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On Modular Invariance Equations

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Modular invariance equations for Bose-Mesner algebras were introduced by Eiichi Bannai [1] in connection with fusion algebras in conformal field theory.

Modular invariance equations are deeply related to spin models. Some spin models have been constructed from solutions of modular invariance equations [2, 3, 5]. Basic theory of modular invariance equations was given by Eiichi Bannai, Etsuko Bannai and François Jaeger [4]. Recently it turned out that every spin model can be obtained from a solution of the modular invariance equation of some self-dual Bose-Mesner algebra [6, 8, 7].

Here we consider the modular invariance equation in the case $d = 3$. We will give formulas which represent the entries of P by three parameters.

Let $\mathcal{A}$ be a Bose-Mesner algebra of a symmetric association scheme of class d on a finite set X, $|X| = n$. We assume that $\mathcal{A}$ is self-dual so that the eigenmatrix P satisfies $P^2 = nI$. The *modular invariance equation* for $\mathcal{A}$ takes the form:

$$(PT)^3 = t_0(\sqrt{n})^3 I, \tag{1}$$

where $T = \mathrm{diag}[t_0, t_1, \ldots, t_d]$ with $d+1$ unknowns t_0, t_1, $\ldots$, t_d.

First we represent the entries of P by four parameters for any self-dual Bose-Mesner algebra.

Proposition 1 *Let P denote the eigenmatrix of a Bose-Mesner algebra such that $P^2 = nI$. Then the entries of P are represented by four parameters $\theta_i := P_{i1}$ $(i = 0, 1, 2, 3)$ as follows.*

$$P = \begin{pmatrix} 1 & \theta_0 & k_2 & k_3 \\ 1 & \theta_1 & (k_2/k_1)\theta_2 & (k_3/k_1)\theta \\ 1 & \theta_2 & a & (k_3/k_2)c \\ 1 & \theta_3 & c & b \end{pmatrix},$$

where

$$\begin{aligned} k_2 &= \frac{\theta_0(1+\theta_1)(\theta_0 - \theta_3 + \theta_3(\theta_1 - \theta_3))}{(\theta_3 - \theta_2)(\theta_0 + \theta_2\theta_3)}, \\ k_3 &= \frac{\theta_0(1+\theta_1)(\theta_0 - \theta_2 + \theta_2(\theta_1 - \theta_2))}{(\theta_2 - \theta_3)(\theta_0 + \theta_2\theta_3)}, \\ a &= \frac{\theta_0 - \theta_3 + \theta_2(\theta_1 - \theta_3)}{\theta_3 - \theta_2}, \\ b &= \frac{\theta_0 - \theta_2 + \theta_3(\theta_1 - \theta_2)}{\theta_2 - \theta_3}, \\ c &= \frac{\theta_0 - \theta_3 + \theta_3(\theta_1 - \theta_3)}{\theta_3 - \theta_2}, \end{aligned}$$

unless the denominators are not zero.

Conversely, the matrix given by the above satisfies $P^2 = nI$ for any θ_i $(i = 0, 1, 2, 3)$ unless the denominators are nonzero.

Next suppose that there exists a diagonal matrix T with diagonal entries t_i $(i = 0, 1, 2, 3)$ which satisfies the modular invariance equation (1).

Proposition 2 *Set* $s = t_1 t_2 t_3$ *and*

$$L = s(t_1^{-1} + t_2^{-1} + t_3^{-1}) - s^{-1}(t_1 + t_2 + t_3).$$

Then

$$\begin{aligned}
t_0 &= s^3, \\
\sqrt{n} &= \frac{(s^3 - t_1)(s^3 - t_2)(s^3 - t_3)}{s^5 L}. \\
P_{01} &= \frac{t_1(s^2 - t_1^2)(s^4 - 1)(s^3 - t_2)(s^3 - t_3)(s^3 t_2^3 + 1)(s^3 t_3^3 + 1)}{s^9 (t_1 - t_2)(t_1 - t_3)(s t_2 + 1)(s t_3 + 1) L^2}, \\
P_{11} &= \frac{t_1(s^3 - t_2)(s^3 - t_3)((s^2 - s^{-2}) + (s t_1^{-1} - s^{-1} t_1) - L)}{s^3 (t_1 - t_2)(t_1 - t_3) L}, \\
P_{12} &= \frac{(s^3 - t_1)(s^3 - t_3)(s^2 - t_2^2)(s^3 t_1^3 + 1)}{t_1 s^5 (t_1 - t_2)(t_3 - t_2)(s t_1 + 1) L}.
\end{aligned}$$

P_{0i} $(i \in \{2, 3\})$ are obtained by exchanging t_1 and t_i in P_{01}. P_{ii} $(i \in \{2, 3\})$ are obtained by exchanging t_1 and t_i in P_{11}. For P_{ij} $(i, j \in \{1, 2, 3\}$ and $i \neq j)$, put $\{i, j, k\} = \{1, 2, 3\}$, then P_{ij} is obtaind by permutating $t_1 \to t_i$, $t_2 \to t_j$, $t_3 \to t_k$ in P_{12}.

Conversely, the matrix P with above entries satisfies $P^2 = nI$ and $(PT)^3 = t_0(\sqrt{n})^3 I$ for any non-zero value of t_i $(i = 1, 2, 3)$ unless the denominators are nonzero.

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