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# Starlike or convex of complex order functions with negative coefficients

SHIGEYOSHI OWA* GRIGORE STEFAN SĂLĂGEAN†

**Abstract.** In this paper we study some relations between classes of analytic functions with negative coefficients and which are starlike or convex of complex order and other classes of analytic functions with negative coefficients. In the same time we give an answer to a conjecture due to S. Owa [3, p.163-164]. In the particular case when $n \in \mathbb{N}$ and $m = 0$ we obtain the same results as in [4].

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Let $U$ denote the unit disc, $U = \{z \in \mathbb{C}; |z| < 1\}$, let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N} = \{1, 2, 3, ...\}$, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $H(U)$ denote the set of functions which are holomorphic in $U$.

For $m \in \mathbb{N}_0$ we define the differential operator $D^m$ by $D^m : H(U) \to H(U)$, $D^0 f = f$, $D^1 f(z) = Df(z) = zf'(z)$ and $D^m f(z) = D(D^{m-1} f(z))$, $m \geq 1$ (see [5]).

We denote by $T_{n,m}$ the classes

$$T_{n,m} =$$

$$\left\{ f \in H(U); \frac{D^m f(z)}{z} \neq 0, (z \in \mathbb{C} - \{0\}), f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, a_k \geq 0, (k \in \mathbb{N}, k > n) \right\}$$

where $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

For $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $b \in \mathbb{C} - \{0\}$ we define the next subclasses of $T_{n,m}$

*Kinki University, Department of Mathematics, Higashi-Osaka, Osaka 577, Japan
†Babes-Bolyai University, Faculty of Mathematics and Computer Science, str. M. Kogalniceanu nr. 1, 3400 Cluj-Napoca, Romania

$$T_{n,m}(b)=\left\{f\in T_{n,m}:\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{D^{m+1}f(z)}{D^m f(z)}-1\right)\right\}>0,(z\in U)\right\},$$

$$O_{n,m}(b)=\left\{f\in T_{n,m}:\sum_{k=n+1}^{\infty}k^m(k-1+|b|)a_k\leq|b|\right\}$$

and

$$P_{n,m}(b)=\left\{f\in T_{n,m}:\sum_{k=n+1}^{\infty}k^m\left[(k-1)\frac{\mathrm{Re}\,b}{|b|}+|b|\right]a_k\leq|b|\right\}.$$

The functions in $T_{n,0}(b)$ are the starlike of the complex order $b$ functions with negative coefficients (see [1, 2]).

The classes $T_{1,0}(1-\alpha)$ and $T_{1,1}(1-\alpha)$, $\alpha\in[0,1)$ ($\alpha$ is real) are the classes of starlike and convex of order $\alpha$ functions with negative coefficients introduced and studied by H. Silverman [6].

The class $O_{n,0}(b)$ and $O_{n,1}(b)$ were introduced by S. Owa in [3, p.163-164], where he conjectured that $T_{n,0}(b)=O_{n,0}(b)$ and $T_{n,1}(b)=O_{n,1}(b)$. In this paper we give an answer to this conjecture in the more general case of $T_{n,m}(b)$ and $O_{n,m}(b)$. In the particular case when $n\in\mathbb{N}$ and $m=0$ we obtain the same results as in [4].

THEOREM . *Let* $n\in\mathbb{N}$, $m\in\mathbb{N}_0$ *and let* $b\in\mathbb{C}-\{0\}$; *then*

1) $O_{n,m}(b)\subseteq T_{n,m}(b)$;

2) $T_{n,m}(b)\subseteq P_{n,m}(b)$;

3) *if* $b\in(0,\infty)$ *(b is a positive real number), then*

$$O_{n,m}(b)=T_{n,m}(b)=P_{n,m}(b);$$

4) *if* $b\in(-\infty,0)$ *or* $-n/2<Re\,b\leq 0$, *then* $P_{n,m}(b)\not\subseteq T_{n,m}(b)$;

5) *if* $b\in(-\infty,0)$, *then* $T_{n,m}(b)\not\subseteq O_{n,m}(b)$.

*Proof.* 1). Let $f\in O_{n,m}(b)$. We prove that

$$\left|\frac{D^{m+1}f(z)}{D^m f(z)}-1\right|<|b|,\quad z\in U. \tag{1}$$

If $f$ has the series expansion

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{2}$$

then

$$\left| \frac{D^{m+1} f(z)}{D^m f(z)} - 1 \right| - |b| \leq \frac{\sum_{k=n+1}^{\infty} k^m (k-1) a_k |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} k^m a_k |z|^{k-1}} - |b|. \tag{3}$$

We use the fact that $D^m f(z)/z \neq 0$ when $z \in U - \{0\}$ and $\lim_{z \to 0}[D^m f(z)/z] = 1$; these imply

$$1 - \sum_{k=n+1}^{\infty} k^m a_k |z|^{k-1} > 0, \tag{4}$$

when $z \in U$.

From (3) and (4) we deduce

$$\left| \frac{D^{m+1} f(z)}{D^m f(z)} - 1 \right| - |b| < \frac{\sum_{k=n+1}^{\infty} k^m (k - 1 + |b|) a_k - |b|}{1 - \sum_{k=n+1}^{\infty} k^m a_k}.$$

By using the definition of $O_{n,m}(b)$ from this last inequality we obtain (1) and this implies

$$\mathrm{Re}\left\{ \frac{1}{b} \left( \frac{D^{m+1} f(z)}{D^m f(z)} - 1 \right) \right\} > -1, \quad z \in U, \tag{5}$$

hence $f \in T_{n,m}(b)$.

2). Let $f$ be in $T_{n,m}(b)$. Then (5) holds and, by using (2), this is equivalent to

$$\mathrm{Re}\left\{ \frac{1}{b} \frac{\sum_{k=n+1}^{\infty} k^m (1-k) a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} k^m a_k z^{k-1}} \right\} > -1 \quad (z \in U). \tag{6}$$

For $z = t \in [0,1)$ if $t \to 1^-$, from (6) we obtain

$$\frac{\sum_{k=n+1}^{\infty} k^m (1-k) a_k}{1 - \sum_{k=n+1}^{\infty} k^m a_k} \frac{\mathrm{Re}\, b}{|b|^2} \geq -1$$

wich is equivalent to

$$\sum_{k=n+1}^{\infty} k^m \left[ (k-1) \frac{\mathrm{Re}\, b}{|b|} + |b| \right] a_k \leq |b|,$$

hence $f \in P_{n,m}(b)$.

3). If $b$ is a real positive number, then the definitions of $O_{n,m}(b)$ and $P_{n,m}(b)$ are equivalent, hence $O_{n,m}(b) = P_{n,m}(b)$. By using 1) and 2) from this theorem we obtain 3).

4). *Case* I: $b \in [-n, 0)$.

Let $f = f_{n,\alpha}$, where

$$f_{n,\alpha}(z) = z - \alpha(n+1)^{-m} z^{n+1} \tag{7}$$

and let $\alpha > 0$. We have

$$\sum_{k=n+1}^{\infty} k^m \left[|b| + \frac{(k-1)\operatorname{Re} b}{|b|}\right] a_k = (n+1)^m \left[-b + n\frac{b}{-b}\right] \alpha(n+1)^{-m}$$

or

$$\sum_{k=n+1}^{\infty} k^m \left[|b| + \frac{(k-1)\operatorname{Re} b}{|b|}\right] a_k = -(n+b)\alpha \le 0 < |b| \tag{8}$$

and then $f_{n,\alpha} \in P_{n,m}(b)$ (see the definition of $P_{n,m}(b)$).

Let now

$$F(z) = 1 + \frac{1}{b}\left(\frac{D^{m+1} f_{n,\alpha}(z)}{D^m f_{n,\alpha}(z)} - 1\right), \quad z \in U.$$

Then, by a simple computation and by using the fact that

$$D^m f_{n,\alpha}(z) = z - (n+1)^m \alpha (n+1)^{-m} z^{n+1} = z - \alpha z^{n+1}$$

we obtain

$$F(z) = 1 + \frac{n\alpha z^n}{b(\alpha z^n - 1)} = 1 + \varphi(\zeta),$$

where $\zeta = z^n$ and

$$\varphi(\zeta) = \frac{n\alpha\zeta}{b(\alpha\zeta - 1)}. \tag{9}$$

For $\alpha > 1$ we have $\varphi(U) = \mathbb{C}_\infty - U(c, \rho)$, where $U$ is the disc with the center

$$c = \frac{n\alpha^2}{b(\alpha^2 - 1)} \tag{10}$$

and the radius

$$\rho = \frac{n\alpha}{b(1-\alpha^2)}. \tag{11}$$

We have $F(U) = \mathbb{C}_\infty - U(c+1,\rho)$ and we deduce that $\operatorname{Re} F(z) > 0$ for all $z \in U$ does not hold.

We have obtained that for $\alpha > 1$, $f_{n,\alpha} \in P_{n,m}(b)$, but $f_{n,\alpha} \notin T_{m,n}(b)$ and in this case $P_{n,m}(b) \not\subseteq T_{n,m}(b)$.

*Case* II: $b \in (-\infty, -n)$.

We consider the function $f_{n,\alpha}$ defined by (7) for $\alpha \in (1, b/(n+b))$. In this case the inequality (8) holds too and this implies that $f_{n,\alpha} \in P_{n,m}(b)$.

We also obtain that $f \notin T_{n,m}(b)$ like in Case I.

*Case* III: $\operatorname{Re} b \in (-n/2, 0)$.

Let now $f = f_{n,1}$, that is

$$f_{n,1}(z) = z - (n+1)^{-m} z^{n+1}.$$

Then $f_{n,1} \in P_{n,m}(b)$ because the inequality

$$\sum_{k=n+1}^{\infty} k^m \left[|b| + (k-1)\operatorname{Re} b/|b|\right] a_k = |b| + n \operatorname{Re} b/|b| \leq |b|$$

holds for all $b$ when $\operatorname{Re} b < 0$.

Now let $r = \operatorname{Re} b < 0$ and let $s$ be a negative real number such that

$$n + 2r(1-s) > 0$$

for $n \in \mathbb{N}$ fixed. If we choose $z_o$ one of the rooth of the equation

$$z^n = \frac{b(1-s)}{n + b(1-s)},$$

then $z_0 \in U$ and for $f_{n,1}$ we have

$$1 + \frac{1}{b}\left(\frac{D^{m+1} f_{n,1}(z_0)}{D^m f_{n,1}(z_0)} - 1\right) = s < 0$$

hence $f_{n,1} \notin T_{n,m}(b)$.

5). Let $f = f_{n,\alpha}$ be given by (7), where $\alpha > |b|/(n+|b|)$. Then

$$\sum_{k=n+1}^{\infty} (n+1)^m (k-1+|b|) a_k = (n+|b|)\alpha > |b|$$

and this implies

$$f_{n,\alpha} \notin O_{n,m}(b) \text{ for } n \in \mathbb{N},\ m \in \mathbb{N}_0 \text{ and } b \in (-\infty, 0).$$

We have

$$F(z) = 1 + \frac{1}{b}\left(\frac{D^{m+1} f_{n,\alpha}(z)}{D^m f_{n,\alpha}(z)} - 1\right) = 1 + \varphi(\zeta),$$

where $\varphi$ is given by (9).

From $\varphi(U) = U(c,\rho)$ where $c$ and $\rho$ are given by (10) and (11), we obtain

$$\operatorname{Re} F(z) \geq \frac{(n+b)\alpha + b}{b(\alpha+1)}. \tag{12}$$

If

$$b \in (-\infty, -n) \text{ and } \alpha \in \left(\frac{|b|}{n+|b|}, 1\right),$$

then

$$\frac{(n+b)\alpha + b}{b(\alpha+1)} > 0 \tag{13}$$

and if

$$b \in (-n, 0) \text{ and } \alpha \in \left(\frac{|b|}{n+|b|}, \frac{|b|}{|n-|b||}\right) \cap (0,1),$$

then (13) also holds. By combining (13) with (12) and the definition of $T_{n,m}(b)$, we obtain that

$$f_{n,\alpha} \in T_{n,m}(b) \text{ for } \alpha \in \left(\frac{|b|}{n+|b|}, \frac{|b|}{|n-|b||}\right) \cap (0,1) \text{ and } b \in (-\infty, 0).$$

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