数理解析研究所講究録
1061巻 1998年 142-152

Well-posedness of capillary-gravity waves with large initial data

Tatsuo IGUCHI (井口 達雄)

Graduate School of Mathematics, Kyushu University
Hakozaki, Fukuoka 812-81, Japan

1 Introduction

In this communication we are concerned with a free boundary problem for the two-dimensional and irrotational flow of incompressible ideal fluid. It is assumed that the domain occupied by the fluid has infinite extent and finite depth. We take the surface tension on the free surface and the gravity into account.

For gravity waves, V. I. Nalimov [3] considered the initial value problem in the case of infinite depth. He showed that the problem is well-posed in suitable Sobolev spaces of finite smoothness under the restriction that the initial data are close to the equiliblium rest state. Then, H. Yosihara [5] extended the Nalimov's result to the case of presence of an almost flat bottom. Yosihara [6] also showed the well-posedness of the initial value problem for capillary-gravity waves and discussed the convergence of solutions when the surface tension coefficient tends to zero. Recently, T. Iguchi, N. Tanaka and A. Tani [1] extended their results to the two-phase problem. Both of Nalimov and Yosihara used the Lagrangian coordinates to reduce the problem to a problem on the free surface. However, for two-phase problem we can not introduce the Lagrangian coordinates because the velocities of upper and lower fluids on the interface do not coincide. Moreover, the well-posedness in a Sobolev space of finite smoothness does not hold for two-phase problem of gravity waves. The result in [1] also implies that we do not have to use the Lagrangian coordinates if we take the surface tension into account, and that the proof becomes simpler if we use the Euler coordinates.

In all the results mentioned above, it is assumed that the initial data are close to the equiliblium rest state. Now, our purpose of this paper is to show the well-posedness of initial value problem for capillary-gravity waves without assuming the initial data to be small.

2 Formulation and result

Assume that the domain Ω_t occupied by the fluid at time $t \geq 0$, the free surface Γ_t and the bottom Σ are of the forms

$$\Omega_t = \{(x,y)\,;\, b(x) < y < \eta(t,x), x \in \mathbf{R}^1\},$$
$$\Gamma_t = \{(x,y)\,;\, y = \eta(t,x), x \in \mathbf{R}^1\},$$
$$\Sigma = \{(x,y)\,;\, y = b(x), x \in \mathbf{R}^1\},$$

where b is a given function, while η is the unknown. The motion of the fluid is described by the velocity $v = (v_1, v_2)$ and the pressure p satisfying the equations

$$\rho(\frac{\partial v}{\partial t} + (v \cdot \nabla)v) + \nabla p = -\rho(0,g) \quad \text{in} \quad \Omega_t, \quad t > 0, \tag{1}$$

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0, \quad \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \quad \text{in} \quad \Omega_t, \quad t > 0, \tag{2}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$, $v \cdot \nabla = v_1(\partial/\partial x) + v_2(\partial/\partial y)$, ρ is a constant density, g is a gravitational constant. In this paper g is not necessarily positive. The dynamical and kinematical boundary conditions on the free surface are given by

$$p = p_0 - \sigma H \qquad \text{on} \qquad \Gamma_t, \quad t > 0 \tag{3}$$

and

$$\Big(\frac{\partial}{\partial t} + v \cdot \nabla\Big)(y - \eta(t,x)) = 0 \qquad \text{on} \qquad \Gamma_t, \quad t > 0, \tag{4}$$

respectively, where p_0 is an external pressure assumed to be constant, σ is a surface tension coefficient and H is the curvature of the free surface. The boundary condition on the bottom is given by

$$v \cdot N = 0 \qquad \text{on} \qquad \Sigma, \quad t > 0, \tag{5}$$

where N is the unit normal vector to Σ. Finally, we impose the initial conditions

$$\eta(0,x) = \eta_0(x), \qquad v(0,x,y) = v_0(x,y). \tag{6}$$

The initial velocity v_0 is assumed to satisfy the compatibility conditions.

Accoding to [1], we reformulate the initial value problem (1)–(6) as a problem on the free surface. Put

$$V(t,x) = v(t,x,\eta(t,x)), \quad x \in \mathbf{R}^1, \ t > 0. \tag{7}$$

Then η and $V = (V_1, V_2)$ satisfy the following system of equations:

$$\begin{aligned} &\frac{\partial V_1}{\partial t} + V_1 \frac{\partial V_1}{\partial x} + \frac{\partial \eta}{\partial x}\Big(\frac{\partial V_2}{\partial t} + V_1 \frac{\partial V_2}{\partial x} + g\Big) \\ &= \mu \frac{\partial^2}{\partial x^2}\Big((1 + (\frac{\partial \eta}{\partial x})^2)^{-1/2} \frac{\partial \eta}{\partial x}\Big) \qquad \text{for} \qquad t > 0, \end{aligned} \tag{8}$$

$$\frac{\partial \eta}{\partial t} + V_1 \frac{\partial \eta}{\partial x} - V_2 = 0 \qquad \text{for} \qquad t > 0, \tag{9}$$

$$V_2 = K V_1 \qquad \text{for} \qquad t > 0, \tag{10}$$

$$\eta = \eta_0, \qquad V_1 = U_0 \qquad \text{at} \qquad t = 0, \tag{11}$$

where $\mu = \sigma/\rho$ and $K = K(\eta, b)$ is a linear operator depending on η and b. These are derived as follows. Using the kinematical boundary condition (4) and the momentum equation (1), we see that

$$
\text{(12)} \qquad \begin{aligned} \frac{\partial}{\partial t}V(t,x) + V_1(t,x)\frac{\partial}{\partial x}V(t,x) &= v_t + (v\cdot\nabla)v|_{\Gamma_t} \\ &= -\frac{1}{\rho}\nabla p|_{\Gamma_t} - (0,g). \end{aligned}
$$

On the other hand, by the dynamical boundary condition (3), we have

$$
p(t,x,\eta(t,x)) = p_0 - \sigma\frac{\partial}{\partial x}\Big((1+(\frac{\partial\eta}{\partial x})^2)^{-1/2}\frac{\partial\eta}{\partial x}\Big).
$$

Differentiating this with respect to x, we get

$$
\text{(13)} \qquad (1, \frac{\partial\eta}{\partial x})\cdot\nabla p|_{\Gamma_t} = -\sigma\frac{\partial^2}{\partial x^2}\Big((1+(\frac{\partial\eta}{\partial x})^2)^{-1/2}\frac{\partial\eta}{\partial x}\Big).
$$

Eliminating $\nabla p|_{\Gamma_t}$ from two relations (12) and (13), we obtain (8). (9) comes from the kinematical boundary condition (4) directly. (2) and (5) implies that $v_1|_{\Gamma_t}$ and $v_2|_{\Gamma_t}$ are not independent. (10) represents such a dependence. The linear operator K can be written explicitly as

$$
\text{(14)} \qquad \begin{cases} K = -(1-B_1)^{-1}B_2, \\ B_1 = A_2 + (A_5 - A_6 b')(1 + A_4 + A_3 b')^{-1}A_7, \\ B_2 = A_1 - (A_5 - A_6 b')(1 + A_4 + A_3 b')^{-1}A_8, \end{cases}
$$

where

$$
\begin{cases} A_j u(x) = \dfrac{1}{\pi}\text{v.p.}\displaystyle\int_{\mathbf{R}^1} a_j(x,y)u(y)dy, \quad j = 1,2,3,4, \\ a_1(x,y) = \dfrac{y - x + (\eta(y)-\eta(x))\eta'(y)}{(y-x)^2 + (\eta(y)-\eta(x))^2}, \\ a_2(x,y) = \dfrac{\eta(y)-\eta(x) - (y-x)\eta'(y)}{(y-x)^2 + (\eta(y)-\eta(x))^2}, \\ a_3(x,y) = \dfrac{y - x + (b(y)-b(x))b'(y)}{(y-x)^2 + (b(y)-b(x))^2}, \\ a_4(x,y) = \dfrac{b(y)-b(x) - (y-x)b'(y)}{(y-x)^2 + (b(y)-b(x))^2}, \end{cases}
$$

and

$$\begin{cases} A_j u(x) = \dfrac{1}{\pi}\displaystyle\int_{\mathbf{R}^1} a_j(x,y)u(y)dy, \quad j = 5,6,7,8, \\ a_5(x,y) = \dfrac{y - x + (b(y) - \eta(x))b'(y)}{(y-x)^2 + (b(y) - \eta(x))^2}, \\ a_6(x,y) = \dfrac{b(y) - \eta(x) - (y-x)b'(y)}{(y-x)^2 + (b(y) - \eta(x))^2}, \\ a_7(x,y) = \dfrac{y - x + (\eta(y) - b(x))\eta'(y)}{(y-x)^2 + (\eta(y) - b(x))^2}, \\ a_8(x,y) = \dfrac{\eta(y) - b(x) - (y-x)\eta'(y)}{(y-x)^2 + (\eta(y) - b(x))^2}. \end{cases}$$

Although Yosihara [5] has already expressed the operator K, this expression is different from the Yosihara's. Finally, the initial datum U_0 is calculated from η_0 and v_0 by utilizing (12).

Let $H^s = H^s(\mathbf{R}^1)$ be the usual Sobolev space of order $s \in \mathbf{R}^1$ and $W^{1,\infty} = \{ b \in L^\infty(\mathbf{R}^1)\,;\, b' \in L^\infty(\mathbf{R}^1) \}$. The following is our main result of this paper.

Theorem *Let μ be a positive constant and g a constant not necessarily positive. Suppose that*

$$\begin{cases} \eta_0 \in H^{s+3/2}, \quad U_0 \in H^s, \quad s \geq 6 + 1/2, \\ b \in W^{1,\infty}, \quad \inf\limits_{x \in \mathbf{R}^1} (\eta_0(x) - b(x)) > 0. \end{cases} \tag{15}$$

Then, problem (8)–(11) *has a unique solution* (η, V) *on some time interval* $[0,T]$ *satisfying*

$$\begin{cases} \eta \in C^j([0,T]; H^{s+3/2-3j/2}), \\ V \in C^j([0,T]; H^{s-3j/2}), \quad j = 0,1. \end{cases}$$

Moreover, the solution (η, V) *depends continuously on the initial data* (η_0, U_0) *in the indicated spaces.*

Remark 1 In the previous results, it was assumed, in addition to (15), that

$$\begin{cases} \|\eta_0\|_5 + \|U_0\|_3 \leq \delta, \quad \delta = \delta(h) \ll 1, \\ b(x) = -h + \beta(x), \quad h > 0, \ \beta \in H^{s+3/2}, \ \|\beta\|_3 \leq \delta, \end{cases}$$

where $\| \cdot \|_s$ is the norm of H^s.

Remark 2 Similar results are valid for the case of infinite depth and for the two-phase problem.

3 Outline of the proof

The initial value problem (8)–(11) can be reduced to the following quasi-linear system of equations for (η, V_1)

$$\begin{cases} V_{1tt} + 2V_1V_{1tx} + B(\eta, V_1, V_{1t})V_{1xx} \\ \qquad + (A(\eta, V_1, V_{1t})|D|^3 - i(A(\eta, V_1, V_{1t}))_x D|D|)V_1 = f_1(\eta, V_1, V_{1t}), \\ \eta_t = f_2(\eta, V_1, V_{1t}) \end{cases}$$

under the initial conditions

$$\eta = \eta_0, \ V_1 = U_0, \ V_{1t} = U_1 \quad \text{at} \quad t = 0,$$

where $A(\eta, V_1, V_{1t}), B(\eta, V_1, V_{1t})$ and $f_j(\eta, V_1, V_{1t}), j = 1, 2$ are lower order terms. The initial datum U_1 should be determined from η_0 and U_0 by means of (8)–(11). Now, let us recall the definition of the operator K (see (14)). In the definition we use two inverse operators

$$(16) \qquad (1 + A_4 + A_3 b')^{-1} \quad \text{and} \quad (1 - B_1)^{-1}.$$

Moreover, in the reduction of problem (8)–(11) to the above quasi-linear system, we also use an inverse operator

$$(17) \qquad (1 + K\eta_x)^{-1}.$$

In the previous works, these inverse operators were defined by the Neumann series. In order to guarantee the convergence of such Neumann series, we imposed η and β to be small. In this paper, we report that these inverse operators can be defined without use of the Neumann series. Actually, we have the following lemma.

Lemma 1 *Suppose that*

$$\eta, b \in W^{1,\infty}, \quad \inf_{x\in\mathbf{R}^1}(\eta(x) - b(x)) > 0.$$

Then the inverse operators in (16) *and* (17) *are well-defined and bounded in* $L^2(\mathbf{R}^1)$.

Once we obtain this lemma, we can show our theorem by using the technique used in the previous workes together with this lemma and some new estimates for the integral operators A_j, $j = 5, 6, 7, 8$. We prove this lemma by making use of the idea due to G. C. Verchota [4]. In the following, we concentrate our argument on the operator $(1 + A_4 + A_3 b')^{-1}$ and use the following identification

$$f(Q) = f(x) \quad \text{for} \quad Q = (x, b(x)) \in \Sigma.$$

Then, we can rewrite the operators A_3 and A_4 as

$$\begin{cases} A_3 f(P) = \dfrac{1}{\pi}\text{v.p.}\displaystyle\int_\Sigma \frac{\langle Q-P, T(Q)\rangle}{|Q-P|^2} f(Q)d\sigma(Q), \\ A_4 f(P) = \dfrac{1}{\pi}\text{v.p.}\displaystyle\int_\Sigma \frac{\langle Q-P, N(Q)\rangle}{|Q-P|^2} f(Q)d\sigma(Q), \end{cases}$$

where $d\sigma(Q)$ denotes the surface element of Σ, $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbf{R}^2$, $T(Q)$ and $N(Q)$ denote the tangential and the normal vector to Σ, respectively. More precisely,

$$\begin{cases} N(Q) = (N_1(Q), N_2(Q)) = \Big(\dfrac{-b'(x)}{(1+(b'(x))^2)^{-1/2}}, \dfrac{1}{(1+(b'(x))^2)^{-1/2}}\Big), \\ T(Q) = (N_2(Q), -N_1(Q)) \qquad \text{for} \qquad Q = (x, b(x)) \in \Sigma. \end{cases}$$

For arbitrary but fixed $f \in L^2(\mathbf{R}^1)$, we define the boundary layer potential $v(X) = (v_1(X), v_2(X))$, $X = (x, y)$ by

$$\begin{cases} v_1(X) = -\frac{1}{\pi}\int_\Sigma \frac{\langle X-Q, N(Q)\rangle}{|X-Q|^2}(b'f)(Q)d\sigma(Q) \\ \qquad\qquad + \frac{1}{\pi}\int_\Sigma \frac{\langle X-Q, T(Q)\rangle}{|X-Q|^2}f(Q)d\sigma(Q), \\ v_2(X) = \frac{1}{\pi}\int_\Sigma \frac{\langle X-Q, N(Q)\rangle}{|X-Q|^2}f(Q)d\sigma(Q) \\ \qquad\qquad + \frac{1}{\pi}\int_\Sigma \frac{\langle X-Q, T(Q)\rangle}{|X-Q|^2}(b'f)(Q)d\sigma(Q). \end{cases}$$

Then, it holds that

$$(18) \qquad \begin{cases} \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0, \quad \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \quad \text{in} \quad \mathbf{R}^2 \setminus \Sigma, \\ v \text{ vanishes at } \infty \text{ in a suitable sense.} \end{cases}$$

Concerning the boundary value of the boundary layer potential v, we define non-tangential cones Γ_+ and Γ_- by

$$\begin{cases} \Gamma_+(P) = \{(x, y)\,;\, y - b(z) > M|x - z|\}, \\ \Gamma_-(P) = \{(x, y)\,;\, y - b(z) < -M|x - z|\} \end{cases}$$

for $P = (z, b(z)) \in \Sigma$, where M is a constant satisfying the inequality $\|b'\|_{L^\infty} < M$. Since the non-tangential maximal function of v, i.e.

$$v_+^*(P) = \sup_{X \in \Gamma_+(P)} |v(X)| \quad \text{and} \quad v_-^*(P) = \sup_{X \in \Gamma_-(P)} |v(X)|$$

belong to L^2 and satisfy $\|v_\pm^*\|_{L^2} \le C\|f\|_{L^2}$ with a constant C independent of f,

$$v^+(P) = \lim_{X \in \Gamma_+(P), X \to P} v(X) \quad \text{and} \quad v^-(P) = \lim_{X \in \Gamma_-(P), X \to P} v(X)$$

exist for almost every $P \in \Sigma$. Moreover, by the well-known properties of double layer and single layer potentials, we see that

$$(19) \qquad \begin{cases} v_2^+(P) = (1 - A_4 - A_3 b')f(P), \\ v_2^-(P) = -(1 + A_4 + A_3 b')f(P), \\ \langle T(P), v^+(P)\rangle = \langle T(P), v^-(P)\rangle. \end{cases}$$

On the other hand, we have the following lemma.

Lemma 2 *If v satisfies* (18), *then it holds that*

$$C^{-1}\int_\Sigma v_1^2 d\sigma \le \int_\Sigma v_2^2 d\sigma \le C\int_\Sigma v_1^2 d\sigma,$$

$$C^{-1}\int_\Sigma \langle T, v\rangle^2 d\sigma \le \int_\Sigma \langle N, v\rangle^2 d\sigma \le C\int_\Sigma \langle T, v\rangle^2 d\sigma,$$

where C is a constant depending only on the bound of $\|b'\|_{L^\infty}$ and v is either v^+ the boundary value from above or v^- the boundary value from below. Particularly, we have

$$(20) \qquad \begin{cases} \int_\Sigma (v_2^+)^2 d\sigma \le C\int_\Sigma \langle T, v^+\rangle^2 d\sigma, \\ \int_\Sigma \langle T, v^-\rangle^2 d\sigma \le C\int_\Sigma (v_2^-)^2 d\sigma. \end{cases}$$

Formally, this lemma is a simple consequence of application of the divergence theorem. In fact, it follows from (18) that

$$\frac{\partial}{\partial y}v_1^2 = \frac{\partial}{\partial y}v_2^2 + 2\frac{\partial}{\partial x}(v_1 v_2).$$

Therefore,

$$(21) \qquad \int_\Sigma N_2 v_1^2 d\sigma = \int_\Sigma N_2 v_2^2 d\sigma + 2\int_\Sigma N_1 v_1 v_2 d\sigma.$$

This and the estimate $(1 + \|b'\|_{L^\infty}^2)^{-1/2} \le N_2(Q)$ imply the first estimate of the lemma. Putting

$$\begin{cases} v_1 = \langle T, v\rangle N_2 + \langle N, v\rangle N_1, \\ v_2 = \langle N, v\rangle N_2 - \langle T, v\rangle N_1 \end{cases}$$

into (21), we obtain

$$\int_\Sigma N_2\langle T, v\rangle^2 d\sigma = \int_\Sigma N_2\langle N, v\rangle^2 d\sigma - 2\int_\Sigma N_1\langle T, v\rangle\langle N, v\rangle d\sigma.$$

Hence, we get the second estimate. This argument can be justified by the same way as in [4].

Applying the estimates in (20) to the previous boundary layer potential v and using the relations in (19), we see that

$$\|(1 - A_4 - A_3 b')f\|_{L^2} \leq C\|(1 + A_4 + A_3 b')f\|_{L^2},$$

which impies that

$$\|f\|_{L^2} \leq C\|(1 + A_4 + A_3 b')f\|_{L^2}, \tag{22}$$

where the constant C depends only on the bound of $\|b'\|_{L^\infty}$. Next, we define the operator L_t by

$$L_t = 1 + A_4(t) + A_3(t)(tb') \quad \text{for} \quad 0 \leq t \leq 1,$$

where the operators $A_3(t)$ and $A_4(t)$ are defined by replacing the function b with tb in the definition of the operators A_3 and A_4, respectively. Then, by the results of R. R. Coifman, A. McIntosh and Y. Meyer [2] and (22), we obtain the properties

(i) $C^{-1}\|f\|_{L^2} \leq \|L_t f\|_{L^2} \leq C\|f\|_{L^2}$,
(ii) $\|L_t f - L_s f\|_{L^2} \leq C|t - s|\|f\|_{L^2}$,
(iii) $L_1 = 1 + A_4 + A_3 b', \quad L_0 = 1$

for $t, s \in [0, 1]$ and $f \in L^2(\mathbf{R}^1)$, where C is a constant depending only on the bound of $\|b'\|_{L^\infty}$. Therefore, by the method of continuity we see that the operator $1 + A_4 + A_3 b'$ map $L^2(\mathbf{R}^1)$ onto itself and that the inverse operator of it satisfies

$$\|(1 + A_4 + A_3 b')^{-1} f\|_{L^2} \leq C\|f\|_{L^2} \quad \text{for} \quad f \in L^2(\mathbf{R}^1).$$

Similarly, we can show that the other operators in (16) and (17) are well-defined and bounded in $L^2(\mathbf{R}^1)$.

The details of the proof will be published elsewhere.

References

[1] Iguchi, T., Tanaka, N. and Tani, A., On the two-phase free boundary problem for two-dimensional water waves, to appear in Math. Annalen.

[2] Coifman, R. R., McIntosh, A. and Meyer, Y., L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes, Ann. of Math., **116** (1982), 361–387.

[3] Nalimov, V. I., The Cauchy-Poisson problem, Dinamika Splošn. Sredy, **18** (1974), 104–210 [Russian].

[4] Verchota, G. C., Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains, Thesis, Univ. of Minnesota, 1982, J. Funct. Anal., **59** (1984), 572–611.

[5] Yosihara, H., Gravity waves on the free surface of an incompressible perfect fluid of finite depth, Publ. RIMS Kyoto Univ., **18** (1982), 49–96.

[6] Yosihara, H., Capillary-gravity waves for an incompressible ideal fluid, J. Math. Kyoto Univ., **23** (1983), 649–694.