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ON SUFFICIENT CONDITIONS FOR MEROMORPHIC STARLIKE FUNCTIONS

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ABSTRACT. The object of the present paper is to show certain sufficient conditions for starlikeness and close-to-convexity of meromorphic functions in the punctured unit disk.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the punctured unit disk $D=\{z:0<|z|<1\}$. For f and g which are analytic in $U=\{z:|z|<1\}$, we say that f is subordinate to g, written $f\prec g$ or $f(z)\prec g(z)$, if g is univalent, $f(0)=g(0)$ and $f(U)\subset g(U)$.

For $0<\alpha\leq 1$, let $\mathcal{SMS}(\alpha)$ denote the class of functions $f\in\Sigma$ which are starlike of order α ; that is, which satisfy

$$-\frac{zf'(z)}{f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\alpha}\quad(z\in U). \tag{1.2}$$

We note that the equation (1.2) can be rewritten by the following form ;

$$\left|\arg\left(-\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{2}\alpha\quad(z\in U).$$

Also, we note that if $\alpha=1$, $\mathcal{SMS}(\alpha)$ coincides with Σ^*, the well known class of meromorphic starlike(univalent) functions with respect to origin.

In [1], Bajpai and Mehrok proved that the functions of the form (1.1) satisfying the condition

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$$\mathrm{Re}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-(\alpha+\beta)\frac{zf'(z)}{f(z)}\right\}>0 \quad (z\in U)$$

are univalent and meromorphic starlike, where α and β are real numbers. For various other interesting developments involving analytic functions in the open unit disk U, the reader may be referred(for example) to the recent work of Nunokawa[3].

In this paper, we investigate some sufficient conditions for starlikeness and close-to-convexity of functions belonging to Σ.

2. Main results

In proving our theorems, we need the following lemma due to Nunokawa [2].

Lemma 2.1 *Let p be analytic in U, $p(0)=1$ and $p(z)\neq 0$ in U. Suppose that there exists a point $z_0\in U$ such that*

$$|\arg\ p(z)| \ < \ \frac{\pi}{2}\delta \quad \text{for} \quad |z|<|z_0| \tag{2.1}$$

and

$$|\arg\ p(z_0)| \ = \ \frac{\pi}{2}\delta \ \ (0<\delta\leq 1). \tag{2.2}$$

Then we have

$$\frac{z_0p'(z_0)}{p(z_0)} \ = \ i\delta k, \tag{2.3}$$

where

$$k\geq\frac{1}{2}\left(a+\frac{1}{a}\right) \ \text{ when } \ \arg\ p(z_0)=\frac{\pi}{2}\delta, \tag{2.4}$$

$$k\leq-\frac{1}{2}\left(a+\frac{1}{a}\right) \ \text{ when } \ \arg\ p(z_0)=-\frac{\pi}{2}\delta, \tag{2.5}$$

and

$$\{p(z_0)\}^{\frac{1}{\delta}}=\pm\, ia \ \ (a>0). \tag{2.6}$$

Applying Lemma 2.1, we have the following

Theorem 2.1. *Let p be analytic in U with $p(0)=1$. If*

$$\left|\arg\left(\beta p(z)+\alpha\frac{zp'(z)}{p(z)}\right)\right|<\frac{\pi}{2}\gamma(\alpha,\beta,\delta)\ (\alpha,\beta>0, 0<\delta<1, z\in U), \tag{2.7}$$

where

$$\gamma(\alpha,\beta,\delta)=\frac{2}{\pi}\tan^{-1}\left\{\tan\frac{\pi}{2}\delta+\frac{\alpha\delta}{\beta(1+\delta)^{\frac{1+\delta}{2}}(1-\delta)^{\frac{1-\delta}{2}}\cos\frac{\pi}{2}\delta}\right\}, \tag{2.8}$$

then

$$|\arg\ p(z)|<\frac{\pi}{2}\delta.$$

Proof. If there exists a point $z_0\in U$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.1) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

From (2.7), we note that $p(z)\neq 0$ in U. In fact, if p has a zero of order m at $z=z_1\in U$, then p can be written as

$$p(z)=(z-z_1)^m q(z)\ \ (m\in N=\{1,2,\cdots\}),$$

where q is analytic in U and $q(z_1)\neq 0$. Hence we have

$$\beta p(z)+\alpha\frac{zp'(z)}{p(z)}=\frac{\alpha mz}{z-z_1}+\alpha\frac{zq'(z)}{q(z)}+\beta(z-z_1)^m q(z). \tag{2.9}$$

But choosing $z\to z_1$ suitably, the argument of the right hand side of (2.9) can take any value between 0 and 2π. This contradicts (2.7). Hence we have $p(z)\neq 0\ \ (z\in U)$. Then we obtain

$$\begin{aligned}\beta p(z_0)+\alpha\frac{z_0p'(z_0)}{p(z_0)}&=\beta(\pm ia)^\delta+i\alpha\delta k\\&=\beta a^\delta\cos\frac{\pi}{2}\delta+i\left\{\beta a^\delta\sin\frac{\pi}{2}\delta+\alpha\delta k\right\}.\end{aligned}$$

Now we suppose that

$$\{p(z_0)\}^{\frac{1}{\delta}}=ia\ \ (a>0).$$

Then we have

$$\arg\left(\beta p(z_0)+\alpha\frac{z_0p'(z_0)}{p(z_0)}\right)=\tan^{-1}\left\{\tan\frac{\pi}{2}\delta+\frac{\alpha\delta k}{\beta\cos\frac{\pi}{2}\delta}\right\},$$

where

$$ka^{-\delta}\geq\frac{1}{2}\left(a^{1-\alpha}+a^{-1-\alpha}\right)\equiv g(a)\ (a>0).$$

Hence, by a simple calculation, we see that the function $g(a)$ takes the minimum value at $a=\sqrt{\frac{1+\alpha}{1-\alpha}}$. Hence we have

$$\begin{aligned}\arg\left(\beta p(z_0)+\alpha\frac{z_0p'(z_0)}{p(z_0)}\right)&\leq\tan^{-1}\left\{\tan\frac{\pi}{2}\delta+\frac{\alpha\delta}{\beta(1+\delta)^{\frac{1+\delta}{2}}(1-\alpha)^{\frac{1-\alpha}{2}}\cos\frac{\pi}{2}\delta}\right\}\\&=\frac{\pi}{2}\gamma(\alpha,\beta,\delta),\end{aligned}$$

where $\gamma(\alpha,\beta,\delta)$ is given by (2.8). This evidently contradicts the assumption of Theorem 2.1.

Next, we suppose that

$$\{p(z_0)\}^{\frac{1}{\delta}}=-ia\quad(a>0).$$

Applying the same method as the above, we have

$$\begin{aligned}\arg\left(\beta p(z_0)+\alpha\frac{z_0p'(z_0)}{p(z_0)}\right)&\geq-\tan^{-1}\left\{\tan\frac{\pi}{2}\delta+\frac{\alpha\delta}{\beta(1+\delta)^{\frac{1+\delta}{2}}(1-\alpha)^{\frac{1-\alpha}{2}}\cos\frac{\pi}{2}\delta}\right\}\\&=-\frac{\pi}{2}\gamma(\alpha,\beta,\delta),\end{aligned}$$

where $\gamma(\alpha,\beta,\delta)$ is given by (2.8), which is a contradiction to the assumption of Theorem 2.1. Therefore, we complete the proof of Theorem 2.1.

Taking $p(z)=-\frac{zf'(z)}{f(z)}$ in Theorem 2.1, we have

Corollary 2.1. *If $f\in\Sigma$ satisties the condition*

$$\left|\arg\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-(\alpha+\beta)\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\pi}{2}\gamma(\alpha,\beta,\delta)\ (\alpha,\beta>0,0<\delta<1,z\in U),$$

where $\gamma(\alpha,\beta,\delta)$ is given by (2.8), then $f \in \mathcal{SMS}(\delta)$.

Next, we prove

Theorem 2.2. *Let $\alpha \geq 0$ or $\alpha \leq -2\beta(\beta > 0)$. If p satisfies the condition*

$$\beta p(z) + \alpha \frac{zp'(z)}{p(z)} \neq ik \quad (z \in U), \tag{2.10}$$

where k is a real number with $|k| \geq \sqrt{(\alpha+2\beta)\alpha}$. Then Re $p(z) > 0 (z \in U)$.

Proof. For the case $\alpha = 0$, it is obvious and so we suppose $\alpha \neq 0$. By using the same method of the proof in Theorem 2.1, we can see easily that $p(z) \neq 0$ in U. Suppose that there exists a point $z_0 \in U$ such that

$$\mathrm{Re}\ p(z) > 0 \ \text{ for } \ |z| < |z_0|,$$

$$\mathrm{Re}\ p(z_0) = 0 \ \text{ and } \ p(z_0) = ia \ (a \neq 0).$$

For the case $\alpha > 0$, from Lemma 2.1 with $\delta = 1$, we have

$$\beta p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} = i(\beta a + \alpha k),$$

and

$$\beta a + \alpha k \geq \frac{1}{2}\left((\alpha+2\beta)a + \frac{\alpha}{a}\right) \geq \sqrt{(\alpha+2\beta)\alpha} \text{ when } a > 0,$$

and

$$\beta a + \alpha k \leq -\frac{1}{2}\left((\alpha+2\beta)|a| + \frac{\alpha}{|a|}\right) \leq -\sqrt{(\alpha+2\beta)\alpha} \text{ when } a < 0,$$

which contradict (2.10). Therefore we have Re $p(z) > 0$ in U. For the case $a \leq -2b$, applying the same method as the above, we easily have the same conclusion. This completes the proof of our theorem.

Letting $p(z) = -\frac{zf'(z)}{f(z)}$ in Theorem 2.2, we easily have the following

Corollary 2.2. *Let $\alpha \geq 0$ or $\alpha \leq -2\beta(\beta > 0)$. If $f \in \Sigma$ satisfies the condition*

$$\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - (\alpha+\beta)\frac{zf'(z)}{f(z)} \neq ik \quad (z \in U),$$

where k is real number with $|k| \geq \sqrt{(\alpha+2\beta)\alpha}$, then $f \in \Sigma^*$.

Making $\alpha = \beta = 1$ in Corollary 2.2, we obtain

Corollary 2.3. *Let $f \in \Sigma$ and suppose that there exists a real number R for which*

$$\left|\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)} - R\right| < \sqrt{(R+1)^2+3} \quad (z \in U).$$

Then f is meromorphic starlike in U.

Putting $p(z) = -z^2 f'(z)$ in Theorem 2.2, we get

Corollary 2.4. *Let $\alpha \geq 0$ or $\alpha \leq -2\beta (\beta > 0)$. If $f \in \Sigma$ satisfies the condition*

$$\alpha\left(2 + \frac{zf''(z)}{f'(z)}\right) - \beta z^2 f'(z) \neq ik \quad (z \in U),$$

where k is given by Corollary 2.2. Then f is meromorphic univalent(or close-to-convex) in U.

Similarly, from Corollary 2.4, we have

Corollary 2.5. *Let $f \in \Sigma$ and suppose that there exists a real number R for which*

$$\left|\frac{zf''(z)}{f'(z)} - z^2 f'(z) - R\right| < \sqrt{(R+2)^2+3} \quad (z \in U).$$

Then f is meromorphic univalent (or close-to-convex) in U.

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