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# SECOND ORDER DIFFERENTIAL SUBORDINATIONS AND CERTAIN SUBORDINATION RELATIONS

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**Abstract.** In the paper connections between certain second order differential subordination and subordination of $f(z)/z$, $f'(z)$ and convexity of the function $f$ are considered. The solution of the second order differential equation is obtained.

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## 1. Introduction

Let $H$ denote the class of functions with normalization $f(0) = f'(0) - 1 = 0$ which are analytic in the open unit disk $U = \{z \in \mathbf{C},\ |z| < 1\}$. Also, let $S$ denote the class of all functions in $H$ which are univalent in the disk $U$. Then, a function $f$ belonging to the class $S$ is said to be *convex* in $U$, if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in U.$$

We denote by $S^c$ the subclass of $H$ consisting of all convex functions in $U$.

If $f$ and $g$ are analytic functions in $U$, and $g \in S$, then we say that the function $f$ is subordinate to $g$ $(f \prec g)$, if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We begin by looking at some well known results concerning the theory of differential subordinations. This theory have been introduced and developed by S. S. Miller and P. T. Mocanu (cf. e.g. [3],[4]).

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Let $\psi : \mathbf{C}^3 \times U \to \mathbf{C}$, and let h be univalent in $U$. If $p$ is analytic in $U$ and satisfies *the second order differential subordination*

$$(1.1) \qquad \psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \; z \in U,$$

then $p$ is called a *solution of the differential subordination.* The univalent function $q$ is called a *dominant of the differential subordination*, if $p \prec q$ for all $p$ satisfying (1.1). A dominant $\bar{q}$ which satisfies $\bar{q} \prec q$ for all dominants $q$ of (1.1) is said to be *the best dominant* of (1.1).

LEMMA 1.1 ([4]) *Let $f$ be analytic in $U$, and $g$ be analytic and univalent on $\bar{U}$, with $f(0) = g(0)$. If $f$ is not subordinate to $g$, then there exist points $z_0 \in U$, $\zeta_0 \in \partial U$, and $m \geq 1$, for which $f(|z| < |z_0|) \subset g(|z| < |z_0|)$, and*

(i) $\quad f(z_0) = g(\zeta_0)$,

(ii) $\quad z_0 f'(z_0) = m\zeta_0 g'(\zeta_0)$, *and*

(iii) $\quad \mathrm{Re}\,[z_0 f''(z_0)/f'(z_0) + 1] \geq m\,\mathrm{Re}\,[\zeta_0 g''(\zeta_0)/g'(\zeta_0) + 1]$.

LEMMA 1.2 ([3]) *Let $p(z) = a + p_n z^n + \ldots$ be analytic in $U$, $\psi : \mathbf{C}^3 \times U \to \mathbf{C}$ be an analytic function in a domain $D \subset \mathbf{C}^3 \times U$, such that $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$ for $z \in U$, where $h$ is analytic and univalent in $U$ with $\psi(p(0), 0, 0, 0) = h(0)$. If*

$$(1.2) \qquad \psi(r, s, t; z) \notin h(U) \quad \text{when } r = q(\zeta_0), \; s = m\zeta_0 q'(\zeta_0),$$

$$\mathrm{Re}\,[t/s + 1] \geq m\,\mathrm{Re}\,[\zeta_0 q''(\zeta_0)/q'(\zeta_0) + 1], \; m \geq n, \; z \in U \quad \text{and} \quad |\zeta_0| = 1,$$

*then $p \prec q$ in $U$.*

LEMMA 1.3 ([1]) *Let $G$ be a convex function in $U$ (not necessary normalized by $G(0) = 0$), and let $\gamma$ be a complex number with $\mathrm{Re}\,\gamma > 0$. If $F$ is analytic in $U$ and $F \prec G$, then*

$$z^{-\gamma} \int_0^z F(w) w^{\gamma-1} dw \prec z^{-\gamma} \int_0^z G(w) w^{\gamma-1} dw.$$

Now, we formulate some simple properties of subordination in a certain class of functions, which will be used in the next part of paper. These properties follow imediately from the definition of subordination, mentioned above, so we omit the proofs.

LEMMA 1.4 *Let $K$, $L$, $N$, $\gamma$, $\delta$ be nonnegative real, fixed numbers, and let*

$$f(z) \prec 1 + Kz, \; g(z) \prec 1 + Lz, \; h(z) \prec Nz, \; z \in U.$$

*Then*

$$\gamma f(z) + \delta g(z) \prec \gamma + \delta + (\gamma K + \delta L)z, \ z \in U, \tag{1.3}$$

$$\gamma f(z) + \delta h(z) \prec \gamma + (\gamma K + \delta N)z, \ z \in U. \tag{1.4}$$

## 2. SECOND-ORDER DIFFERENTIAL SUBORDINATION

Our goal is to find conections between certain second-order differential subordination and some subordination of the expressions: $f(z)/z$, $f'(z)$ and $1+zf''(z)/f'(z)$. Obtained results extend the results of Kanas and Stankiewicz from [2]. Results of a similar type, but mainly of the first order, have been investigated by numerous authors (cf. e.g. [5], [6], [7], [8]).

THEOREM 2.1 *Let $\alpha$ and $\beta$ be real numbers, such that $\beta \geq 0$, $\alpha + 2\beta \geq 0$. If $f \in H$, and*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta\, zf''(z) \prec 1 + Mz, \quad \text{for } z \in U, \tag{2.1}$$

*for some $M > 0$, then*

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{\alpha + 2\beta + 1} = q(z), \quad \text{for } z \in U, \tag{2.2}$$

*and the result is the best as possible.*

**P r o o f.** Denote by $p(z) = \dfrac{f(z)}{z}$. Of course $p(0) = 1 = q(0)$, and (2.1) can be rewritten in the following form

$$\beta z^2 p''(z) + (\alpha + 2\beta) z p'(z) + p(z) \prec 1 + Mz, \ z \in U. \tag{2.3}$$

The case $\beta = 0$ and $\alpha = 0$ is obvious, assume then $\alpha > 0$. First, let consider $\beta = 0$. This case is evidently true in view of Lemma 1.3 with $G(z) = 1 + Mz$, $F(z) = p(z) + \alpha z p'(z)$, and $\gamma = 1/\alpha$. Then we shall study only the case, for which $\beta \neq 0$.

Suppose, that $p \not\prec q$. Then, on account of Lemma 1.1, there exist $z_0 \in U$, and $\zeta_0 \in \partial U$, and $m \geq 1$ such that

$$p(z_0) = q(\zeta_0), \ z_0 p'(z_0) = m\, \zeta_0 q'(\zeta_0), \text{ and}$$

$$\operatorname{Re}\{z_0 p''(z_0)/p'(z_0)+1\} \geq m \operatorname{Re}\{\zeta_0 q''(\zeta_0)/q'(\zeta_0)+1\}.$$

In this case we have

$$q(\zeta_0)=1+\frac{M\zeta_0}{\alpha+2\beta+1}, \quad \zeta_0 q'(\zeta_0)=\frac{M\zeta_0}{\alpha+2\beta+1},$$

and $q''(\zeta_0)=0$. Consequently, for $\zeta_0=e^{i\theta}$ we get

$$\operatorname{Re}[e^{-i\theta}{z_0}^2 p''(z_0)] \geq m(m-1)\frac{M}{\alpha+2\beta+1}.$$

Thus

$$\begin{aligned}
&\left|\beta {z_0}^2 p''(z_0)+(\alpha+2\beta)z_0 p'(z_0)+p(z_0)-1\right| \\
&=\left|\beta {z_0}^2 p''(z_0)+m\frac{\alpha+2\beta}{\alpha+2\beta+1}Me^{i\theta}+1+\frac{M}{\alpha+2\beta+1}e^{i\theta}-1\right| \\
&=|e^{i\theta}|\left|\beta e^{-i\theta}{z_0}^2 p''(z_0)+m\frac{\alpha+2\beta}{\alpha+2\beta+1}M+\frac{M}{\alpha+2\beta+1}\right| \\
&\geq \operatorname{Re}\left[\beta e^{-i\theta}{z_0}^2 p''(z_0)+m\frac{\alpha+2\beta}{\alpha+2\beta+1}M+\frac{M}{\alpha+2\beta+1}\right] \\
&=\operatorname{Re}\left[\beta e^{-i\theta}{z_0}^2 p''(z_0)\right]+m\frac{\alpha+2\beta}{\alpha+2\beta+1}M+\frac{M}{\alpha+2\beta+1} \\
&\geq \beta m(m-1)\frac{M}{\alpha+2\beta+1}+m\frac{\alpha+2\beta}{\alpha+2\beta+1}M+\frac{M}{\alpha+2\beta+1} \\
&=[\beta m^2+m(\alpha+\beta)+1]\frac{M}{\alpha+2\beta+1} \geq M
\end{aligned}$$

for $m \geq 1$, $\beta \geq 0$ and $\alpha+2\beta \geq 0$.

Above inequality contradicts the assumption (2.1), then we must have $p \prec q$ in $U$.

Moreover, it is easy to check, that the function

$$q(z)=1+\frac{Mz}{\alpha+2\beta+1}$$

realizes equality in the differential subordination (2.3), thus $q(z)$ is the best dominant of (2.3), and obtained result is the best as possible. □

REMARK 2.1 From Theorem 2.1 follows that the function

$$f(z) = z + \frac{Mz^2}{\alpha + 2\beta + 1}$$

is a solution of differential equation

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta\, zf''(z) = 1 + Mz, \ z \in U.$$

THEOREM 2.2 *Let $\alpha$ and $\beta$ be real numbers, such that $\beta \geq 0$, $\alpha \geq 1$. If $f \in H$, and the differential subordination*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta\, zf''(z) \prec 1 + Mz, \quad \text{for } z \in U, \tag{2.4}$$

*holds true for some $M > 0$, then*

$$f'(z) \prec 1 + \frac{2Mz}{\alpha + 2\beta + 1}, \quad \text{for } z \in U, \tag{2.5}$$

**Proof.** Denote $P(z) = f'(z)$. Then $P(0) = 1$ and subordination (2.4) can be rewritten in the following form

$$\beta zP'(z) + \alpha P(z) + (1-\alpha)\frac{f(z)}{z} \prec 1 + Mz, \ z \in U.$$

From Theorem 2.1 (all the assumptions are satisfied), we get

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{\alpha + 2\beta + 1}, \ z \in U.$$

Making use Lemma 1.4 (condition (1.3)), we obtain for $\alpha \geq 1$

$$\beta zP'(z) + \alpha P(z) \prec \alpha + \frac{2(\alpha + \beta)}{\alpha + 2\beta + 1}Mz, \ z \in U.$$

Suppose now, that

$$P(z) \not\prec 1 + \frac{2Mz}{\alpha + 2\beta + 1}, \ z \in U.$$

Then, in view of Lemma 1.2, it is enough to show that

$$\left|\beta\frac{2mMe^{i\theta}}{\alpha+2\beta+1}+\alpha\left(1+\frac{2Me^{i\theta}}{\alpha+2\beta+1}\right)-\alpha\right|\geq\frac{2(\alpha+\beta)M}{\alpha+2\beta+1},\ z\in U,\ \theta\in\mathbf{R}. \tag{2.6}$$

But, above condition is obvious under the assumptions concerning the parametres $\alpha$ and $\beta$, and for $m\geq 1$. Consequently (2.6) and Lemma 1.2 yields

$$P(z)\prec 1+\frac{2Mz}{\alpha+2\beta+1},\ z\in U,$$

it means the condition (2.5). □

THEOREM 2.3 *Let $\alpha$ and $\beta$ be real numbers, such that $\alpha\geq 1$, and $\beta\geq 0$. If $f\in H$ and for $0<M\leq M(\alpha,\beta)$ where*

$$M(\alpha,\beta)=\frac{\beta(\alpha+2\beta+1)}{2(\sqrt{(\alpha+\beta)^2+\alpha}+|\alpha-\beta|},\tag{2.7}$$

*the differential subordination*

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\beta\, zf''(z)\prec 1+Mz,\quad \textit{for } z\in U,\tag{2.8}$$

*holds true, then $f(z)\in S^c$*

**Proof.** Assume, that for $M\leq M(\alpha,\beta)$ the subordination (2.7) holds. From Theorem 2.1 and 2.2 we have

$$\frac{f(z)}{z}\prec 1+\frac{Mz}{\alpha+2\beta+1},\ z\in U,\tag{2.9}$$

and

$$f'(z)\prec 1+\frac{2Mz}{\alpha+2\beta+1},\ z\in U.\tag{2.10}$$

Since

$$\frac{2M}{\alpha+2\beta+1}<1,\ \text{for } M\leq M(\alpha,\beta),$$

then $f'(z)\neq 0$ in $U$.

Let $Q(z) = 1 + \frac{zf''(z)}{f'(z)}$. In this case (2.7) can be rewritten in the form

$$\beta f'(z)(Q(z)-1) + \alpha f'(z) + (1-\alpha)\frac{f(z)}{z} \prec 1 + Mz, \ z \in U,$$

and, by (2.9) and condition (1.4) from Lemma 1.3, we obtain

$$f'(z)(\alpha - \beta + \beta Q(z)) \prec \alpha + \frac{2M(\alpha+\beta)z}{\alpha+2\beta+1}, \ z \in U.$$

Suppose, on the contrary, that there exists $z_0 \in U$, such that Re $Q(z_0) = 0$. Then $Q(z_0) = ix$, where $x \in \mathbf{R}$. Hence, we obtain the contradiction of the assumption, if we show that

$$|\beta f'(z)ix + (\alpha-\beta)f'(z) - \alpha| \geq \frac{2M(\alpha+\beta)}{\alpha+2\beta+1},$$

for all real $x$. Applying Lemma 2.1 ([2]) we need to prove that

$$\frac{2M(\alpha+\beta)}{(\alpha+2\beta+1)|f'(z)|} \leq \left|\alpha - \beta - \alpha \mathrm{Re}\, \frac{1}{f'(z)}\right|$$

holds true. Above inequality is equivalent to

$$\frac{4M^2(\alpha+\beta)^2}{(\alpha+2\beta+1)^2} + \frac{\alpha^2(\mathrm{Im}\, f'(z))^2}{|f'(z)|^2} \leq |(\alpha-\beta)f'(z) - \alpha|^2. \tag{2.11}$$

Taking into account (2.10) we have

$$\frac{|\mathrm{Im}\, f'(z)|}{|f'(z)|} \leq \frac{2M}{\alpha+2\beta+1}$$

and

$$|(\alpha-\beta)f'(z) - \alpha| = |(\alpha-\beta)(f'(z)-1) - \beta| \geq$$
$$\beta - |\alpha-\beta||f'(z)-1| \geq \beta - |\alpha-\beta|\frac{2M}{\alpha+2\beta+1},$$

so (2.11) is satisfied if

$$\frac{4\alpha(4\beta+1)}{(\alpha+2\beta+1)^2}M^2 + \frac{4\beta|\alpha-\beta|}{\alpha+2\beta+1}M - \beta^2 \leq 0.$$

Above is fulfilled for $M \leq M(\alpha, \beta)$, where $M(\alpha, \beta)$ is given by (2.7). This completes the proof. □

## 3. COROLLARIES AND SPECIAL CASES

In the case, when $\beta = 0$ from Theorems 2.1 - 2.3 we get

COROLLARY 3.1 *Let $\alpha$ be real number, such that $\alpha \geq 0$. If $f \in H$ and the differential subordination*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec 1 + Mz, \quad for\ z \in U,$$

*holds true for some $M > 0$, then*

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{\alpha+1}, \quad z \in U.$$

Let $\alpha = 0$. Then Theorems 2.1 reduces to

COROLLARY 3.2 *Let $\beta$ be real number, such that $\beta \geq 0$. If $f \in H$ and the differential subordination*

$$\frac{f(z)}{z} + \beta\, zf''(z) \prec 1 + Mz, \quad for\ z \in U,$$

*holds true for some $M > 0$, then*

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{2\beta+1}, \quad z \in U.$$

In the special case $\alpha = \beta$, we have from Theorems 2.1 - 2.2

COROLLARY 3.3 *Let $\alpha$ be real number, such that $\alpha \geq 0$. If $f \in H$ and the differential subordination*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \alpha\, zf''(z) \prec 1 + Mz, \quad for\ z \in U,$$

*holds true for some $M > 0$, then*

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{3\alpha+1}, \quad f'(z) \prec 1 + \frac{2Mz}{3\alpha+1}, \quad z \in U.$$

For $\alpha = \beta = 1/2$, we obtain

COROLLARY 3.4 *Let $f \in H$. If the differential subordination*

$$\frac{1}{2}\left[\frac{f(z)}{z} + f'(z) + zf''(z)\right] \prec 1 + Mz, \quad for\ z \in U,$$

*holds true for some $M > 0$, then*

$$\frac{f(z)}{z} \prec 1 + \frac{2}{5}Mz, \ z \in U.$$

In the case when $\alpha = 1$ we have

COROLLARY 3.5. ([2]) *Let $f \in H$ and $\beta \geq 0$. If*

$$f'(z) + \beta z f''(z) \prec 1 + M z, \ z \in U,$$

*holds for some $M > 0$, then*

$$\frac{f(z)}{z} \prec 1 + \frac{M z}{2(\beta + 1)}, \ f'(z) \prec 1 + \frac{M z}{\beta + 1}, \ z \in U.$$

*If moreover $0 < M < \beta(\beta + 1)/(\sqrt{(\beta + 1)^2 + 1} + |\alpha - 1|)$ then $f \in S^c$.*

## REFERENCES


[1] D. J. Hallenbeck, S. Ruscheweyh, *Subordination by convex functions,* Proc. Amer. Math. Soc., **52**, 1975, 191-195.

[2] S. Kanas, J. Stankiewicz, *Arithmetic means with reference to convexity condition,* Bulletin de la Société des Sciences et des Letters de Łódź, vol. XLVII, 1997, 73-82.

[3] S.S. Miller, P.T. Mocanu, *The theory and applications of second-order differential subordinations,* Studia Univ. Babes-Bolyai, Mathematica, XXXIV, **4**, 1989, 3-32.

[4] S.S. Miller, P.T. Mocanu, *Second order differential inequalities in the complex plane,* J. Math. Anal. Appl. 65 **2**, 1978, 289-305.

[5] P. T. Mocanu, *Some starlikeness condition for analytic functions* Rev. Roumaine Math. Pures Appl., **33**, 1988, 117-124.

[6] P. T. Mocanu, *On a starlikeness condition,* Mathematica, **35(58)**, 1993, 175-178.

[7] S. Ponnusamy, *Differential subordinations concerning starlike functions,* Proc. of Indian Acad. Sciences, (Math. Sci.), **104**, 1994, 397-411.

[8] S. Ponnusamy, S. Rajasekaran, *New sufficient conditions for starlike and univalent functions,* J. of Math., **104**, 1995.