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Starlikeness of Libera transformation II

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1. Introduction.

Let A denote the class of function

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

that are analytic in $E=\{z:|z|<1\}$.

A function $f(z)\in A$ is called to be starlike if and only if

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0 \qquad in \quad E.$$

Similarly, $f(z)\in A$ is called to be convex if and only if

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}>0 \qquad in \quad E.$$

The following interesting results are due to Libera [2].

Theorem A. If $f(z)$ is starlike in E, then so does the function $F(z)$, defined by

$$F(z)=\frac{2}{z}\int_0^z f(t)dt.$$

Theorem B. If $f(z)$ is convex in E, then so does the function $F(z)$, defined by

$$F(z)=\frac{2}{z}\int_0^z f(t)dt.$$

On the other hand, S.Singh and R.Singh [5, Theorem 1] and [4, Theorem 1] proved the following Theorem C and D and Nunokawa [3] proved Theorem E.

Theorem C. If $f(z)\in A$ and $\mathrm{Re}f'(z)>0$ in E, then the function

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt,$$

is starlike in E for all c, -1 $<$ c $\leq$ 0.

Theorem D. If $f(z) \in A$ and $| zf''(z)/f'(z) | < 3/2$ in E, then the function

$$F(z) = \frac{2}{z} \int_0^z f(t)dt.$$

is convex.

Theorem E. Let $f(z) \in A$ and suppose that

$$\mathrm{Re} f'(z) > \frac{1}{12}(\log \frac{4}{e})(\tan^2 \alpha^* \frac{\pi}{2} - 3) \quad in \quad E.$$

where

$$-0.01759 < \frac{1}{12}(\log \frac{4}{e})(\tan^2 \alpha^* \frac{\pi}{2} - 3) < -0.01751.$$

Then $F(z)$ is starlike in E, where

$$F(z) = \frac{2}{z} \int_0^z f(t)dt.$$

2. Prliminary.

Lemma. Let $f(z) \in A$ and $f(z) \neq 0$ in $0 < |z| < 1$ and suppose that

$$|\arg f'(z)| < \frac{\pi}{2}\alpha \quad in \quad E$$

where

$$\alpha = \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta}{2} \quad and \quad \beta > 0.$$

Then we have

$$|\arg g'(z)| < \frac{\pi}{2}\alpha \quad in \quad E$$

and

$$|\arg \frac{g(z)}{z}| < \frac{\pi}{2}\gamma \quad in \quad E,$$

where

$$0 < \gamma, \quad \beta = \gamma + \frac{2}{\pi} \tan^{-1} \gamma$$

and

$$g(z) = \frac{2}{z} \int_0^z f(t)dt.$$

Proof. From the hypothesis, it follows that

$$f(z) = g(z) + zg'(z),$$

$$f'(z) = 2g'(z) + zg''(z)$$

and so

$$\arg f'(z) = \arg g'(z) + \arg(2 + \frac{zg''(z)}{g'(z)}).$$

Now, if there exists a point $z_0 \in E$ such that

$$|\arg g'(z)| < \frac{\pi}{2}\beta \quad for \quad |z| < |z_0|$$

and

$$|\arg g(z_0)| = \frac{\pi}{2}\beta$$

Then, from [3, Lemma], we have

$$\frac{z_0 g''(z_0)}{g'(z_0)} = i\beta k$$

where

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \quad when \quad \arg g'(z_0) = \frac{\pi}{2}\beta$$

and

$$k \leq -\frac{1}{2}(a + \frac{1}{a}) \quad when \quad \arg g'(z_0) = -\frac{\pi}{2}\beta$$

where

$$p(z_0)^{\frac{1}{\beta}} = \pm ia, \qquad a > 0.$$

If $\arg g'(z_0) = \pi\beta/2$, then we have

$$\arg f'(z_0) = \arg g'(z_0) + \arg(2 + \frac{zg''(z_0)}{g'(z_0)})$$

$$= \frac{\pi}{2}\beta + \arg(2 + i\beta k)$$

$$\geq \frac{\pi}{2}(\beta + \frac{2}{\pi}\tan^{-1}\frac{\beta}{2}).$$

This contradicts the hypothesis. On the other hand, if $\arg g'(z_0) = -\pi\beta/2$, then we have

$$\begin{aligned}\arg f'(z_0) &= -\frac{\pi}{2}\beta + \arg(2+i\beta k)\\ &\leq -\frac{\pi}{2} - \tan^{-1}\frac{\beta}{2}\\ &= -\frac{\pi}{2}(\beta + \frac{2}{\pi}\tan^{-1}\frac{\beta}{2}).\end{aligned}$$

This also contradicts the hypothesis. Therefore we must have

$$|\arg g(z)| < \frac{\pi}{2}\beta \quad in \quad E.$$

Putting

$$p(z) = \frac{g(z)}{z}, \quad p(0) = 1,$$

then we have

$$g'(z) = p(z) + zp'(z)$$

and

$$\arg g'(z) = \arg p(z) + \arg(1 + \frac{zp'(z)}{p(z)}).$$

If there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi}{2}\gamma \quad for \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\gamma,$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\gamma k$$

where

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \quad when \quad \arg p(z_0) = \frac{\pi}{2}\gamma$$

and

$$k \leq -\frac{1}{2}(a + \frac{1}{a}) \quad when \quad \arg p(z_0) = -\frac{\pi}{2}\gamma$$

where

$$p(z_0)^{\frac{1}{\gamma}} = \pm ia, \qquad a > 0.$$

Then, applying the same methed as the above, we have

$$|\arg g'(z_0)| < \frac{\pi}{2}(\gamma + \frac{2}{\pi}\tan^{-1}\gamma).$$

This contradicts the hypothesis. Therefore we must have

$$|\arg \frac{g(z)}{z}| < \frac{\pi}{2}\gamma \quad in \quad E.$$

3. Main result.

Theorem. Let $f(z) \in A$, $f(z) \neq 0$ in $0 < |z| < 1$,

(1) $$|\arg f'(z)| < \frac{\pi}{2}(\frac{5}{3} - \gamma_0) \quad in \quad E$$

where γ_0 is the smallest positive root of the equation

$$\frac{5}{6} = 2\gamma + \frac{2}{\pi}\tan^{-1}\gamma + \frac{2}{\pi}\tan^{-1}\frac{1}{2}(\gamma + \frac{2}{\pi}\tan^{-1}\gamma)$$

and $0.266 < \gamma_0 < 0.267$.

Let us put

(2) $$g(z) = \frac{2}{z}\int_0^z f(t)dt.$$

Then $g(z)$ is starlike in E.

Proof. From (1) and Lemma, we easily have

$$|\arg \frac{g(z)}{z}| < \frac{\pi}{2}\gamma_0 \quad in \quad E.$$

From(2), we have

$$2f(z) = g(z) + zg'(z)$$

and

$$2f'(z) = 2g'(z) + zg''(z)$$

Let us put

$$\frac{zg'(z)}{g(z)} = \frac{1 + w(z)}{1 - w(z)}, \qquad w(0) = 0$$

then $w(z)$ is analytic in E and $w(z) \neq 1$ in E.

Then it follows that

$$2f'(z) = 2g'(z) + zg''(z)$$

$$= \frac{zg'(z)}{g(z)}[(\frac{1+w(z)}{1-w(z)})^2 + \frac{2zw'(z)}{(1-w(z))^2} + \frac{1+w(z)}{1-w(z)}].$$

If there exists a point $z_0 \in E$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1,$$

then from Jack's lemma [1 ,Lemma], we have

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \geq 1$$

Putting $w(z_0) = e^{i\theta}$, $0 \leq \theta < 2\pi$, we have

$$\begin{aligned} 2f'(z_0) &= 2g'(z_0) + z_0 g''(z_0) \\ &= \frac{g(z_0)}{z_0}[(\frac{1+e^{i\theta}}{1-e^{i\theta}})^2 + \frac{2ke^{i\theta}}{(1-e^{i\theta})^2} + \frac{1+e^{i\theta}}{1-e^{i\theta}}] \\ &= \frac{g(z_0)}{z_0}[-\frac{\sin^2\theta}{(1-\cos\theta)^2} - \frac{k}{1-\cos\theta} + \frac{i\sin\theta}{1-\cos\theta}]. \end{aligned}$$

Then we have

$$\begin{aligned} \arg f'(z_0) &\geq \arg(-\frac{\sin^2\theta + k(1-\cos\theta)}{(1-\cos\theta)^2} - \frac{i\sin(1-\cos\theta)}{(1-\cos\theta)^2}) - |arg(\frac{g(z_0)}{z_0})| \\ &\geq \arg(-\frac{sin^2\theta + 1 - \cos\theta}{(1-\cos\theta)^2} - \frac{i\sin\theta(1-\cos\theta)}{(1-\cos\theta)^2}) - \frac{\pi}{2}\gamma_0 \\ &= \pi - \tan^{-1}\frac{|\sin\theta|(1-\cos\theta)}{\sin^2\theta + 1 - \cos\theta} - \frac{\pi}{2}\gamma_0 \\ &= \pi - \tan^{-1}\frac{|\sin\theta|}{2+\cos\theta} - \frac{\pi}{2}\gamma_0 \\ &\geq \pi - \tan^{-1}\frac{1}{\sqrt{3}} - \frac{\pi}{2}\gamma_0 \\ &= \frac{5}{6}\pi - \frac{\pi}{2}\gamma_0 \\ &= \frac{\pi}{2}(\frac{5}{3} - \gamma_0). \end{aligned}$$

This contradicts (1). Therefore we must have

$$|w(z)| < 1 \quad in \quad E.$$

This show that

$$Re\frac{zg'(z)}{g(z)} > 0 \quad in \quad E.$$

or $g(z)$ is starlike in E.

References

[1] I. S. Jack, Functions starlike and convx of order α. J. London Math. Soc., 3, 469-474(1971).

[2] R. J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16, 755-758(1965).

[3] M. Nunokawa, On Starlikeness of Libera Transformation, Complex Variables, 17, 79-83(1991).

[4] R. Singh and S. Singh, Starlikeness and convexity of certain integrals, Ann Univ. Mariac Curie-Sklodowska Sect. A 35, 45-47(1981).

[5] R. Singh and S. Singh, Starlikeness of close-to-convex functions. Indian J. Pure Appl. Math., 13, 190-194(1982).

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