

APPLICATIONS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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1. Introduction

Let \mathcal{A} denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in \mathcal{A} is said to be starlike in U if it satisfies

$$(1.2) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in U).$$

We denote by \mathcal{S}^* the subclass of \mathcal{A} consisting of all starlike functions in U . Further, a function $f(z)$ belonging to \mathcal{A} is said to be convex in U if it satisfies

$$(1.3) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in U).$$

Also, we denote by \mathcal{K} the subclass of \mathcal{A} consisting of all such functions.

It is well-known that

$$(1.4) \quad f(z) \in \mathcal{S}^* \implies |a_n| \leq n \quad (n \geq 2)$$

and

$$(1.5) \quad f(z) \in \mathcal{K} \implies |a_n| \leq 1 \quad (n \geq 2) \quad (\text{cf. [4]}).$$

A function $f(z) \in \mathcal{A}$ is called to be uniformly starlike in U if it satisfies

$$(1.6) \quad \operatorname{Re} \left\{ \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} \right\} \geq 0$$

for all $(z, \zeta) \in U \times U$. The subclass of \mathcal{A} consisting of all functions which are uniformly starlike in U is denoted by UST . Furthermore, a function $f(z) \in \mathcal{A}$ is said to be uniformly convex in U if it satisfies

$$(1.7) \quad \operatorname{Re} \left\{ 1 + (z - \zeta) \frac{f''(z)}{f'(z)} \right\} \geq 0$$

for all $(z, \zeta) \in U \times U$. We also denote by UCV the subclass of \mathcal{A} consisting of all uniformly convex functions in U . The classes UST and UCV were introduced by Goodman ([1], [2]).

The generalized hypergeometric series ${}_pF_q(z)$ defined by

$$(1.8) \quad {}_pF_q(z) \equiv {}_pF_q \left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n}{\prod_{i=1}^q (b_i)_n} \frac{z^n}{(1)_n} \quad (p \leq q + 1),$$

where p and q are non-negative integers, a_j ($j = 1, 2, \dots, p$) and b_i ($i = 1, 2, \dots, q$) are complex numbers with $b_i \neq 0, -1, -2, \dots$. Here $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(1.9) \quad (\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in N). \end{cases}$$

If we set

$$(1.10) \quad \omega = \sum_{i=1}^q b_i - \sum_{j=1}^p a_j,$$

we see that the series ${}_pF_q(z)$, with $p = q + 1$, is

- (i) absolutely convergent for $|z| = 1$ if $\operatorname{Re}(\omega) > 0$,
- (ii) conditionally convergent for $|z| = 1$, $z \neq 1$ if $-1 < \operatorname{Re}(\omega) \leq 0$, and
- (iii) divergent for $|z| = 1$ if $\operatorname{Re}(\omega) \leq -1$.

If $p < q + 1$ and $\operatorname{Re}(\omega) > 0$, then the ${}_pF_q(z)$ series (1.8) is absolutely convergent for $|z| < \infty$ (cf. Srivastava [6]).

2. Uniformly starlikeness

We discuss the uniform starlikeness of the generalized hypergeometric functions with some conditions.

Lemma 2.1 ([3]) *If a function $f(z) \in \mathcal{A}$ satisfies*

$$(2.1) \quad \sum_{n=2}^{\infty} (3n - 2) |a_n| \leq 1,$$

then $f(z) \in UST$.

Now, we derive

Theorem 2.1 Let $f(z) \in S^*$, $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \operatorname{Re} b_i > \sum_{j=1}^p |a_j| + 2$.
If

$$(2.2) \quad {}_3F_{q+2} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2, 2 \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q, 1, 1 \end{matrix} ; 1 \right) - 2 {}_pF_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2 \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q, 1 \end{matrix} ; 1 \right) \leq 2,$$

then $(f * z {}_pF_q)(z) \in UST$, where $(s * t)(z)$ denotes the Hadamard product of $s(z)$ and $t(z)$.

Proof. By virtue of Lemma 2.1, if

$$(2.3) \quad \sum_{n=2}^{\infty} (3n - 2) \left| \frac{\prod_{j=1}^p (a_j)_{n-1}}{\prod_{i=1}^q (b_i)_{n-1} (1)_{n-1}} a_n \right| \leq 1$$

is satisfied, then $(f * z {}_pF_q)(z) \in UST$. Indeed, we have

$$(2.4) \quad \begin{aligned} & \sum_{n=2}^{\infty} (3n - 2) \left| \frac{\prod_{j=1}^p (a_j)_{n-1}}{\prod_{i=1}^q (b_i)_{n-1} (1)_{n-1}} a_n \right| \\ & \leq 3 \sum_{n=0}^{\infty} \left(\frac{\prod_{j=1}^p (|a_j|)_{n+1} (n+2)^2}{\prod_{i=1}^q (\operatorname{Re} b_i)_{n+1} (1)_{n+1}} \right) - 2 \sum_{n=2}^{\infty} \left(\frac{\prod_{j=1}^p (|a_j|)_{n+1} (n+2)}{\prod_{i=1}^q (\operatorname{Re} b_i)_{n+1} (1)_{n+1}} \right) \\ & = 3 \sum_{n=0}^{\infty} \left(\frac{\prod_{j=1}^p (|a_j|)_n ((2)_n)^2}{\prod_{i=1}^q (\operatorname{Re} b_i)_n ((1)_n)^3} \right) - 2 \sum_{n=0}^{\infty} \left(\frac{\prod_{j=1}^p (|a_j|)_n (2)_n}{\prod_{i=1}^q (\operatorname{Re} b_i)_n ((1)_n)^2} \right) - 1 \\ & = 3 {}_pF_{q+2} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2, 2 \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q, 1, 1 \end{matrix} ; 1 \right) \\ & \quad - 2 {}_pF_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2 \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q, 1 \end{matrix} ; 1 \right) - 1, \end{aligned}$$

because $|a_n| \leq n$ ($n \geq 2$) for $f(z) \in S^*$. This completes the proof of Theorem 2.1.

Applying the Gauss summation theorem given by

$$(2.5) \quad {}_2F_1 \left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} ; 1 \right) = \frac{\Gamma(b_1)\Gamma(b_1 - a_1 - a_2)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)}$$

with $\operatorname{Re}(b_1 - a_1 - a_2) > 0$, we have

Corollary 2.1 Let $f(z) \in \mathcal{S}^*$, $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), $\operatorname{Re} b_m > |a_m| + 1$ ($m = 1, 2, \dots, q-3$), $\operatorname{Re} b_{q-2} > |a_{q-2}| + 2$, and $\operatorname{Re} b_q > |a_q| + |a_{q+1}|$. If

$$(2.6) \quad \frac{(\operatorname{Re} b_{q-2} - 2)\Gamma(\operatorname{Re} b_q - 1)\Gamma(\operatorname{Re} b_q - |a_q| - |a_{q+1}|)}{(\operatorname{Re} b_{q-1} - |a_{q-1}| - 2)\Gamma(\operatorname{Re} b_q - |a_q| - 1)\Gamma(\operatorname{Re} b_q - |a_{q+1}|)} \\ \times \left\{ \prod_{m=1}^q \left(\frac{\operatorname{Re} b_m - 1}{\operatorname{Re} b_m - |a_m| - 1} \right) \right\} \left(3 \frac{\operatorname{Re} b_{q-2} - 2}{\operatorname{Re} b_{q-2} - |a_{q-2}| - 2} - 2 \right) \leq 2$$

is satisfied, then $(f * z_{q+1}F_q)(z) \in UST$.

Similarly, we have

Theorem 2.2 Let $f(z) \in \mathcal{K}$, $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \operatorname{Re} b_i > \sum_{j=1}^p |a_j| + 1$. If

$$(2.7) \quad {}_3 {}_{p+1}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2 \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q, 1 \end{matrix} ; 1 \right) \\ - {}_2 {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q \end{matrix} ; 1 \right) \leq 2,$$

is satisfied, then $(f * z_p F_q)(z) \in UST$.

Corollary 2.2 Let $f(z) \in \mathcal{K}$, $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), $\operatorname{Re} b_m > |a_m| + 1$ ($m = 1, 2, \dots, q-2$), $\operatorname{Re} b_{q-1} > |a_{q-1}| + 2$, and $\operatorname{Re} b_q > |a_q| + |a_{q+1}|$. If

$$(2.8) \quad \frac{\Gamma(\operatorname{Re} b_q - 1)\Gamma(\operatorname{Re} b_q - |a_q| - |a_{q+1}|)}{\Gamma(\operatorname{Re} b_q - |a_q| - 1)\Gamma(\operatorname{Re} b_q - |a_{q+1}|)} \\ \times \left\{ \prod_{m=1}^q \left(\frac{\operatorname{Re} b_m - 1}{\operatorname{Re} b_m - |a_m| - 1} \right) \right\} \left(3 \frac{\operatorname{Re} b_{q-1} - 2}{\operatorname{Re} b_{q-1} - |a_{q-1}| - 2} - 2 \right) \leq 2$$

is satisfied, then $(f * z_{q+1}F_q)(z) \in UST$.

If we take $f(z) = \frac{z}{1-z} \in \mathcal{K}$ in Theorem 2.2, then we obtain

Corollary 2.3 Let $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \operatorname{Re} b_i > \sum_{j=1}^p |a_j| + 1$. If

$$(2.9) \quad {}_3 {}_{p+1}F_{q+1} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2 \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q, 1 \end{matrix} ; 1 \right) \\ - {}_2 {}_pF_q \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p| \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q \end{matrix} ; 1 \right) \leq 2,$$

is satisfied, then $z_p F_q(z) \in UST$.

3. Uniformly convexity

For uniformly convex functions Rønning [5] proved that $f(z) \in UCV$ if and only if $f(z) \in \mathcal{A}$ satisfies

$$(3.1) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) \geq \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in U).$$

From this fact, we have

Lemma 3.1 *If $f(z) \in \mathcal{A}$ satisfies*

$$(3.2) \quad \sum_{n=2}^{\infty} n^2 |a_n| \leq 1,$$

then $f(z) \in UCV$.

With the above lemma, we get

Theorem 3.1 *Let $f(z) \in \mathcal{S}^*$, $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \operatorname{Re} b_i > \sum_{j=1}^p |a_j| + 3$. If*

$$(3.3) \quad {}_{p+3}F_{q+3} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2, 2, 2 \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q, 1, 1, 1 \end{matrix} ; 1 \right) \leq 2$$

*is satisfied, then $(f * z {}_pF_q)(z) \in UCV$.*

Corollary 3.1 *Let $f(z) \in \mathcal{S}^*$, $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), $\operatorname{Re} b_m > |a_m| + 1$ ($m = 1, 2, \dots, q-4$), $\operatorname{Re} b_{q-3} > |a_{q-3}| + 2$, and $\operatorname{Re} b_{q-2} > |a_{q-2}| + 2$, $\operatorname{Re} b_{q-1} > |a_{q-1}| + 2$, and $\operatorname{Re} b_q > |a_q| + |a_{q+1}|$. If*

$$(3.4) \quad \frac{(\operatorname{Re} b_{q-3} - 2)(\operatorname{Re} b_{q-2} - 2)(\operatorname{Re} b_{q-1} - 2)}{(\operatorname{Re} b_{q-3} - |a_{q-3}| - 2)(\operatorname{Re} b_{q-2} - |a_{q-2}| - 2)(\operatorname{Re} b_{q-1} - |a_{q-1}| - 2)} \\ \times \frac{\Gamma(\operatorname{Re} b_q - 1)\Gamma(\operatorname{Re} b_q - |a_q| - |a_{q+1}|)}{\Gamma(\operatorname{Re} b_q - |a_q| - 1)\Gamma(\operatorname{Re} b_q - |a_{q+1}|)} \left\{ \prod_{m=1}^q \left(\frac{\operatorname{Re} b_m - 1}{\operatorname{Re} b_m - |a_m| - 1} \right) \right\} \leq 2$$

*is satisfied, then $(f * z {}_{q+1}F_q)(z) \in UCV$.*

Theorem 3.2 *Let $f(z) \in \mathcal{K}$, $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \operatorname{Re} b_i > \sum_{j=1}^p |a_j| + 2$. If*

$$(3.5) \quad {}_{p+2}F_{q+2} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2, 2 \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q, 1, 1 \end{matrix} ; 1 \right) \leq 2$$

*is satisfied, then $(f * z {}_pF_q)(z) \in UCV$.*

Corollary 3.2 Let $f(z) \in \mathcal{K}$, $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), $\operatorname{Re} b_m > |a_m| + 1$ ($m = 1, 2, \dots, q-3$), $\operatorname{Re} b_{q-2} > |a_{q-2}| + 2$, $\operatorname{Re} b_{q-1} > |a_{q-1}| + 2$, and $\operatorname{Re} b_q > |a_q| + |a_{q+1}|$. If

$$(3.6) \quad \frac{(\operatorname{Re} b_{q-2} - 2)(\operatorname{Re} b_{q-1} - 2)\Gamma(\operatorname{Re} b_q - 1)\Gamma(\operatorname{Re} b_q - |a_q| - |a_{q+1}|)}{\Gamma(\operatorname{Re} b_q - |a_q| - 1)\Gamma(\operatorname{Re} b_q - |a_{q+1}|)} \\ \times \left\{ \prod_{m=1}^q \left(\frac{\operatorname{Re} b_m - 1}{\operatorname{Re} b_m - |a_m| - 1} \right) \right\} \leq 2$$

is satisfied, then $(f * z_{q+1}F_q)(z) \in UCV$.

Corollary 3.3 Let $\operatorname{Re} b_i > 0$ ($i = 1, 2, \dots, q$), and $\sum_{i=1}^q \operatorname{Re} b_i > \sum_{j=1}^p |a_j| + 2$. If

$$(3.7) \quad {}_{p+2}F_{q+2} \left(\begin{matrix} |a_1|, |a_2|, \dots, |a_p|, 2, 2 \\ \operatorname{Re} b_1, \operatorname{Re} b_2, \dots, \operatorname{Re} b_q, 1, 1 \end{matrix} ; 1 \right) \leq 2$$

is satisfied, then $z {}_pF_q(z) \in UCV$.

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