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A LOCAL EXISTENCE THEOREM FOR THE NAVIER-STOKES FLOW IN THE EXTERIOR TO A ROTATING OBSTACLE

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ABSTRACT. Let us consider the three dimensional Navier-Stokes initial value problem in the exterior to a rotating obstacle. It is proved that a unique solution exists locally in time when the initial data a possess some regularity in the space L^2 (similarly to the assumption given by Fujita and Kato [4]) and satisfy $(\omega \times x) \cdot \nabla a \in H^{-1}$, where ω stands for the angular velocity of the rotating obstacle. An essential step for the proof is to deduce a certain smoothing property together with estimates near $t = 0$ of the semigroup (it is not an analytic one) generated by the operator $\mathcal{L}u = -P[\Delta u + (\omega \times x) \cdot \nabla u - \omega \times u]$, where P denotes the projection associated with the Helmholtz decomposition.

It is one of important problems in fluid mechanics to study the Navier-Stokes flow past a rotating obstacle. In order to understand the rotation effect mathematically, we will limit ourselves to a problem under the following simple situation; the angular velocity is constant and the translation is absent. In this article we discuss the locally in time existence of a unique solution to such a problem.

Let $\mathcal{O} \subset \mathbb{R}^3$ be a compact, isolated rigid obstacle which is bounded by a smooth surface Γ, and $\Omega = \mathbb{R}^3 \setminus \mathcal{O}$ the exterior domain occupied by a viscous incompressible fluid. Assume that the obstacle $\mathcal{O}$ is rotating about the x_3-axis with angular velocity $\omega = (0,0,1)^T$. Here and hereafter, super-T denotes the transpose and all vectors are column ones; $x = (x_1, x_2, x_3)^T, \nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^T$ and so on. Set

$$\Omega(t) = \{y = O(t)x;\ x \in \Omega\}, \qquad \Gamma(t) = \{y = O(t)x;\ x \in \Gamma\},$$

which actually vary as time t goes on (this is the situation under consideration) unless $\mathcal{O}$ is axisymmetric, where

$$O(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We now consider the fluid motion around $\mathcal{O}$, which is governed by the initial boundary value problem for the Navier-Stokes equation

$$\text{(NS.1)} \qquad \begin{cases} \partial_t w + w \cdot \nabla_y w = \Delta_y w - \nabla_y q, & y \in \Omega(t),\ t > 0, \\ \nabla_y \cdot w = 0, & y \in \Omega(t),\ t \geq 0, \\ w = \omega \times y, & y \in \Gamma(t),\ t > 0, \\ w \to 0, & |y| \to \infty,\ t > 0, \\ w(y, 0) = a(y), & y \in \Omega, \end{cases}$$

where $w = (w_1(y,t), w_2(y,t), w_3(y,t))$ and $q = q(y,t)$ denote, respectively, unknown velocity and pressure of the fluid. The boundary condition on $\Gamma(t)$ is the non-slip one since $dy/dt = \dot{O}(t)O(t)^T y = \omega \times y$, where $\dot{O}(t) = (d/dt)O(t)$. It is natural to reduce (NS.1) to the problem in the fixed domain Ω by using the coordinate system $x = O(t)^T y$ attached to the rotating obstacle. There are two ways to make the change of the fluid velocity. The one is

$$u(x,t) = O(t)^T w(y,t),$$

and the other is

$$v(x,t) = O(t)^T \left[w(y,t) - \omega \times y \right] = u(x,t) - \omega \times x.$$

We also make the change of the pressure by

$$p(x,t) = q(y,t).$$

Then we have

$$
\begin{aligned}
\partial_t w &= O(t)\left[\partial_t u + \left(\dot O(t)^T O(t)x\right)\cdot\nabla_x u + O(t)^T\dot O(t)u\right]\\
&= O(t)\left[\partial_t u - (\omega\times x)\cdot\nabla_x u + \omega\times u\right]\\
&= O(t)\left[\partial_t v - (\omega\times x)\cdot\nabla_x v + \omega\times v\right],\\
\Delta_y w &= O(t)\Delta_x u = O(t)\Delta_x v,\\
\nabla_y q &= O(t)\nabla_x p,\\
\nabla_y\cdot w &= \nabla_x\cdot u = \nabla_x\cdot v,
\end{aligned}
$$

and

$$
\begin{aligned}
w\cdot\nabla_y w &= O(t)\left[u\cdot\nabla_x u\right]\\
&= O(t)\left[v\cdot\nabla_x v + (\omega\times x)\cdot\nabla_x v + \omega\times v + \omega\times(\omega\times x)\right].
\end{aligned}
$$

The problem (NS.1) is thus reduced to the following (NS.2) and (NS.3) for $\{v,p\}$ and $\{u,p\}$, respectively. The former is the problem with not only the Coriolis force $2\,\omega\times v$ but also the growing boundary condition at space infinity:

$$
\text{(NS.2)}\quad
\begin{cases}
\partial_t v + v\cdot\nabla_x v = \Delta_x v - 2\,\omega\times v - \omega\times(\omega\times x) - \nabla_x p, & x\in\Omega,\ t>0,\\
\nabla_x\cdot v = 0, & x\in\Omega,\ t\ge 0,\\
v = 0, & x\in\Gamma,\ t>0,\\
v + \omega\times x \to 0, & |x|\to\infty,\ t>0,\\
v(x,0) = a(x) - \omega\times x, & x\in\Omega.
\end{cases}
$$

The latter is the problem with the convection term having the coefficient $\omega\times x$ which is understood as the rigid motion rotating about the x_3-axis:

(NS.3)
$$\begin{cases} \partial_t u + u\cdot\nabla_x u = \Delta_x u + (\omega\times x)\cdot\nabla_x u - \omega\times u - \nabla_x p, & x\in\Omega,\ t>0, \\ \nabla_x\cdot u = 0, & x\in\Omega,\ t\geq 0, \\ u = \omega\times x, & x\in\Gamma,\ t>0, \\ u\to 0, & |x|\to\infty,\ t>0, \\ u(x,0) = a(x), & x\in\Omega. \end{cases}$$

Up to now the mathematical theory for the existence and uniqueness of solutions to the problem (NS.1) has been little developed. In his Habilitationsschrift [2] Borchers first attacked this problem, including the case where the angular velocity depends on time t. He dealt with the problem (NS.2) and proved the existence of weak solutions of class

$$v+\omega\times x\ (=u)\in L^\infty\left(0,T;\ L^2(\Omega)\right)\cap L^2\left(0,T;\ H^1(\Omega)\right),\qquad \forall\, T>0,$$

with the energy inequality provided that $a\in L^2(\Omega)$ satisfies

$$\text{(1)}\qquad \nabla\cdot a = 0 \quad\text{in}\quad \Omega, \qquad \nu\cdot(a-\omega\times x) = 0 \quad\text{on}\quad \Gamma,$$

where ν is the unit exterior normal vector to Γ. We donot know the uniqueness of weak solutions and this feature is the same as the standard Navier-Stokes theory. Later on, in [3] Chen and Miyakawa have treated (NS.3) for $\Omega = \mathbb{R}^3$, that is, the Cauchy problem. They have discussed the existence of weak solutions with the so-called strong energy inequality and some decay properties of the constructed solutions.

The purpose of the present article is to prove that there exists a unique local solution to the problem (NS.3) whenever the initial data $a\in L^2(\Omega)$ satisfying (1) possess some regularity and fulfill $(\omega\times x)\cdot\nabla a\in H^{-1}(\Omega)$.

To state our results precisely, we introduce notation. We use the same symbols for denoting the spaces of scalar and vector functions if there is no confusion. By $C_0^\infty(\Omega)$ we denote the class of all C^∞ functions with compact supports in Ω. Let $H^s(\Omega)$ for $s \geq 0$ be the usual L^2 Sobolev spaces. If s is not an integer, then the space $H^s(\Omega)$ is defined via the complex interpolation (see Lions and Magenes [11, Chapter 1]), that is,

$$H^s(\Omega) = \left[L^2(\Omega), H^m(\Omega)\right]_\theta, \quad s = \theta m, \quad m > 0 \text{ (integer)}, \quad 0 < \theta < 1.$$

The scalar product and the norm of $L^2(\Omega) = H^0(\Omega)$ are respectively denoted by $(\cdot,\cdot)$ and $\|\cdot\|$. The space $H_0^s(\Omega), s > 0$, is the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, and $H^{-s}(\Omega)$ stands for the dual space of $H_0^s(\Omega)$. Let $C_{0,\sigma}^\infty(\Omega)$ be the class of all solenoidal (that is, divergence free) vector functions whose components are in $C_0^\infty(\Omega)$. By $L_\sigma^2(\Omega)$ we denote the completion of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)$. Then the space $L^2(\Omega)$ of vector functions admits the following orthogonal decomposition, the Helmholtz decomposition (Temam [13, Chapter I]):

$$L^2(\Omega) = L_\sigma^2(\Omega) \oplus L_\pi^2(\Omega),$$

where

$$L_\pi^2(\Omega) = \left\{\nabla p \in L^2(\Omega);\ p \in L_{\mathrm{loc}}^2\left(\overline{\Omega}\right)\right\}.$$

Let P be the projection (the Fujita-Kato projection) from $L^2(\Omega)$ onto $L_\sigma^2(\Omega)$ associated with the decomposition above. Then the Stokes operator $A : L_\sigma^2(\Omega) \to L_\sigma^2(\Omega)$ is defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \cap L_\sigma^2(\Omega), \qquad Au = -P\Delta u.$$

In view of (NS.3), the linear operator $\mathcal{L} : L_\sigma^2(\Omega) \to L_\sigma^2(\Omega)$ we should consider is as follows:

$$\begin{cases} D(\mathcal{L}) = \{u \in D(A);\ (\omega \times x) \cdot \nabla u \in L^2(\Omega)\}, \\ \mathcal{L}u = Au - P\left[(\omega \times x) \cdot \nabla u - \omega \times u\right]. \end{cases}$$

It is proved that the operator $\mathcal{L}$ is m-accretive, so that $-\mathcal{L}$ generates a (C_0) semigroup $\{e^{-t\mathcal{L}}; t \geq 0\}$ of contractions on $L^2_\sigma(\Omega)$. Furthermore, we have

$$\|u\|_{H^2(\Omega)} + \|P\left[(\omega \times x) \cdot \nabla u\right]\| \leq C\, \|(1 + \mathcal{L})u\|, \tag{2}$$

for all $u \in D(\mathcal{L})$ (see [8]). On account of unboundedness of the coefficient of $\mathcal{L}$, the elliptic regularity estimate (2) is no longer trivial. It is thus not so easy to show the closedness of $\mathcal{L}$ directly. But the accretivity implies that $\mathcal{L}$ is closable. So, we prove that $1 + \overline{\mathcal{L}}$ is surjective, where $\overline{\mathcal{L}}$ is the closure of $\mathcal{L}$. For the proof, we solve the corresponding stationary problem by using the solutions in $\mathbb{R}^3$ and in a bounded domain near the boundary Γ together with cut-off functions. For the recovery of the solenoidal condition in the localization, we make use of the result of Bogovskiĭ [1] on a continuous right-inverse of the divergence operator with zero boundary condition in bounded domains. At the next step, we show $\overline{\mathcal{L}} = \mathcal{L}$ together with estimate (2). The fractional powers of $\mathcal{L}$ are also well defined as closed operators in $L^2_\sigma(\Omega)$, and we see that $D(\mathcal{L}^\alpha) \subset D(A^\alpha)$ with estimate

$$\|A^\alpha u\| \leq C_\alpha\, \|(1 + \mathcal{L})^\alpha u\|, \tag{3}$$

for all $u \in D(\mathcal{L}^\alpha)$ and $0 < \alpha \leq 1$. Indeed, (3) for the case $\alpha = 1$ is equivalent to (2), and the Heinz-Kato inequality for m-accretive operators (Tanabe [12, Chapter 2]) implies (3) for $0 < \alpha < 1$.

Our method to solve (NS.3) is to make use of the semigroup $e^{-t\mathcal{L}}$ together with the fractional powers of A and $\mathcal{L}$. Although this approach itself is, in principle, standard (see Fujita and Kato [4], Giga and Miyakawa [6]), the semigroup $e^{-t\mathcal{L}}$ is not a usual one. The essential difficulty is the growth at space infinity of the coefficient $\omega \times x$ of the operator $\mathcal{L}$, so that the convection term $(\omega \times x) \cdot \nabla$ is not a perturbation of the Stokes operator A. In fact, the associated semigroup for the Cauchy problem in $\mathbb{R}^3$ is explicitly given by

$$(4) \qquad [U(t)f](x) = O(t)^T \left[e^{t\Delta} f\right](O(t)x), \qquad x \in \mathbb{R}^3,\ t > 0,$$

where

$$\left[e^{t\Delta} f\right](x) = (4\pi t)^{-3/2} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4t}} f(y)dy,$$

and it is proved that the semigroup $U(t)$ is never analytic on $L^2_\sigma(\mathbb{R}^3)$ (see [9]). This is a different feature caused by the convection term $(\omega \times x) \cdot \nabla$. Thus, we cannot expect that $e^{-t\mathcal{L}}$ is analytic. However, it has the remarkable smoothing effect. The following theorem asserts that $e^{-t\mathcal{L}} f$ is in $D(A)$ for all $t > 0$ whenever f is slightly smooth, and that $e^{-t\mathcal{L}} f$ is in $D(\mathcal{L})$ for all $t > 0$ under the additional assumption $(\omega \times x) \cdot \nabla f \in H^{-\infty}(\Omega) \equiv \bigcup_{s \geq 0} H^{-s}(\Omega)$.

Theorem 1. *(i) Suppose that $f \in D(A^\delta)$ for some $0 < \delta \leq 1/2$. Then $e^{-t\mathcal{L}} f \in D(A)$ for all $t > 0$. Furthermore, there is a constant $C = C(\delta) > 0$ such that*

$$(5) \qquad \left\| A e^{-t\mathcal{L}} f \right\| \leq C\, t^{-1+\delta} \|f\|_{D(A^\delta)},$$

for all $0 < t \leq 1$.

(ii) Suppose that $f \in D(A^\delta)$ for some $0 < \delta < 1$, and that $(\omega \times x) \cdot \nabla f \in H^{-s}(\Omega)$ for some $s \geq 0$. Then $e^{-t\mathcal{L}} f \in D(\mathcal{L})$ for all $t > 0$ and

$$\mathcal{L} e^{-t\mathcal{L}} f \in C\left(0, \infty;\ L^2_\sigma(\Omega)\right), \qquad e^{-t\mathcal{L}} f \in C^1\left(0, \infty;\ L^2_\sigma(\Omega)\right),$$

with

$$\frac{d}{dt}e^{-t\mathcal{L}}f + \mathcal{L}e^{-t\mathcal{L}}f = 0, \qquad t > 0,$$

in $L^2_\sigma(\Omega)$. *Furthermore, there are constants* $C = C(\delta) > 0$ *and* $C' = C'(s) > 0$ *such that*

$$\begin{aligned} \left\|\mathcal{L}e^{-t\mathcal{L}}f\right\| \le\, & C\,(t \wedge 1)^{-1+\delta}\|f\|_{D(A^\delta)} \\ & + C'\,(t\wedge 1)^{-s/2}\left\{\|(\omega \times x)\cdot\nabla f\|_{H^{-s}(\Omega)} + \|f\|\right\}, \end{aligned} \tag{6}$$

for all $t > 0$, *where* $t \wedge 1 = \min\{t, 1\}$.

(iii) Let $0 < \delta < 1/2$. *Then*

$$\lim_{t\to 0} t^{1-\delta}\left\|Ae^{-t\mathcal{L}}f\right\| = 0,$$

for all $f \in D(A^\delta)$. *For the same* δ *as above, let* $0 \le s < 2(1-\delta)$. *Then*

$$\lim_{t\to 0} t^{1-\delta}\left\|\mathcal{L}e^{-t\mathcal{L}}f\right\| = 0,$$

for all $f \in D(A^\delta)$ *satisfying* $(\omega \times x)\cdot\nabla f \in H^{-s}(\Omega)$.

In Theorem 1 the case $\delta = 0$ (namely, $f \in L^2_\sigma(\Omega)$) is excluded on account of a technical difficulty caused by the solenoidal constraint. Indeed, in [7, Theorem 4] sharper results including $\delta = 0$ have been established for the realization of a model operator $\Delta + (\omega \times x)\cdot\nabla$ with the homogeneous Dirichlet boundary condition in $L^2(\Omega)$. But estimates (5) and (6) near $t = 0$ together with the fractional powers of A and $\mathcal{L}$ are very useful for the proof of local existence of a unique solution to (NS.3). The strategy for the proof of Theorem 1 is as follows. We first derive the similar smoothing effect to Theorem 1 for the semigroup $U(t)$ given by (4). We next employ the method based on a refinement of the cut-off procedure developed in the proof of Theorem 4 of [7] combined with the result of Bogovskiĭ [1] mentioned above. For the details, see [9].

We now fix $\zeta \in C^\infty(\mathbb{R}^3)$ such that $0 \le \zeta \le 1, \zeta = 1$ near Γ and $\zeta = 0$ for large $|x|$, and put

(7) $$b(x) = -\frac{1}{2}\,\nabla \times \{\zeta(x)|x|^2\omega\}.$$

Then $\nabla \cdot b = 0$ in Ω, $b = \omega \times x$ on Γ and $b = 0$ for large $|x|$. We set

$$\widetilde{u}(x,t) = u(x,t) - b(x),$$

in (NS.3) and apply the projection P to the equation of motion to obtain the integral equation

(NS.4) $$\widetilde{u}(t) = e^{-t\mathcal{L}}[a-b] - \int_0^t e^{-(t-s)\mathcal{L}} P\,[\widetilde{u}\cdot\nabla\widetilde{u} + B\widetilde{u}]\,(s)\,ds, \qquad t \ge 0,$$

in $L^2_\sigma(\Omega)$, where

$$B\widetilde{u} = \widetilde{u}\cdot\nabla b + b\cdot\nabla\widetilde{u} + F[b],$$
$$F[b] = \Delta b + (\omega \times x)\cdot\nabla b - \omega\times b - b\cdot\nabla b.$$

The main theorem then reads as follows.

Theorem 2. *Suppose that $a - b \in D(\mathcal{L}^\gamma)$ for some $1/4 < \gamma < 1/2$ and that $(\omega \times x)\cdot\nabla a \in H^{-s}(\Omega)$ for some $1 \le s < 2(1-\gamma)$. Then there exist $T > 0$ and a unique solution $\widetilde{u}$ to (NS.4) on the interval $[0,T]$, which is of class*

$$\widetilde{u} \in C\left([0,T];\ L^2_\sigma(\Omega)\right),$$

and possesses the regularity $\widetilde{u}(t) \in D(A), 0 < t \le T$, with the properties:

(8) $$\lim_{t\to 0}\|\widetilde{u}(t) - (a-b)\|_{D(A^\gamma)} = \lim_{t\to 0}\|u(t) - a\|_{D(A^\gamma)} = 0,$$

$$\lim_{t\to 0} t^{\alpha-\gamma}\,\|\widetilde{u}(t)\|_{D(A^\alpha)} = 0, \qquad \gamma < \alpha \le 1, \tag{9}$$

$$\|\widetilde{u}(t)\|_{D(A^\alpha)} \le C_\alpha K_0\, t^{-\alpha+\gamma}, \qquad 0 < t \le T,\ \gamma \le \alpha \le 1, \tag{10}$$

where

$$K_0 = \|a-b\|_{D(\mathcal{L}^\gamma)} + \|(\omega \times x)\cdot \nabla a\|_{H^{-s}(\Omega)} + \||x|b\| + \|F[b]\|_{H^1(\Omega)}.$$

The proof is given in [9]. We conclude this article with some comments on Theorem 2.

Remark. (i) In view of (7), the assumption $a-b \in D(\mathcal{L}^\gamma) \subset D(A^\gamma)$ (see (3)) with $\gamma > 1/4$ implies that $a = \omega \times x$ on Γ (cf. Fujiwara [5]).

(ii) The critical case $\gamma = 1/4$ is the well known exponent of Fujita and Kato [4]. If Theorem 1 for $\delta = 0$ were deduced, then we could show Theorem 2 for the case $\gamma = 1/4$.

(iii) Under the assumption $(\omega \times x)\cdot\nabla a \in H^{-2(1-\gamma)}(\Omega)$, it is also possible to construct a unique solution. But the behavior (9) of such a solution is not clear.

(iv) The solution obtained in Theorem 2 is the so-called mild solution. Since we find the solution $\widetilde{u}(t)$ with values in $D(A)$ and it does not belong to $D(\mathcal{L})$ in general, it seems to be difficult to derive the differentiability of $\widetilde{u}$ with respect to time t.

(v) Theorem 2 holds true with $\omega = (0,0,1)^T$ replaced by $\omega = (0,0,\omega_0)^T$ for every $\omega_0 \in \mathbb{R}$. The existence interval $T = T(|\omega_0|) > 0$ is then monotonically decreasing with respect to $|\omega_0|$.

(vi) When the obstacle $\mathcal{O}$ is not rotating, that is $\omega = 0$, the problem (NS.3) possesses a unique local strong solution for $a \in L^3_\sigma(\Omega) \supset D(A^{1/4})$, where $L^3_\sigma(\Omega)$ denotes the completion of $C^\infty_{0,\sigma}(\Omega)$ in $L^3(\Omega)$. If $\|a\|_{L^3(\Omega)}$ is sufficiently small, then the solution is extended globally in time. This is the result of Iwashita [10].

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