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C^1 APPROXIMATIONS OF INERTIAL MANIFOLDS VIA FINITE DIFFERENCES AND APPLICATIONS

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1. INTRODUCTION

We shall present a method for the construction of approximate inertial manifolds by means of finite differences. The theory of inertial manifolds (IM for short) is a useful tool for reducing the long-time behavior of PDEs to that of finite-dimensional dynamical systems. (See [1-7] and [13]). To compute the reduced finite dynamical system, one would need to know the explicit form of the IM. Howevwer, even when existence of an IM can be established, the theory does not provide us with an explicit form of IMs. In this paper, from the point of finite differences we construct such an approximate IM that reflects the true dynamics of the original PDE.

Each of the PDEs can be viewed as an evolution equation in a Hilbert space. To be more specific, let X and Y be Hilbert spaces with norms $\|\cdot\|$ and $|\cdot|$, respectively, such that X is continuously embeded in Y. Let $\{S(t); t \geq 0\}$ be a C_0- semigroup on Y and $F \in \mathrm{Lip}(X, Y) \cap C^1(X, Y)$, the set of Lipschitz and continuously differentiable mappings from X into Y. The evolution equations take the form

$$du(t)/dt = Au(t) + Fu(t), \qquad t \geq 0 \tag{1.1}$$

$$u(0) = x_0 \tag{1.2}$$

where $x_0 \in X$ and A is the infinitesimal generator of $\{S(t); t \geq 0\}$ satisfying $|S(t)y| \leq Me^{\omega t}|y|$ for $t \geq 0$ and $y \in Y$.

We assume the following conditions:

(S1) $S(t)Y \subset X$ for $t > 0$ and $S(t)x \in C([0, \infty); X)$ for $x \in X$.
(S2) $Y = Y_1 \oplus Y_2$ and $P_i S(t) = S(t) P_i$ for $i = 1, 2$ and $t \geq 0$, where Y_i is a closed linear subspace and P_i is a projection from X onto Y_i.
(S3) $\{S(t)P_1; t \geq 0\}$ forms a uniformly continuous semigroup on Y_1.
(S4) There exist constants $\alpha, \beta > 0, \gamma \in [0, 1), \eta < -\max\{\alpha, \beta\}$ and $M_1, M_2, M_3, M_4, M_5 \geq 0$ such that

$$\|y\| \leq M_1 |y|, \qquad y \in Y_1, \tag{1.3}$$

$$|e^{-\eta t} S(t) P_1| \leq M_2 e^{\alpha t} |y|, \qquad t \leq 0, y \in Y, \tag{1.4}$$

$$\|e^{-\eta t} S(t) P_2 x\| \leq M_3 e^{-\beta t} \|x\|, \qquad t \geq 0, x \in X, \tag{1.5}$$

$$\|e^{-\eta t} S(t) P_2 y\| \leq (M_4 t^{-\gamma} + M_5) e^{-\beta t} |y|, \qquad t > 0, y \in Y. \tag{1.6}$$

The above assumptions ensure the unique mild solution $u(t;x_0) \in C([0,\infty);X)$ of (1.1) and (1.2) for each $u_0 \in X$ (e.g., see [14]). It is known ([1],[5]) that we obtain the existence of IMs for (1.1) under the above conditions.

Theorem 1.1. *Let (S1)-(S4) be satisfied. In addition we assume*

$$K(\alpha,\beta)Lip(F) < 1 \quad \textit{and} \quad \frac{M_2M_3K(\alpha,\beta)Lip(F)}{1-K(\alpha,\beta)Lip(F)} < 1, \tag{1.7}$$

where

$$K(\alpha,\beta) = M\{M_1M_2\alpha^{-1} + M_4\Gamma(1-\gamma)\beta^{\gamma-1} + M_5\beta^{-1}\} \tag{1.8}$$

$Lip(F)$ the Lipschitz constant of $F: X \to Y$, Γ the gamma function. Then there exists $h \in C^1(Y_1, P_2X)$ whose graph $\mathcal{M} = \{y + h(y) : y \in Y_1\}$ is an IM for (1.1), that is,

(a) If $x_0 \in \mathcal{M}$, then $u(t;x_0)$, the mild solution of (1.1) and (1.2), belongs to $\mathcal{M}$ for all $t > 0$.

(b) For each $x_0 \in X$ there exists a unique element $x_0^* \in \mathcal{M}$ such that

$$\sup_{t\geq 0} e^{-\eta t}\|u(t;x_0) - u(t;x_0^*)\| < \infty.$$

Since the solution on $\mathcal{M}$ must be of the form $u(t) = p(t) + h(p(t))$ with $p(t) = P_1u(t)$, the restriction of (1.1) to $\mathcal{M}$ yields

$$dp/dt = Ap + P_1F(p + h(p)), \qquad p \in Y_1, \tag{1.9}$$

whose long-time behavior is equivalent to that of (1.1) because by virtue of (b) the IM $\mathcal{M}$ attracts every orbit at an exponential rate. (1.9) is called an inertial form for (1.1).

2. Approximations of IMs

We approximate (1.1) by the following finite difference scheme of the form

$$x_\ell^n = C(\lambda_\ell)x_\ell^{n-1} + \lambda_\ell K_\ell F_\ell(x_\ell^{n-1}), \quad n, \ell \in \mathbb{N} \tag{2.1}$$

in a space Y_ℓ approximating Y in some sense, where $\lambda_\ell \downarrow 0$ as $\ell \to \infty$, $C(\lambda_\ell)$ and K_ℓ are given operators in $B(Y_\ell, Y_\ell)$ and F_ℓ is a given nonlinear operator in Y_ℓ stated below. We denote by $B(W,Z)$ the space of bounded linear operators from a Banach space W into a Banach space Z. The norm in $B(W,Z)$ will be denoted by $\|\cdot\|_{W,Z}$. We make the following assumptions.

(C1) Let X and Y are reflexive Banach spaces such that X is densely and continuously embedded in Y and that $Y = Y_1 \oplus Y_2$, the direct sum of a finite dimensional subspace Y_1 and a closed subspace Y_2.

(C2) For each $\ell \in \mathbb{N}$ let X_ℓ and Y_ℓ be Banach spaces with norms $\|\cdot\|_\ell$ and $|\cdot|_\ell$, respectively, such that X_ℓ is continuously embedded in Y_ℓ. Moreover, there exist $V_\ell \in B(Y,Y_\ell) \cap B(X,X_\ell)$ and $W_\ell \in B(Y_\ell,Y) \cap B(X_\ell,X)$ such that $\lim_{\ell\to\infty}|V_\ell y|_\ell =$

$|y|$, $\lim_{\ell\to\infty}\|V_\ell x\|_\ell = \|x\|$, $\lim_{\ell\to\infty}|W_\ell V_\ell y - y| = 0$ and $V_\ell W_\ell y = y$ for $x \in X, y \in Y$ and that both $\|W_\ell\|_{Y_\ell,Y}$ and $\|W_\ell\|_{X_\ell,X}$ are bounded in ℓ.

(C3) There exist closed subspaces $Y_{\ell 1}$ and $Y_{\ell 2}$ such that $Y_\ell = Y_{\ell 1} \oplus Y_{\ell 2}$, $V_\ell P_i = P_{\ell i} V_\ell$ and $W_\ell P_{\ell i} = P_i W_\ell$ for $i = 1, 2$, where P_i (resp. $P_{\ell i}$) denotes a projection from Y onto Y_i (resp. Y_ℓ onto $Y_{\ell i}$).

(C4) The linear operators $C(\lambda_\ell)$ and K_ℓ satisfy: (i) there exist $M \geq 0$ and $\omega \geq 0$ such that $|C(\lambda_\ell)^n y|_\ell \leq M e^{\omega n \lambda_\ell}|y|_\ell$ and $|K_\ell y|_\ell \leq M e^{\omega\lambda_\ell}|y|_\ell$ for $\ell, n \in \mathbb{N}, y \in Y_\ell$; (ii) $\lim_{\ell\to\infty}|(K_\ell - I)V_\ell y|_\ell = 0$ for $y \in Y$; (iii) for each $\ell, \ell' \in \mathbb{N}$ and $i = 1, 2$, $C(\lambda_\ell)$ commutes with $P_{\ell i}$, $C(\lambda_\ell)$ with K_ℓ, K_ℓ with $P_{\ell i}$, $\tilde{C}(\lambda_\ell)$ with $\tilde{C}(\lambda_{\ell'})$ and $\tilde{C}(\lambda_\ell)$ with $\tilde{K}_{\ell'}$, respectively, where $\tilde{C}(\lambda) = W_\ell C(\lambda) V_\ell$ and $\tilde{K}_\ell = W_\ell K_\ell V_\ell$.

(C5) A is a densely defined linear operator in Y such that $Y_1 \subset D(A)$, the range of $I - \lambda_0 A$ is dense in Y for some $\lambda_0 > 0$ and

$$\lim_{\ell\to\infty} |\lambda_\ell^{-1}(C(\lambda_\ell) - I)V_\ell y - V_\ell A y|_\ell = 0 \quad \text{for } y \in D(A).$$

(C6) The inverse of $C(\lambda_\ell)P_{\ell 1}$ exists in $B(Y_{\ell 1})$ and there exist constants $\alpha, \beta > 0$, $\gamma \in [0, 1), \eta < -\max\{\alpha, \beta\}$ and $M_1, \cdots, M_5 \geq 0$ such that

$$\|P_{\ell 1} y\|_\ell \leq M_1 |P_{\ell 1} y|_\ell \tag{2.2}$$

$$|[C(\lambda_\ell)P_{\ell 1}]^{-n} P_{\ell 1} y|_\ell \leq M_2 e^{-(\alpha+\eta)n\lambda_\ell}|y|_\ell \tag{2.3}$$

$$\|C(\lambda_\ell)^n P_{\ell 2} x\|_\ell \leq M_3 e^{(\eta-\beta)n\lambda_\ell}\|x\|_\ell \tag{2.4}$$

$$\|C(\lambda_\ell)^n P_{\ell 2} K_\ell y\|_\ell \leq \{M_4((n+1)\lambda_\ell)^{-\gamma} + M_5\} e^{(\eta-\beta)n\lambda_\ell}|y|_\ell \tag{2.5}$$

$$\text{for} \quad n \geq 0, \ell \geq 1, x \in X_\ell, y \in Y_\ell.$$

(C7) $F_\ell \in C^1(X_\ell, Y_\ell)$ and there exists a constant $L_F \geq 0$ satisfying

$$|F_\ell(\xi_1) - F_\ell(\xi_2)|_\ell \leq L_F \|\xi_1 - \xi_2\|_\ell \quad \text{for } \ell \in \mathbb{N},\ \xi_1, \xi_2 \in X_\ell.$$

(C8) For each $x, z \in X$ and each positive sequence $\{\nu_\ell\}$ convergent to 0 we have

$$\lim_{\ell\to\infty} |F_\ell(V_\ell x) - V_\ell F(x)|_\ell = 0,$$
$$\lim_{\ell\to\infty} |DF_\ell(V_\ell x)V_\ell z - V_\ell DF(x) z|_\ell = 0, \quad \text{and}$$
$$\lim_{\ell\to\infty} \Big(\sup_{\|\xi\|_\ell \leq \nu_\ell} |(DF_\ell(V_\ell x + \xi) - DF_\ell(V_\ell x))V_\ell z|_\ell \Big) = 0.$$

To construct an IM for (2.1) we introduce the Banach space c_η^- of sequences $\tilde{x} = \{x_n\}_{n\leq 0}$ in X_ℓ with the norm $\|\tilde{x}\|_\ell^{(\eta)} = \sup_{n\leq 0} e^{-\eta n\lambda_\ell}\|x_n\|_\ell$. Let B_ℓ be a bounded subset of $Y_{\ell 1}$. We denote by $BC(B_\ell, c_\ell^-)$ the Banach space consisting of bounded and continuous functions $\psi : B_\ell \to c_\eta^-$ with the norm $\|\psi\|_{B_\ell}^{(\eta)} = \sup_{\xi\in B_\ell} \|\psi(\xi)\|_\ell^{(\eta)}$. We shall write $\psi \in BC(B_\ell, c_\eta^-)$ as

$$\psi(\xi) = \{\psi(\xi, n)\}_{n\leq 0} \in c_\eta^- \qquad \text{for} \qquad \xi \in B_\ell.$$

Then we define the mapping H_ℓ from $BC(B_\ell, c_\eta^-)$ into itself by

$$(2.6)\qquad (H_\ell\psi)(\xi,n) = R_\ell^n\xi - \lambda_\ell \sum_{i=-1}^{n} R_\ell^{n-i-1}P_{\ell 1}K_\ell F_\ell(\psi(\xi,i)) + \lambda_\ell \sum_{i=n+1}^{\infty} Q_\ell^{i-n-1}P_{\ell 2}K_\ell F_\ell(\psi(\xi,i))$$

for $\xi \in B_\ell$ and $n \le 0$. Here $R_\ell = C(\lambda_\ell)P_{\ell 1}$ and $Q_\ell = C(\lambda_\ell)P_{\ell 2}$. Furthermore we define

$$(2.7)\qquad h_{\ell k}(\xi) = ((H_\ell)^k\psi_0)(\xi,0) - \xi$$

with

$$\psi_0(\xi,n) = \xi \qquad \text{for} \qquad n \le 0.$$

Then we have ([10])

Theorem 2.1. *Let (C1)-(C7) be satisfied. In addition we assume*

$$(2.8)\qquad K(\alpha,\beta)L_F < 1 \quad \textit{and} \quad \frac{M_2M_3'K(\alpha,\beta)L_F}{1-K(\alpha,\beta)L_F} < 1$$

where

$$(2.9)\qquad k(\alpha,\beta) = M\{M_1M_2\alpha^{-1} + M_4'\Gamma(1-\gamma)\beta^{\gamma-1} + M_5'\beta^{-1}\},$$

and

$$M_i' = M_i \max\{1, \varliminf_{\ell\to\infty} \|W_\ell\|_{X_\ell,X}\}, \qquad i = 3,4,5$$

Then, for every $\ell \in \mathbb{N}$ there exists $h_\ell \in C^1(Y_{\ell 1}, c_\ell^-)$ whose graph $\mathcal{M}_\ell = \{\xi + h_\ell(\xi); \xi \in Y_{\ell 1}\}$ is an IM for (2.1). Moreover, we have for each bounded set $B_\ell \subset Y_{\ell 1}$

$$(2.10)\qquad \lim_{k\to\infty} \sup_{\xi\in B_\ell} \|h_{\ell k}(\xi) - h_\ell(\xi)\|_\ell = 0$$

and

$$(2.11)\qquad \lim_{k\to\infty} \sup_{\xi\in B_\ell} \|Dh_{\ell k}(\xi) - Dh_\ell(\xi)\|_{B(Y_{\ell 1},X_{\ell 1})} = 0.$$

From this theorem the inertial form for (2.1) is described by the system of equations

$$(2.12)\qquad \begin{aligned} p_\ell^{n+1} &= C(\lambda_\ell)p_\ell^n + \lambda_\ell K_\ell P_{\ell 1}F_\ell(p_\ell^n + h_\ell(p_\ell^n)) \\ p_\ell^n &\in Y_{\ell 1}, \qquad n,\ell \in \mathbb{N} \end{aligned}$$

Furthermore, as an approximate inertial form for (2.1) we may employ the following system of equations with some k

$$(2.13)\qquad \begin{aligned} p_\ell^{n+1} &= C(\lambda_\ell)p_\ell^n + \lambda_\ell K_\ell P_{\ell 1}F_\ell(p_\ell^n + h_{\ell k}(p_\ell^n)) \\ p_\ell^n &\in Y_{\ell 1}, \qquad n,\ell \in \mathbb{N} \end{aligned}$$

We emphasize that (2.13) can be solved for p_ℓ^n explicitly.

Now we have our main result which is proved in [12].

Theorem 2.2. *Let (C1)-(C8) and (2.7) are satisfied. Then, conditions (S1)-(S4) and (1.7) hold true with the semigroup generated by the operator A in (C5). Consequently, there exists $h \in C^1(Y_1, P_2X)$ whose graph is an IM for (1.1). Moreover we have for each bounded set $B \subset Y_1$*

$$\lim_{\ell\to\infty} \sup_{y\in B} \|h_\ell(V_\ell y) - V_\ell h(y)\|_\ell = 0 \tag{2.14}$$

and

$$\lim_{\ell\to\infty} \sup_{y\in B} \|Dh_\ell(V_\ell y) - V_\ell Dh(y)\|_{B(Y_1,X_\ell)} = 0. \tag{2.15}$$

From this theorem we can employ (2.13) as an explicit C^1- approximation of the inertial form (1.9). The C^1 closeness would be a necessary and important step toward establishing a relationship between the dynamics of the PDE and its approximation.

3. Kuramoto-Sivashinsky equations

We consider the renormalized Kuramoto-Sivashinsky equation with periodic boundary condition, with period L

$$\begin{cases} u_t + D^4u + D^2u + uDu = 0 & (x,t) \in \mathbf{R} \times \mathbf{R}^+, \\ u(x,t) = u(x+L,t) & (x,t) \in \mathbf{R} \times \mathbf{R}^+, \\ u(x,0) = u_0(x) & x \in \mathbf{R}. \end{cases} \tag{3.1}$$

Here D denotes $\partial/\partial x$ or d/dx. Let $H^m_{per}(0,L)$ denote the subspace of the Sobolev space $H^m(0,L)$ consisting of functions which, along with all their derivatives up to order $m-1$, are periodic with period L. A function u defined a.e. on $(0,L)$ is said to be odd whenever $u(x) = -u(L-x)$ a.e. in $(0,L)$. Following Foias et al. [4] and Foias and Titi [6] we set

$$\begin{aligned} Y &= \{u \in L^2_{per}(0,L); u \quad \text{is odd}\} \\ < u,v > &= \int_0^L u(x)v(x)dx \qquad \text{for} \quad u,v \in Y \\ |u| &= \sqrt{< u,u >} \qquad \text{for} \quad u \in Y \\ X &= \{u \in H^2_{per}(0,L); u \quad \text{is odd}\} \\ \|u\| &= |D^2u| \qquad \text{for} \quad u \in X \\ Au &= -D^4u \qquad \text{for} \quad u \in D(A) \equiv H^4_{per}(0,L) \cap Y \end{aligned}$$

and

$$Ru = -D^2u - uDu \qquad \text{for} \quad u \in X.$$

Then (3.1) is written as the following evolution equation in the Hilbert space Y

$$\begin{cases} du(t)/dt = Au(t) + Ru(t), & t \geq 0 \\ u(0) = u_0 \end{cases} \tag{3.2}$$

It is known (see [4]) that for every $u_0 \in Y$ there exists a unique solution $u(t)$ of (3.2). Moreover, for every $r > 0$ there exists a time $T^*(r) > 0$ such that $\|u(t)\| \leq r_0$ for all $t \geq T^*(r)$ and $u_0 \in Y$ with $|u_0| \leq r$, , where r_0 is a constant which is independent of r. Hence, the study of asymptotic behavior of solutions to (3.2) can be reduced to the study of the prepared equation

$$du/dt = Au + Fu, \qquad t \geq 0 \tag{3.3}$$

where

$$\begin{cases} Fu = -D^2u - \rho(\|u\|)uDu, \\ \rho \in C_0^\infty(\mathbf{R}), 0 \leq \rho \leq 1, \\ \rho(s) = 1 \quad \text{for} \quad |s| \leq r_0, \quad \rho(s) = 0 \quad \text{for} \quad |s| \geq 2r_0 \end{cases}$$

The operator $-A$ is a positive selfadjoint operator in Y and the functions

$$e_k(x) = \sin(2\pi kx/L)$$

are eigenfunctions of the operator A with corresponding eigenvalues $\nu_k = (2\pi k/L)^4$ for $k = 1, 2, \cdots$. $\{\sqrt{2/L}e_k\}_{k=1}^\infty$ forms an orthonomal basis for Y. We can easily see that the conditions (S1)-(S4) and (1.7) in Section 1 are satisfied with $Y_1 = \text{span}\{e_1, e_2, \cdots, e_N\}$, $Y_2 = \text{span}\{e_{N+1}, e_{N+2}, \cdots\}, \alpha' = \beta' = (\nu_{N+1} - \nu_N)/2, \eta' = -(\nu_{N+1} + \nu_N)/2, \gamma' = 1/2, \tilde{M}' = M_2' = M_3' = 1, M_1' = \sqrt{\nu_N}$ and $M_5' = \sqrt{\nu_{N+1}}$ if N is sufficiently large. Therefore, (3.3) has an inertial manifold.

We shall approximate (3.1) by finite difference schemes. Following Foias and Titi [6], we introduce the set $S_{odd,per}^\ell$ consisting of ℓ-dimensional vectors $\xi = (\xi_0, \cdots, \xi_{\ell-1})$ which satisfy

$$\xi_j = -\xi_{\ell-j} \quad \text{for} \quad j = 1, 2, \cdots, \ell - 1, \quad \xi_0 = 0$$

and are extended periodically to a double infinite sequemce such that

$$\xi_{j+\ell} = \xi_j, \qquad j = 0, \pm 1, \pm 2, \cdots.$$

For $\ell \geq 1$ we set

$$Y_\ell = X_\ell = S_{odd,per}^\ell, \qquad <\xi, \zeta>_\ell = \frac{L}{\ell}\sum_{k=0}^{\ell-1}\xi_k\zeta_k,$$

$$|\xi|_\ell = \sqrt{<\xi, \xi>_\ell} \quad \text{for} \quad \xi, \zeta \in Y_\ell; \qquad \text{and}$$

$$\|\xi\|_\ell = |\Delta_\ell \xi|_\ell \quad \text{for} \quad \xi \in X_\ell,$$

where

$$\Delta_\ell = -\frac{\ell^2}{L^2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$$

Define $\theta_\ell : C([0,L]) \to \mathbf{R}^\ell$ by

$$\theta_\ell(u) = (u(x_0), u(x_1), \cdots, u(x_{\ell-1})),$$

where $x_j = jh$ for $j = 0, 1, \cdots, \ell - 1$, and $h = L/\ell$.

Lemma 3.1. *Let $\ell_0 = [(\ell-1)/2]$, the integer part of $(\ell-1)/2$. Y_ℓ is an ℓ_0-dimensional Banach space with the norm $|\cdot|_\ell$. $\{\theta_\ell(e_1), \theta_\ell(e_2), \cdots, \theta_\ell(e_{\ell_0})\}$ forms an orthogonal basis for Y_ℓ with $|\theta_\ell(e_j)|_\ell = \sqrt{L/2}$.*

Lemma 3.2. *$\theta_\ell(e_k)$ are eigenvectors of $\Delta_\ell : Y_\ell \to Y_\ell$ with corresponding eigenvalue $-(2/h)^2 \sin^2(\pi k/\ell)$ for $1 \le k \le \ell_0$.*

In what follows we set

$$\mu_k^\ell = (2/h)^4 \sin^4(\pi k/\ell), \quad k = 1, 2, \cdots, \ell_0.$$

Notice that $(2/\pi)^4 \nu_k \le \mu_k^\ell \le \nu_k$ for $1 \le k \le \ell_0$.

Define linear operators $V_\ell : Y \to Y_\ell$ and $W_\ell : Y_\ell \to Y$ as follows.

$$V_\ell u = \theta_\ell(u_\ell) \qquad \text{for} \quad u \in Y,$$

where $u_\ell = \sum_{i=1}^{\ell_0} \alpha_i e_i$ with $\alpha_i = 2L^{-1} < u, e_i >$. Next, thanks to Lemma 3.1, every $\xi \in Y_\ell$ can be written uniquely as

$$\xi = \alpha_1 \theta_\ell(e_1) + \cdots + \alpha_{\ell_0} \theta_\ell(e_{\ell_0}).$$

We then set

$$W_\ell \xi = \alpha_1 e_1 + \cdots + \alpha_{\ell_0} e_{\ell_0}.$$

Finally, we set

$$\begin{aligned} Y_{\ell 1} &= \mathrm{span}\{\theta_\ell(e_1), \cdots, \theta_\ell(e_N)\}, \quad \text{and} \\ Y_{\ell 2} &= \mathrm{span}\{\theta_\ell(e_{N+1}), \cdots, \theta_\ell(e_{\ell_0})\} \quad \text{for} \quad N < \ell_0. \end{aligned}$$

It is easy to see that conditions (C1)-(C3) in Section 2 hold true in this case.

We here consider the following semi-implicit discrete scheme for (3.1):

$$\frac{\xi^{i+1} - \xi^i}{\lambda_\ell} + \Delta_\ell^2((1-\theta)\xi^i + \theta\xi^{i+1}) + F_\ell(\xi^i) = 0, \quad \xi^i \in Y_\ell \tag{3.4}$$

where $\lambda_\ell \to +0$ as $\ell \to \infty$, $2^{-1} < \theta \leq 1$, $F_\ell(\xi) = -\rho(\|\xi\|_\ell^2)(\Delta_\ell \xi + B^\ell(\xi,\xi))$ and $B^\ell : Y_\ell \times Y_\ell \to Y_\ell$ is defined as follows: For every $\xi, \hat{\xi} \in Y_\ell$ the k-th element $B_k^\ell(\xi,\hat{\xi})$ of $B^\ell(\xi,\hat{\xi})$ is given by

$$B_k^\ell(\xi,\hat{\xi}) = \frac{1}{6h}\{\xi_k(\hat{\xi}_{k+1} - \hat{\xi}_{k-1}) + \xi_{k+1}\hat{\xi}_{k+1} - \xi_{k-1}\hat{\xi}_{k-1}\}.$$

To apply the preceding results put

$$C(\lambda_\ell) = (I - (1-\theta)\lambda_\ell \Delta_\ell^2)(I + \theta\lambda_\ell\Delta_\ell^2)^{-1}$$

and

$$K_\ell = (I + \theta\lambda_\ell\Delta_\ell^2)^{-1}.$$

Then (3.4) can be rewritten as (2.1). We have already shown in [2] that conditions (C4)-(C6) hold with $M = M_2 = M_3 = 1$, $\omega = 0$, $M_1 = \sqrt{\mu_N}$, $M_4 = 2$, $M_5 = \sqrt{2\mu_{N+1}}$, $\alpha = \beta = (\nu_{N+1} - \nu_N)/4$, $\eta = (\nu_{N+1} + \nu_N)/2$ and $\gamma = 1/2$.

Finally, to see (C7)and (C8) it suffices to note that

$$\begin{aligned} DF(u)v =& - D^2v - 2\rho'(\|u\|^2) < D^2u, D^2v > uDu \\ & - \rho(\|u\|^2)(uDv + vDu) \qquad \text{for} \quad u, v \in X \end{aligned}$$

and

$$\begin{aligned} DF_\ell(\xi)\eta =& - \Delta_\ell\eta - 2\rho'(\|\xi\|_\ell^2) < \Delta_\ell\xi, \Delta_\ell\eta >_\ell B^\ell(\xi,\xi) \\ & - \rho(\|\xi\|_\ell^2) DB^\ell(\xi,\xi)\eta \end{aligned}$$

for $\xi = (\xi_0, \cdots, \xi_{\ell-1})$, $\eta = (\eta_0, \cdots, \eta_{\ell-1}) \in Y_\ell$, where the k-th element of $DB^\ell(\xi,\xi)\eta$ is defined by

$$\begin{aligned} \{DB^\ell(\xi,\xi)\eta\}_k =& (6h)^{-1}(\xi_{k+1} + \xi_k + \xi_{k-1})(\eta_{k+1} - \eta_{k-1}) \\ & + (6h)^{-1}(\xi_{k+1} - \xi_{k-1})(\eta_{k+1} + \eta_k + \eta_{k-1}). \end{aligned}$$

As a result, one can apply Theorem 2.2 to the Kuramoto-Sivashinsky equation (3.1).

REFERENCES

1. S.N.Chow and K.Lu, *Invariant manifolds for flows in Banach spaces*, J.Differential Equations **74** (1988), 285–317.
2. S.N.Chow, K.Lu and G.R.Sell, *Smoothness of inertial manifolds*, J. Mathe. Anal. Appl. **169** (1992), 238–321.
3. F.Demengel and J.M.Ghidaglia, *Inertial manifolds for partial differential evolution equations under time-discretization: Existence, cnvergence and applications*, J. Math. Anal. Appl. **155** (1991), 177–225.
4. C.Foias, M.S.Jolly, I.G.Kevrekidis and E.S.Titi, *Dissipativity of numerical schemes*, Nonlinearity **4** (1991), 591–613.
5. C.Foias, G.R.Sell and R.Temam, *Inertial manifolds for nonlinear evolutionary equations*, J.Differential Equations **73** (1988), 309–353.

6. C.Foias and E.S.Titi, *Determining nodes, finite difference schemes and inertial manifolds*, Nonlinearity **4** (1991), 135–153.
7. M.S.Hirsch, C.C.Pugh and M.Shub, *Invariant Manifolds*, Lecture Notes in Mathematics Vol. 583, Springer- Verlag, Now York, 1977.
8. D.A.Jones and E.S.TiTi, C^1 *approximation of inertial manifolds for dissipative nonlinear equations*, J. Differential Equations **127** (1996), 54–86.
9. K.Kobayasi, *Convergence and approximation of inertial manifolds for evolution equations*, Differential and Integral Equations **8** (1995), 1117–1134.
10. K.Kobayasi, *Inertial manifolds for discrete approximations of evolution equations: Convergence and applications*, Advances in Math. Sci. and Appl. **3** (1993/1994), 161–189.
11. K.Kobayasi, C^1 *approximations of inertial manifolds for nonlinear evolution equations*, Gakujutsu Kenkyu, School of Education, Waseda Univrtsity, Series of Mathematics **45** (1997), 25–35.
12. K.Kobayasi, C^1 *approximations of inertial manifolds via finite differences*, preprint.
13. J.Mallet-Paret and G.R.Sell, *Inertial manifolds for reaction-diffusion equations in higher space dimensions*, J. Amer. Math. Soc. **1** (1988), 805–866.
14. A.Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer -Verlag, New York, 1983.

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