not seek for the optimal one here.

The following property will be needed in the proof of Theorem 2.

Proposition 3. *Assume that $\beta \geq \frac{1}{n+2}$ and $0 < \mu < \frac{1}{6}$. Let $A_\varepsilon \in \mathcal{C}(n+1)$, with $0 < \varepsilon \leq 1$, be compact and satisfy*

$$A_\varepsilon \subset A_\delta \quad \textit{if } \varepsilon < \delta.$$

Then we have

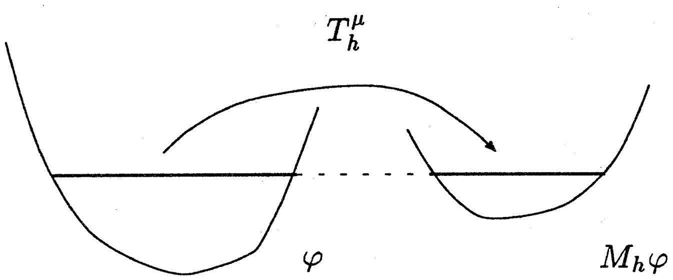
$$T_h^\mu\Big(\bigcap_{0<\varepsilon\le 1} A_\varepsilon\Big) = \bigcap_{0<\varepsilon\le 1} T_h^\mu(A_\varepsilon).$$

§4 **Level-set approach.** We shall take the level-set approach to proving Theorem 2 and here introduce the level-set approach to our approximation scheme.

In what follows we fix $\beta \ge \frac{1}{n+2}$ and $\mu \in \left(0, \frac{1}{6}\right)$. Fix $h > 0$ as well. For any function $\varphi \in C(\mathbf{R}^{n+1})$, following [E], we set

$$M_h\varphi(z) = \inf\{\lambda \in \mathbf{R} \mid z \in T_h^\mu(\{\varphi \le \lambda\})\} \quad (z \in \mathbf{R}^{n+1}). \tag{4.1}$$

Here and henceforth we use the notation $\{P\}$ for $\{z \mid P(z)\}$, where P or $P(z)$ is a proposition concerning z.



We see immediately from (4.1) that for $\varphi \in C(\mathbf{R}^{n+1})$ and $\lambda \in \mathbf{R}$,

$$T_h^\mu(\{\varphi \le \lambda\}) \subset \{M_h\varphi \le \lambda\} \quad \text{and} \quad \{M_h\varphi \le \lambda\} \subset \bigcap_{\gamma > \lambda} T_h^\mu(\{\varphi \le \gamma\}).$$

Loosely speaking, these say that T_h^μ maps the sub-level set of φ of height λ to the sub-level set of $M_h\varphi$ of height λ. In other words, the mapping T_h^μ on sets of $\mathbf{R}^{n+1}$ can be understood by seeing the mapping M_h on functions in $\mathbf{R}^{n+1}$.

Fix $\varphi \in C(\mathbf{R}^{n+1})$. Since $T_h^\mu(\{\varphi \le \lambda\}) \subset \{\varphi \le \lambda\}$ for $\lambda \in \mathbf{R}$, we see that for all $\varphi \in C(\mathbf{R}^{n+1})$,

$$M_h\varphi \ge \varphi \quad \text{in } \mathbf{R}^{n+1}. \tag{4.2}$$

Also, it follows from Proposition 2 that if $\varphi, \psi \in C(\mathbf{R}^{n+1})$ and $\varphi \leq \psi$ in $\mathbf{R}^{n+1}$, then

$$(4.3) \qquad M_h\varphi \leq M_h\psi \quad \text{in } \mathbf{R}^{n+1}.$$

It follows that $M_h\varphi$ is a real-valued function on $\mathbf{R}^{n+1}$.

Proposition 1 has direct consequences for M_h. Indeed, for any $\varphi \in C(\mathbf{R}^{n+1})$ we have

$$(4.4) \qquad (M_h\varphi) \circ U = M_h(\varphi \circ U) \quad \text{for all } U \in O(n+1),$$

where $U \in O(n+1)$ is regarded as a mapping, and

$$(4.5) \qquad M_h \circ \tau_y = \tau_y \circ M_h \quad \text{for all } y \in \mathbf{R}^{n+1},$$

where τ_y denotes the translation by y, i.e., $\tau_y\varphi(z) = \varphi(z-y)$. The proof of these claims are again left to the reader.

Next, we observe that $M_h\varphi \in UC(\mathbf{R}^{n+1})$ for all $\varphi \in UC(\mathbf{R}^{n+1})$. This will be proved as a consequence of (4.3), (4.5), and the following claim.

Let $\theta \in C(\mathbf{R})$ be any nondecreasing function. The claim is:

$$(4.6) \qquad M_h(\theta \circ \varphi) = \theta \circ (M_h\varphi) \quad \text{for } \varphi \in C(\mathbf{R}^{n+1}).$$

The proof of this claim, which is again easy, is left to the reader. (See (2.10) in [I1] for a similar observation.)

To conclude the uniform continuity, let $\varphi \in UC(\mathbf{R}^{n+1})$ and ω denote the modulus of continuity of φ. If $y \in \mathbf{R}^{n+1}$, then

$$\tau_y\varphi \leq \varphi + \omega(|y|),$$

and so, using (4.5), (4.3), and (4.6) with $\theta(r) = r + \omega(|y|)$, we see that

$$M_h\varphi(z-y) = M_h(\tau_y\varphi)(z) \leq M_h\varphi(z) + \omega(|y|) \quad (z \in \mathbf{R}^{n+1}),$$

from which follows the uniform continuity of $M_h\varphi$, i.e.,

$$|M_h\varphi(z) - M_h\varphi(y)| \le \omega(|z-y|) \quad \text{for all } z, y \in \mathbf{R}^{n+1}. \tag{4.7}$$

Similarly, we have

$$\|M_h\varphi - M_h\psi\| \le \|\varphi - \psi\| \quad \text{for } \varphi, \psi \in C(\mathbf{R}^{n+1}), \tag{4.8}$$

where $\|\varphi\| = \sup_{\mathbf{R}^{n+1}} |\varphi| \in [0, \infty]$.

Also, we easily see that if c is a constant function on $\mathbf{R}^{n+1}$ then

$$M_h c = c.$$

Our proof of Theorem 2 will be carried out via the following

Theorem 3. *Let $g \in BUC(\mathbf{R}^{n+1})$ be such that for any $\lambda < \sup_{\mathbf{R}^{n+1}} g$, the set $\{g \le \lambda\}$ is compact and convex. Let $u \in BUC(\mathbf{R}^{n+1} \times [0,\infty))$ be the viscosity solution of* (1.3). *Define $v_h : \mathbf{R}^{n+1} \times [0,\infty) \to \mathbf{R}$ by*

$$v_h(z,t) = M_h^i g(z) \quad \text{if } (i-1)h \le t < ih \text{ and } i \in \mathbf{N}, \tag{4.9}$$

where M_h^i denotes the i times iterates of the mapping M_h. Then for each $0 < T < \infty$, as $h \searrow 0$,

$$v_h(z,t) \to u(z,t) \quad \text{uniformly on } \mathbf{R}^{n+1} \times [0,T]. \tag{4.10}$$

The above definition (4.9) is a reformulation of (2.7) in terms of the level-set approach. (See the next Proposition.)

Let us state here a corollary of Proposition 3, which gives a better connection between (4.9) and (2.7).

Proposition 4. *Let $\gamma \in \mathbf{R}$ and $\varphi \in C(\mathbf{R}^{n+1})$ be such that $\{\varphi \le \gamma\}$ is a compact set. Then, for $z \in \mathbf{R}^{n+1}$, if $M_h\varphi(z) < \gamma$, then we have*

$$M_h\varphi(z) = \min\{\lambda \in \mathbf{R} \mid z \in T_h^\mu(\{\varphi \le \lambda\})\}. \tag{4.11}$$

Note that under the above hypothesis, if $\lambda < \gamma$ then we have

$$\{M_h\varphi \le \lambda\} \subset T_h^\mu(\{\varphi \le \lambda\}).$$

Proof. Assume that $z \in \mathbf{R}^{n+1}$ satisfies $\lambda \equiv M_h\varphi(z) < \gamma$.

It follows that if $t > \lambda$ then

$$z \in M_h(\{\varphi \le t\}). \tag{4.12}$$

Fix any $\eta \in (\lambda, \gamma)$. Note that

$$\{\varphi \le \lambda\} = \bigcap_{\lambda < t \le \eta} \{\varphi \le t\}.$$

Now, from Proposition 3 and (4.12), we have

$$T_h^\mu(\{\varphi \le \lambda\}) = \bigcap_{\lambda < t \le \eta} T_h^\mu(\{\varphi \le t\}) \ni z,$$

whence follows (4.11). □

§5 Approximate derivative of M_h at $h = 0$. In this section we assume that $\beta \ge \frac{1}{n+2}$ and $\mu \in \left(0, \frac{1}{6}\right)$.

The key observation in the proof of Theorem 3 will be stated in this section, which roughly says that the generator of Gauss curvature flow in terms of the level-set approach, i.e., $-G$ in (1.3), (i) "approximates the derivative" of M_h at $h = 0$.

Indeed, we have the following two theorems. The reader who is interested in the proof of these theorems should consult [I2].

Theorem 4. *Let $\varphi \in C^2(\mathbf{R}^{n+1})$ satisfy $D\varphi(\hat{z}) \ne 0$, with $\hat{z} \in \mathbf{R}^{n+1}$. Then for each $\varepsilon > 0$ there is a constant $\delta > 0$ such that*

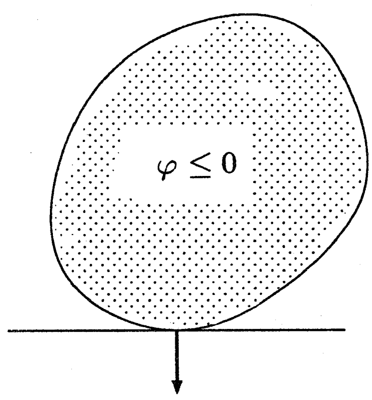
$$M_h\varphi(z) \le \varphi(z) + \left(-G(D\varphi(\hat{z}), D^2\varphi(\hat{z})) + \varepsilon\right) h \quad (z \in B^{n+1}(\hat{z}, \delta),\ h \in (0, \delta)).$$

Theorem 5. *Let $\varphi \in C^2(\mathbf{R}^{n+1})$ satisfy $D\varphi(\hat{z}) \neq 0$, with $\hat{z} \in \mathbf{R}^{n+1}$. Assume that*

$$\begin{cases} \varphi(z) > \varphi(\hat{z}) \quad \text{if} \quad \langle z, D\varphi(\hat{z})\rangle \geq 0 \quad \text{and} \quad z \neq 0, \\ \liminf_{|z|\to\infty} \varphi(z) > \varphi(\hat{z}), \\ -G(D\varphi(\hat{z}), D^2\varphi(\hat{z})) > 0. \end{cases}$$

Then for each $\varepsilon > 0$ there is a constant $\delta > 0$ such that

$$M_h\varphi(z) \geq \varphi(z) + \left(-G\left(D\varphi(\hat{z}), D^2\varphi(\hat{z})\right) - \varepsilon\right) h \quad \left(0 < h \leq \delta,\ z \in B^{n+1}(\hat{z}, \delta)\right).$$



§6 **Convergence of the approximation scheme.** We shall complete the proof of Theorems 2 and 3 in this section.

We will again assume throughout that $\beta \geq \frac{1}{n+2}$ and $0 < \mu < \frac{1}{6}$.

Proof of Theorem 3. Let $g \in BUC(\mathbf{R}^{n+1})$, $u \in BUC(\mathbf{R}^{n+1} \times [0,\infty))$, and $v_h : \mathbf{R}^{n+1} \times [0,\infty) \to \mathbf{R}$ be as in Theorem 3.

Define

$$\overline{v}(z,t) = \lim_{r \searrow 0} \sup\{v_h(y,s) \mid (y,s) \in \mathbf{R}^{n+1} \times [0,\infty) \cap B^{n+2}(z,t;r),\ 0 < h < r\}$$
$$((z,t) \in \mathbf{R}^{n+1} \times [0,\infty)),$$

and

$$\underline{v}(z,t) = \liminf_{r \searrow 0} \{v_h(y,s) \mid (y,s) \in \mathbf{R}^{n+1} \times [0,\infty) \cap B^{n+2}(z,t;r),\ 0 < h < r\}$$
$$((z,t) \in \mathbf{R}^{n+1} \times [0,\infty)).$$

We shall first prove that $u = \overline{v} = \underline{v}$ in $\mathbf{R}^{n+1} \times [0,\infty)$, which guarantees that $v_h(z,t) \to u(z,t)$ uniformly on compact subsets of $\mathbf{R}^{n+1} \times [0,\infty)$ as $h \searrow 0$.

To see this, we prove that $\overline{v}$ and $\underline{v}$ are a viscosity subsolution of (1.3), (i) and a viscosity supersolution of (1.3), (i), and that

$$\lim_{r \searrow 0} \sup\{|\overline{v}(x,t) - \underline{v}(y,s)| \mid |x-y| < r, 0 < t, s < r\} \leq 0.$$

Then we use a comparison theorem, to find that $\overline{v} \leq \underline{v}$ in $\mathbf{R}^{n+1} \times [0,\infty)$. And we see that

$$u = \overline{v} = \underline{v} \quad \text{on } \mathbf{R}^{n+1} \times [0,\infty).$$

This shows that

$$v_h(x,t) \to u(x,t) \quad \text{as } h \searrow 0.$$

Here the convergence holds uniformly on any compact subsets of $\mathbf{R}^{n+1} \times [0,\infty)$.

A simple modification of the above arguments yields the uniform convergence in the whole space $\mathbf{R}^{n+1} \times [0,\infty)$. □

Proof of Theorem 2. Fix any compact convex $V_0 \subset \mathbf{R}^{n+1}$ and choose $g \in BUC(\mathbf{R}^{n+1})$ so that $\{g \leq 0\} = V_0$ and so that for any $\lambda < \sup_{\mathbf{R}^{n+1}} g$ the set $\{g \leq \lambda\}$ is compact and convex. Let $u \in BUC(\mathbf{R}^{n+1} \times [0,\infty))$ be the viscosity solution of (1.3) and $v_h : \mathbf{R}^{n+1} \times [0,\infty) \to \mathbf{R}$ be defined by (4.9).

First we observe in view of Proposition 4 that

$$V_t^{\mu,h} = \{v_h(\cdot,t) \leq 0\} \quad (h > 0,\ t \geq 0).$$

Next, since $G \leq 0$, we infer that for each $z \in \mathbf{R}^{n+1}$ the function $u(z,t)$ of $t \geq 0$ is nondecreasing. By comparison between u and the constant function $\sup_{\mathbf{R}^{n+1}} g$, we see that $u \leq \sup_{\mathbf{R}^{n+1}} g$ in $\mathbf{R}^{n+1} \times [0, \infty)$. By comparing each Gauss curvature flow $\{u(\cdot, t) \leq \lambda\}_{t \geq 0}$ with the Gauss curvature flow of a ball containing the set $\{g \leq \lambda\}$, we then conclude that

$$\lim_{t \to \infty} u(z, t) = \sup_{\mathbf{R}^{n+1}} g$$

uniformly in $\mathbf{R}^{n+1}$.

Fix $\gamma > 0$ so that $\gamma < \sup_{\mathbf{R}^{n+1}} g$, and then $T > 0$ so that

$$u(z, t) > \gamma \quad (z \in \mathbf{R}^{n+1},\ t \geq T).$$

Fix $\varepsilon > 0$ and, in view of the compactness of the set $\{u \leq \gamma\}$ and the continuity of u, choose $\delta \in (0, \gamma)$ so that

$$\{u \leq \delta\} \subset \{u \leq 0\} + B^{n+2}(0, \varepsilon).$$

By Theorem 3, we can choose $\eta > 0$ so that for all $0 < h \leq \eta$,

$$|u(z, t) - v_h(z, t)| \leq \delta \quad (z \in \mathbf{R}^{n+1},\ t \in [0, T]). \tag{6.1}$$

Then, if $0 < h \leq \eta$, $(z, t) \in \mathbf{R}^{n+1} \times [0, T]$, and $v_h(z, t) \leq 0$, we have

$$(z, t) \in \{u \leq \delta\} \subset \{u \leq 0\} + B^{n+2}(0, \varepsilon).$$

If $0 < h \leq \eta$ and $z \in \mathbf{R}^{n+1}$, then, since $u(z, T) > \gamma > \delta$, we see that $v_h(z, T) > 0$. Noting that for each $z \in \mathbf{R}^{n+1}$ the function $v_h(z, t)$ of $t \geq 0$ is nondecreasing, we find that if $0 < h \leq \eta$, then $v_h(z, t) > 0$ for all $(z, t) \in \mathbf{R}^{n+1} \times [T, \infty)$.

Thus we see that

$$\{v_h \leq 0\} \subset \{u \leq 0\} + B^{n+2}(0, \varepsilon),$$

proving (2.9).

The other inclusion is observed with a little more care. Indeed, we need to divide our considerations into two cases.

Case 1: $\operatorname{Int} V_0 = \emptyset$. The extinction time in this case is zero (see [CEI]), i.e., $V_t = \emptyset$ for $t > 0$. Therefore, it is obvious that $\{u \leq 0\} \subset \{v_h \leq 0\}$ for all $h > 0$.

Case 2: $\mathrm{Int}\, V_0 \neq \emptyset$. We may now assume that $g(x) < 0$ for all $x \in \mathrm{Int}\, V_0$. The convexity of V_0 guarantees (see [CEI]) that there does not happen "fattening"in the Gauss curvature flow $\{V_t\}_{t\geq 0}$, from which we can conclude that

$$\{u \leq 0\} = \overline{\bigcup_{s>0} \{u \leq -s\}}.$$

Fix $\varepsilon > 0$, and we see from the above that for some $\delta > 0$,

$$\{u \leq 0\} \subset \{u \leq -\delta\} + B^{n+2}(0, \varepsilon).$$

We may assume that (6.1) holds with current $\delta > 0$ and some $\eta > 0$. Now it is immediate to conclude that for all $0 < h \leq \eta$,

$$\{u \leq 0\} \subset \{v_h \leq 0\} + B^{n+2}(0, \varepsilon),$$

which proves (2.8). $\square$

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