数理解析研究所講究録
1065 巻 1998 年 108-123

The canonical decompositions of some family of compact orientable hyperbolic 3-manifolds with totally geodesic boundary

AKIRA USHIJIMA

This article is a summary of [Us].

1 Introduction

D. B. A. Epstein and R. C. Penner proved in [EP] that every noncompact complete hyperbolic manifold of finite volume admits a canonical decomposition by ideal polyhedra. S. Kojima [Ko1, Ko2] extended the result to the case of complete hyperbolic manifolds of finite volume with non-empty totally geodesic boundary. In fact, he proved that such a manifold is decomposed by partially truncated polyhedra.

For each pair of integers n and k such that $n \geq 3$, $0 \leq k \leq n-1$ and that the greatest common divisor $\gcd(n, 2-k)$ of n and $2-k$ is 1, L. Paoluzzi and B. Zimmermann [PZ] constructed a compact orientable hyperbolic 3-manifold $M_{n,k}$ with totally geodesic boundary (a surface of genus $n-1$) by identifying several faces of a certain hyperbolic polyhedron $\mathcal{P}_n$. Our main purpose is to determine the canonical decomposition $\mathcal{D}_{n,k}$ of $M_{n,k}$ (Theorem 2.2). This gives an alternative proof of the classification theorem of $M_{n,k}$ (Corollary 2.3), which was shown in [PZ], and to determine the isometry group of $M_{n,k}$ (Theorem 2.4).

The manifold $M_{n,1}$ is homeomorphic to the exterior of *Suzuki's Brunnian graph* θ_n of order n (see [Sc, Su] and Figure 1). Hence, the results mentioned above give an alternative proof of the non-triviality of θ_n and lead us to the chirality of θ_n (Corollary 2.6). Furthermore, we determine the symmetry groups of these graphs (Corollary 2.5).

S. Kinoshita and K. Wolcott [Wo] proved that θ_3 is chiral in the 3-sphere S^3, i.e., there does not exist an orientation-reversing homeomorphism of S^3

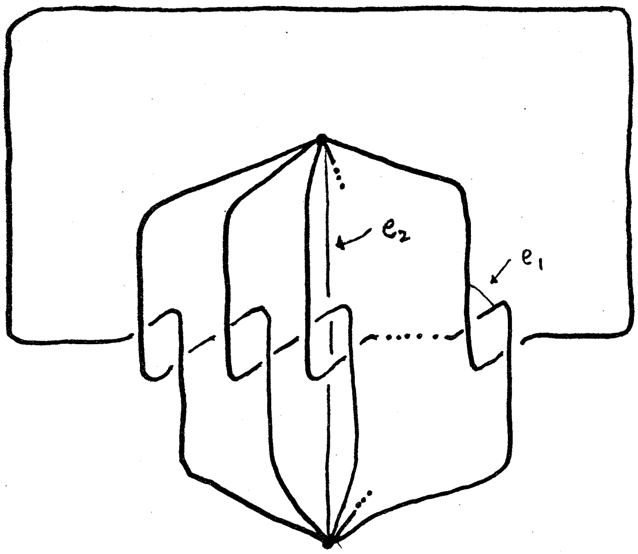

Figure 1: Suzuki's Brunnian graph

that preserves θ_3. Furthermore, S. Kinoshita proposed the following problem (see [KG, Problem 2.3]):

Problem 1.1 *Let F be the boundary of the regular neighborhood of θ_3. Then, is the pair (S^3, F) chiral, i.e., is it true that S^3 does not admit an orientation-reversing homeomorphism which preserves the surface F?*

We give an affirmative answer to this problem (Corollary 2.6).

Finally, we present some observation concerning the Heegaard splittings of $M_{n,k}$ (Theorem 2.8), which supports the conjecture proposed by [Ad, ANS, SW].

The author would like to thank Professor Katsuo Kawakubo for his encouragement. The author also expresses his sincere gratitude to Professor Makoto Sakuma for his helpful comments and advice. This paper grew out of the discussion with him. The author would also like to thank Professor Andrei Vesnin for his comments and advice, especially about the volume of hyperbolic manifolds.

2 Statement of results

Throughout this paper, $N(X)$ denotes the regular neighborhood of X, $\operatorname{int} X$ denotes the interior of X and ∂X denotes the boundary of X.

First of all, we recall the topological construction of the manifold $M_{n,k}$. We start with the polyhedron $\mathcal{CP}_n$; it is a solid double pyramid (or a solid double cone) whose base is a regular n-gon $\mathcal{G}_n$. Figure 3 shows the surface of $\mathcal{CP}_n$ flattened out on the plane where the bottom cone point is at infinity. We imagine that the solid occupies the half-space behind the page. We identify the faces of $\mathcal{CP}_n$ in pairs as indicated in Figure 3: for the fixed integer k such that $0 \leq k \leq n-1$, the face $a_i\, b_{i+1}\, b_i$ gets identified with the face $c_{i+k}\, a_{i+k}\, c_{i+k+1}$ by a transformation (homeomorphisms of faces). These identifications also induce those of the polyhedron $\mathcal{P}_n$, where $\mathcal{P}_n$ denotes the polyhedron $\mathcal{CP}_n$ truncated at all vertices (see Figure 2), and we denote this resulting identification space by $M_{n,k}$.

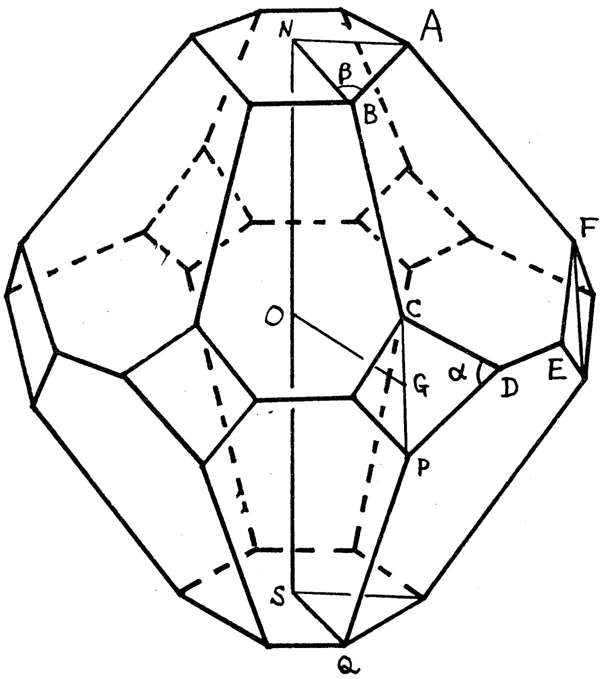

Figure 2: The truncated polyhedron $\mathcal{P}_6$

We call a face of $\mathcal{P}_n$ consisting the boundary of $M_{n,k}$ *external* (*internal* otherwise); we also call an edge of an external face *external* (*internal* otherwise). We note that the faces obtained by truncations are external and that the faces induced by that of $\mathcal{CP}_n$ are internal.

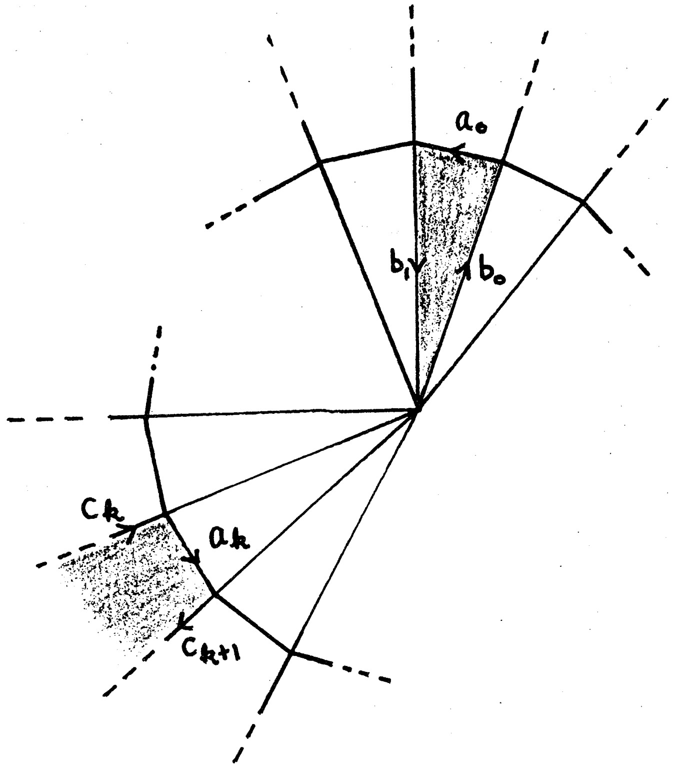

Figure 3: The identification of faces of $\mathcal{CP}_n$

We assume that $\gcd(n, 2-k) = 1$ in the following; then all edges (resp. vertices) of $\mathcal{CP}_n$ become identified to a single edge (resp. vertex). Now $M_{n,k}$ is a compact orientable 3-manifold whose boundary is a closed orientable surface of genus $n-1$.

The following Theorem 2.1 was proved in [PZ] by using Andreev's theorem;

Theorem 2.1 *For each pair of integers n and k such that $n \geq 3$, $0 \leq k \leq n-1$ and that $\gcd(n, 2-k) = 1$, the manifold $M_{n,k}$ admits a hyperbolic structure such that its boundary $\partial M_{n,k}$ is totally geodesic.*

We note that (the mirror image of) $M_{3,1}$ is the hyperbolic manifold constructed by W. P. Thurston ([Th, Example 3.3.12]), and when $n = 3$, $\mathrm{Vol}(M_{3,k}) = 6.45198979\cdots$. This value coincides with the smallest volume among all compact hyperbolic 3-manifolds with totally geodesic boundary shown by S. Kojima and Y. Miyamoto in [KM]. Determining the shape of $\mathcal{P}_n$ concretely, we give a brief review of a proof of this theorem in Section 4.

S. Kojima proved in [Ko1, Ko2] that every complete hyperbolic manifold of finite volume with non-empty totally geodesic boundary admits a canonical decomposition by partially truncated polyhedra. It is isotopic to a dual to the cut locus of the boundary. The cut locus is a subset in the manifold which consists of the points that admit at least two distinct shortest paths to the boundary. In Section 5, we prove Theorem 2.2 below, which is the main theorem of this paper and describes the canonical decomposition of $M_{n,k}$. This is the main theorem of this paper. To present Theorem 2.2, we suppose $\mathcal{CP}_n$ is in the Euclidean 3-space $\mathbf{R}^3$, $\mathcal{G}_n$ is on the x-y plane and that two cone points are on the z-axis.

Theorem 2.2 *The canonical decomposition $\mathcal{D}_{n,k}$ of the manifold $M_{n,k}$, where $n \geq 3$, $0 \leq k \leq n-1$ and $\gcd(n, 2-k) = 1$, is given as follows;*

(1) Suppose $n = 3$. Let Δ_0 and Δ_1 be the truncated tetrahedra obtained from $\mathcal{P}_3$ by cutting along the x-y plane. Then $\mathcal{D}_{3,k}$ consists of Δ_0 and Δ_1.

(2) Suppose $n = 4$. Then $\mathcal{D}_{4,k}$ consists of $\mathcal{P}_4$.

(3) Suppose $n \geq 5$. Let $\Delta_0, \Delta_1, \ldots, \Delta_{n-1}$ be the truncated tetrahedra obtained from $\mathcal{P}_n$ by slicing by half planes, each of which is bounded by the z-axis and contains a vertex of $\mathcal{G}_n$. Then $\mathcal{D}_{n,k}$ consists of $\Delta_0, \Delta_1, \ldots, \Delta_{n-1}$.

We give a shorten proof of this theorem in Section 5.

Using Theorem 2.2, we can solve the classification problem of $M_{n,k}$. In fact, by virtue of Mostow rigidity theorem (cf. [PZ, Proposition 2]) combined with the canonical decomposition, two manifolds $M_{n,k}$ and $M_{n',k'}$ are homeomorphic (or equivalently, isometric) if and only if their canonical decompositions have the same combinatorial structures. Thus we can recover the following classification theorem of $M_{n,k}$ established by [PZ];

Corollary 2.3 *The manifold $M_{n,k}$ and $M_{n',k'}$ are homeomorphic (or equivalently, isometric) if and only if $n = n'$ and $k = k'$.* □

Again using Theorem 2.2, we can also determine the isometry group of $M_{n,k}$. Let $Isom(M_{n,k})$ be the isometry group of $M_{n,k}$, $Aut(\mathcal{D}_{n,k})$ the combinatorial automorphism group of $\mathcal{D}_{n,k}$ and $\mathcal{M}(M_{n,k})$ the mapping class group of $M_{n,k}$, that is, the group consisting of the self-homeomorphisms of $M_{n,k}$ modulo isotopy. Again, by virtue of Mostow rigidity theorem combined with the canonical decomposition, $\mathcal{M}(M_{n,k}) \cong Isom(M_{n,k}) \cong Aut(\mathcal{D}_{n,k})$. Thus, calculating $Aut(\mathcal{D}_{n,k})$ we can determine $Isom(M_{n,k})$ and $\mathcal{M}(M_{n,k})$.

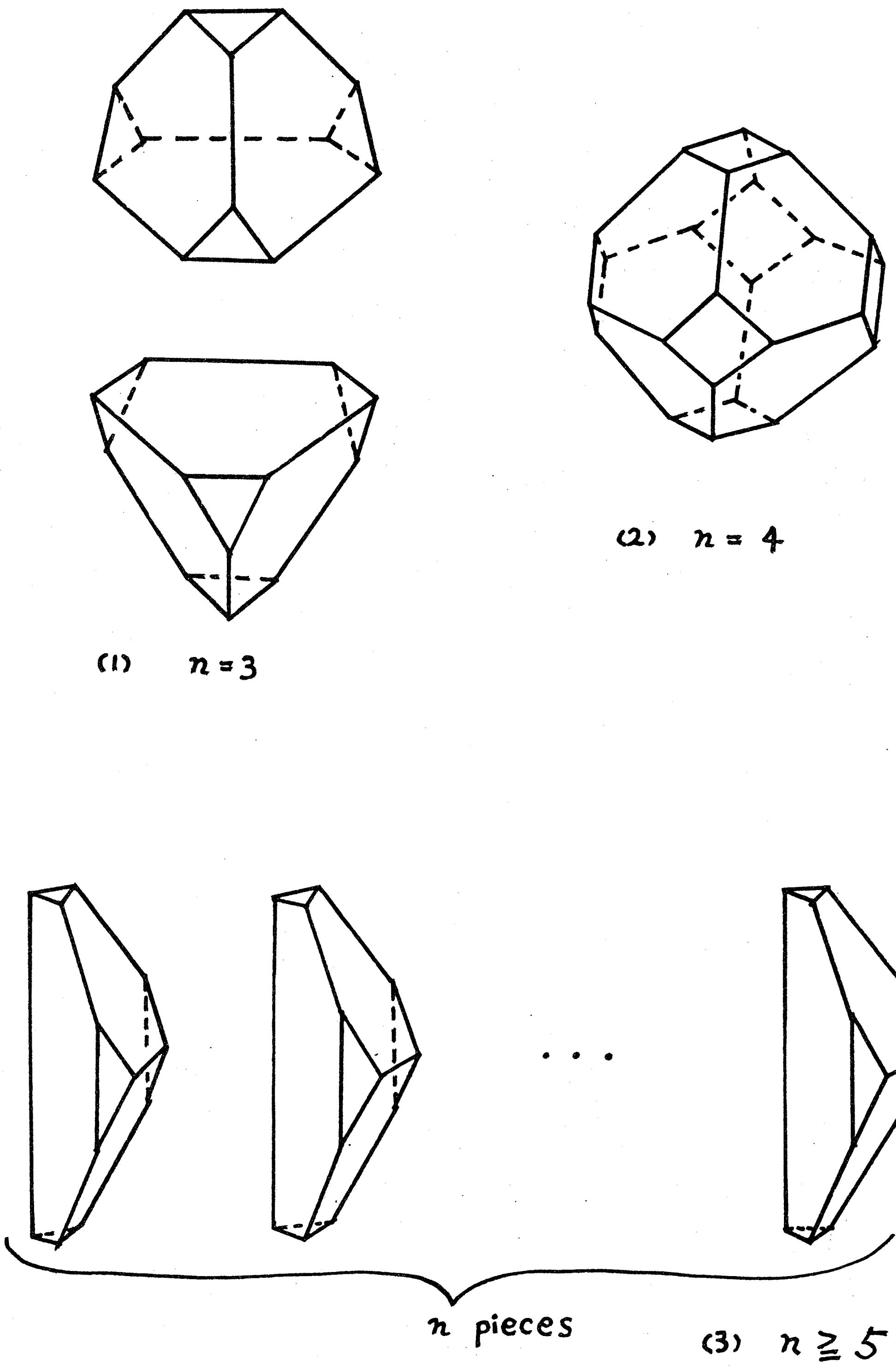

Figure 4: The canonical decomposition $\mathcal{D}_{n,k}$

Theorem 2.4 *For each pair of integers n and k such that $n \geq 3$, $0 \leq k \leq n-1$ and that* $\gcd(n, 2-k) = 1$, *$Isom(M_{n,k})$ (and hence $\mathcal{M}(M_{n,k})$) is isomorphic to D_{2n}, where D_{2n} is the dihedral group of order $2n$, i.e.,*

$$D_{2n} = \left\langle t, r \,|\, t^n = 1, r^2 = 1, r\,t\,r\,t = 1 \right\rangle .$$

Here, t (resp. r) is induced by the $\frac{2\pi}{n}$-rotation of $\mathcal{G}_n$, i.e., the $\frac{2\pi}{n}$-rotation of $\mathcal{P}_n$ around the geodesic arc joining the top and the bottom cone points of $\mathcal{CP}_n$ (resp. a reflection about a line through the origin and a vertex of $\mathcal{G}_n$). □

A graph G embedded in S^3 is called a *spatial θ-curve (of order n)* if it has two vertices and n edges connecting one vertex to the other, and G is said to be *trivial* if it is contained in a 2-sphere in S^3. S. Suzuki introduced a certain family θ_n of spatial θ-curves in [Su] (see Figure 1), which are called *Suzuki's Brunnian graphs* or the *Suzuki's θ_n-curves* (cf. [Sc]). These spatial θ-curves are generalizations of Kinoshita's theta-curve introduced in [Ki], and satisfy the following interesting property: θ_n is not trivial, though every proper subgraph of θ_n is trivial. When $n \geq 3$, this property is obtained from the fact that the handlebody does not admit a hyperbolic structure with totally geodesic boundary. We note that this is an alternative proof of [Sc, Su]. It is shown in [Th, pp. 136–137] that the exterior $E(\theta_3) = S^3 - \operatorname{int} N(\theta_3)$ is homeomorphic to $M_{3,1}$. Using the same method, we can see that the exterior $E(\theta_n)$ of θ_n is homeomorphic to $M_{n,1}$. Since the the exterior $E(\theta_n)$ of θ_n admits a hyperbolic structure such that its boundary $\partial N(\theta_n)$ is totally geodesic, in the same fashion as hyperbolic knots and links, we may say that θ_n is a "hyperbolic" graph. Let $Sym(S^3, \theta_n)$ be the symmetry group of (S^3, θ_n), i.e., the group consisting of the self-homeomorphisms of (S^3, θ_n) modulo pairwise isotopy. Using Theorem 2.4, we can determine $Sym(S^3, \theta_n)$;

Corollary 2.5 *For each $n \geq 3$, $Sym(S^3, \theta_n) \cong D_{2n}$.* □

A graph G (resp. a surface F) in S^3 is *chiral* if there does not exist an orientation-reversing homeomorphism of S^3 that fixes G (resp. F). It has been shown by S. Kinoshita and K. Wolcott [Wo] that θ_3 is chiral. Corollary 2.5 leads us to the following corollary, which generalizes this result and also gives an affirmative answer to Problem 1.1 proposed by S. Kinoshita.

Corollary 2.6 *Both the graph θ_n and the surface $\partial N(\theta_n)$ are chiral for every $n \geq 3$.* □

An *unknotting tunnel* for a compact 3-manifold M is an arc, say τ, properly embedded in M such that $M - \operatorname{int} N(\tau)$ is a handlebody. Two unknotting tunnels τ_1 and τ_2 for M are said to be *isotopic* (resp. *homeomorphic*), if there is an ambient isotopy (resp. homeomorphism) of M carrying τ_1 to τ_2. For a cusped hyperbolic 3-manifold, the following problem is proposed by [Ad, ANS, SW]:

Problem 2.7 *Is an unknotting tunnel for a cusped hyperbolic 3-manifold isotopic to a vertical geodesic, or even to an edge of the canonical decomposition? Is it short?*

Here, a *vertical geodesic* is a geodesic which is perpendicular to the cusp at each of its ends. This problem leads us to the following natural generalization: is an unknotting tunnel for a compact hyperbolic 3-manifold isotopic to a vertical geodesic, or even to an edge of the canonical decomposition? Is it short? Here, a *vertical geodesic* is a geodesic which is perpendicular to the boundary at each of its ends. The following theorem gives the answer to this problem for the manifold $M_{n,k}$. Let e_1 be the geodesic arc in $M_{n,k}$ induced by internal edges of $\mathcal{P}_n$, and e_2 the geodesic arc in $M_{n,k}$ induced by the geodesic arc in $\mathcal{CP}_n$ joining the top and the bottom cone points through its center.

Theorem 2.8 *(1) The geodesic arc e_1 is the only unknotting tunnel for $M_{n,k}$ up to isotopy.*

(2) The geodesic arc e_1 is the shortest vertical geodesic. □

The first statement of this theorem is proved by using [He, Main Theorem 1.5], and the second one is obtained by evaluating the lengths like Section 4.

3 Hyperbolic geometry

We give a brief review of the Minkowski space $\mathbf{E}^{1,n}$. It is the real vector space $\mathbf{R}^{n+1}$ with the inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n$. The set $H^+ = \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0 \}$ forms the *hyperboloid model* (or the *Minkowski space model*) of the hyperbolic n-space $\mathbf{H}^n$. A ray from the origin in the positive light cone $L^+ = \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0, x_0 > 0 \}$ corresponds to a point on the ideal boundary of $\mathbf{H}^n$. The set of such rays forms the sphere at infinity and we denote it by S^{n-1}_∞. Let $H_s = \{ \boldsymbol{x} \in \mathbf{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \}$ be the hyperboloid of one sheet. We call an affine hypersubspace S in $\mathbf{E}^{1,n}$

a *hyperplane*, and a hyperplane through the origin *linear*. Suppose S is a linear hyperplane which contains a time-like vector, and let S^+ be a half space bounded by S. Then we can associate a unique unit vector $\boldsymbol{q} \in H_s$ so that $\langle \boldsymbol{q}, \boldsymbol{v} \rangle \leq 0$ for arbitrary $\boldsymbol{v} \in S^+$. This establishes a duality between half spaces in $\mathbf{E}^{1,n}$ and points on H_s.

Let us denote by Π the projection from $\mathbf{E}^{1,n}$ to the compactification of the plane $\Lambda = \{x_0 = 1\}$, by adding the set of lines on $\{x_0 = 0\}$, along the ray from the origin. The projection Π is a homeomorphism on H^+ to the open unit sphere P^n on Λ, which is the *projective model* of $\mathbf{H}^n$, and ∂P^n is canonically identified with $\mathrm{S}^{n-1}_{\infty}$. Then the projection Π of H_s to the compactified space $\overline{\Lambda}$ becomes precisely a two-to-one map. Suppose a linear hyperplane S intersects P^n, then the dual vector $\boldsymbol{q} \in H_s$ to the half space S^+ is projected to a point on $\overline{\Lambda}$ so that a cone from $\Pi(\boldsymbol{q})$ to $S \cap \partial P^n$ is tangent to ∂P^n. Thus $\Pi(S) = S \cap \Lambda$ becomes a polar hyperplane to $\Pi(\boldsymbol{q})$. The duality in $\mathbf{E}^{1,n}$ induces a duality between points on $\overline{\Lambda} - (P^n \cup \partial P^n)$ and polar hyperplanes in two-to-one manner.

Consider a compact Euclidean polyhedron $\mathcal{R}$ in Λ so that its vertices lie outside ∂P^n and that each edge meets ∂P^n. We can regard $\mathcal{R} \cap P^n$ as an ideal polyhedron in $\mathbf{H}^n$. For each vertex v of $\mathcal{R}$, we have its polar hyperplane L_v intersecting P^n. This plane intersects each face of $\mathcal{R}$ meeting the vertex in question perpendicularly. Truncating a neighborhood of each vertex by a polar hyperplane, we get a polyhedron in P^n. We call it a *truncated* polyhedron of $\mathcal{R}$.

4 Proof of Theorem 2.1

We want to realize $\mathcal{P}_n$ as a hyperbolic polyhedron in the hyperbolic 3-space $\mathbf{H}^3$ such that Poincaré's theorem on fundamental polyhedra ([Ma]) can be applied realizing $M_{n,k}$ as a compact orientable hyperbolic 3-manifold with totally geodesic boundary. To do this, we put $\mathcal{CP}_n$ in $\mathbf{R}^3$ as follows: its center sets at the origin, one of the vertices of $\mathcal{G}_n$ at $(r, 0, 0)$ and the top (resp. bottom) cone point of $\mathcal{CP}_n$ at $(0, 0, h)$ (resp. $(0, 0, -h)$).

We regard the interior of the unit ball B^3 centered at the origin as the projective model P^3 of $\mathbf{H}^3$. Then $\partial\mathrm{B}^3 = \mathrm{S}^2$ is the sphere at infinity ∂P^3 of P^3. Now we suppose that the vertices of $\mathcal{CP}_n$ lie outside ∂P^3 and that each edge of $\mathcal{CP}_n$ intersects ∂P^3 twice. Then we can truncate $\mathcal{CP}_n$ and obtain a compact truncated hyperbolic polyhedron $\mathcal{P}_n$. The assumption that each edge of $\mathcal{CP}_n$ intersects ∂P^3 twice means that all external faces are ultraparallel. Using half planes each of which is bounded by the z-axis and contains a vertex of $\mathcal{G}_n$, we divide $\mathcal{P}_n$ into n mutually isometric truncated tetrahedra $\mathcal{Q}_n$. Let $A, B, \dots$ be the vertices (or the points) of $\mathcal{Q}_n$ as in Figure 2. We denote by $l_E(AB)$ (resp. $l_H(AB)$) the Euclidean (resp. hyperbolic) length of the line AB.

The identifying transformations of $\mathcal{P}_n$ can be chosen as hyperbolic isometries in $\mathbf{H}^3$ if and only if the following equation holds:

$$l_H(BC) = l_H(DE). \tag{4.1}$$

Expressing $l_H(BC)$ and $l_H(DE)$ in r, h and c_n, we get the following relation;

$$h = h_n(r) = \frac{r\sqrt{c_n^2\, r^2 - 2\, c_n + 1}}{1 - c_n\, r^2}.$$

If this condition is satisfied, $M_{n,k}$ has a structure of hyperbolic 3-cone manifold. After this, we find the condition for $M_{n,k}$ to have a complete hyperbolic structure.

For $M_{n,k}$being hyperbolic 3-manifold, the angle along the single edge cycle have to sum up to 2π, so we get the following conditions:

$$\begin{aligned} & n \times \alpha + 2n \times 2\beta = 2\pi \\ \Longleftrightarrow\quad & \cos\alpha\left(2\cos^2 2\beta - 1\right) + 2\sin\alpha\sin 2\beta\cos 2\beta - c_n = 0. \end{aligned} \tag{4.2}$$

Expressing $\cos\alpha$ and $\cos\beta$ in r and c_n, we get the following relation; if $n = 4$, then

$$r = r(4) = \sqrt{4\sqrt{3} - 5},$$

and if $n \neq 4$, then

$$r = r(n) = \frac{1}{c_n\sqrt{6}}\sqrt{(1 + c_n)\, f_1(n) + \frac{(1 - c_n)\, f_2(n)}{\sqrt[3]{f_3(n)}} + (1 - c_n)\sqrt[3]{f_3(n)}},$$

where

$$\begin{aligned} c_n &= \cos\frac{2\pi}{n}, \\ f_1(n) &= 8\, c_n^3 - 12\, c_n^2 + 8\, c_n - 1, \\ f_2(n) &= 64\, c_n^6 + 64\, c_n^5 + 16\, c_n^4 - 80\, c_n^3 - 28\, c_n^2 - 12\, c_n + 1, \\ f_3(n) &= -512\, c_n^9 - 768\, c_n^8 - 384\, c_n^7 + 896\, c_n^6 \\ &\quad + 816\, c_n^5 + 312\, c_n^4 - 240\, c_n^3 - 12\, c_n^2 + 18\, c_n - 1 \\ &\quad + 24\sqrt{3}\, c_n^2\, (1 + c_n)\sqrt{1 - c_n}\sqrt{1 + c_n}\sqrt{32\, c_n^3 + 52\, c_n^2 + 44\, c_n - 3}. \end{aligned}$$

Thus we obtain $\mathcal{P}_n$ satisfying two equations (4.1) and (4.2). By Poincaré's theorem on fundamental polyhedra (see [Ma]), we get the compact orientable hyperbolic 3-manifold $M_{n,k}$ with non-empty totally geodesic boundary by gluing $\mathcal{P}_n$ as above. We have thus completed the proof of Theorem 2.1. □

5 The canonical decomposition

We first give a brief review of the canonical decomposition of a compact hyperbolic n-manifold M with non-empty totally geodesic boundary (cf. [Ko1, Ko2]). We regard the universal cover $\widetilde{M}$ of M as a subset of the hyperboloid model H^+. To each component of $\partial\widetilde{M}$, assign a label. To each component of $\partial\widetilde{M}$ labeled by α, we can associate a unique linear hyperplane S_α in $\mathbf{E}^{1,n}$ including it. The positive half space S_α^+ bounded by S_α will be the side containing $\widetilde{M}$. $\widetilde{M}$ is identified with the intersection of H^+ and $\bigcap_\alpha S_\alpha^+$. To each S_α, we associate a dual space-like vector $\boldsymbol{b}_\alpha \in H_s$ so that $\langle \boldsymbol{b}_\alpha, \boldsymbol{v} \rangle \leq 0$ for all $\boldsymbol{v} \in S_\alpha^+$. Let $\mathcal{A}$ be the set of dual vectors $\{\boldsymbol{b}_\alpha\}$ on H_s. Then $\mathcal{A}$ is invariant under the action of the covering transformation group. Let $\mathcal{H}_\mathcal{A}$ be the closed convex hull of $\mathcal{A}$ in $\mathbf{E}^{1,n}$. The projection $\Pi(\mathcal{H}_\mathcal{A})$ contains P^n ([Ko1, Lemma 4.3]), and the intersection of $\Pi(\mathcal{H}_\mathcal{A})$ with $\Pi(\widetilde{M})$ in Λ defines a $\pi_1(M)$-equivariant polyhedral decomposition on $\widetilde{M}$. It induces a truncated polyhedral decomposition of M ([Ko1, Theorem 4.8]), which we call the *canonical decomposition* of M.

We now recall an intrinsic construction of the canonical decomposition of M. The cut locus $\mathcal{C}$ of $\partial\widetilde{M}$ in $\widetilde{M}$ is a set of points in the interior of $\widetilde{M}$ each of which admits at least two distinct shortest paths to $\partial\widetilde{M}$. The cut locus $\mathcal{C}$ admits a locally finite cell complex structure with respect to its canonical stratification by number of shortest paths. To each vertex of $\mathcal{C}$, assign a label. To each vertex v_β of $\mathcal{C}$ labeled by β, we associate the components of $\partial\widetilde{M}$ closest to v_β. Let $\mathcal{A}^\beta$ be the subset of $\mathcal{A}$ which consists of dual points to these closest components. Then there is a unique elliptic hyperplane S_β so that $S_\beta \cap \mathcal{A} = \mathcal{A}^\beta$ and that S_β is an n-dimensional face of $\mathcal{H}_\mathcal{A}$. Let $\mathcal{H}^\beta$ be the closed convex hull of $\mathcal{A}^\beta$, i.e., $\mathcal{H}^\beta = S_\beta \cap \mathcal{H}_\mathcal{A}$, then the intersection of $\Pi\left(\mathcal{H}^\beta\right)$ with $\Pi(\widetilde{M})$ is a truncated polyhedron, which is a dual to v_β. The family $\left\{\Pi(\mathcal{H}^\beta \cap \widetilde{M})\right\}_\beta$ defines a $\pi_1(M)$-equivariant truncated polyhedral decomposition of M and it coincides with the canonical decomposition of M.

Next, we give a condition for a given truncated polyhedral decomposition $\mathcal{D}$ of M to be canonical. Let σ be a (n-dimensional) truncated polyhedron in $\mathcal{D}$ and $\widetilde{\sigma}$ a lift of σ to $\widetilde{M}$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ be the dual vectors in H_s

corresponding to the faces of $\widetilde{\sigma}$ each of which is included in $\partial\widetilde{M}$. Let $\widehat{\sigma}$ be the convex hull of $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$. We suppose the affine hull $A(\widehat{\sigma})$ of $\widehat{\sigma}$ is an elliptic hyperplane in $\mathbf{E}^{1,n}$. Then we can define the center $\boldsymbol{o}(\widetilde{\sigma})$ of $\widetilde{\sigma}$ as follows: let $\boldsymbol{p}$ be the unit normal to $A(\widehat{\sigma})$ in $\mathbf{E}^{1,n}$, i.e., $\langle \boldsymbol{p}, \boldsymbol{p} \rangle = -1$ and $\langle \boldsymbol{p}, \boldsymbol{x} - \boldsymbol{y} \rangle = 0$ for arbitrary $\boldsymbol{x}, \boldsymbol{y} \in A(\widehat{\sigma})$. Then $\boldsymbol{o}(\widetilde{\sigma}) = \boldsymbol{p} \in H^+$. It should be noted that $\boldsymbol{o}(\widetilde{\sigma})$ is not necessarily contained in $\widetilde{\sigma}$. If we choose a coordinate of $\mathbf{E}^{1,n}$ so that $\widehat{\sigma}$ lies in a horizontal plane $x_0 =$ constant, then $\boldsymbol{p} = (1, 0, \ldots, 0)$. From this fact, we can see that $\boldsymbol{o}(\widetilde{\sigma})$ is characterized by the property that its hyperbolic distance from the linear hyperplanes dual to the unit vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}$ are equal. This means that $\boldsymbol{o}(\widetilde{\sigma})$ corresponds to a vertex of the projective image of the cut locus $\mathcal{C}$ in case $\mathcal{D}$ is the canonical decomposition of M.

Proposition 5.1 *Suppose a truncated polyhedral decomposition $\mathcal{D}$ of M satisfies the following conditions:*

(1) The affine hull $A(\widehat{\sigma})$ of $\widehat{\sigma}$ corresponding to a truncated polyhedron σ in $\mathcal{D}$ is an elliptic hyperplane in $\mathbf{E}^{1,n}$.

(2) $\mathcal{D}$ is the coarsest truncated polyhedral decompositions among those of M having the same truncated polyhedra.

(3) The center $\boldsymbol{o}(\widetilde{\sigma})$ lies in the interior of $\widetilde{\sigma}$ for an arbitrary truncated polyhedron σ in $\mathcal{D}$.

Then $\mathcal{D}$ is the canonical decomposition of M. □

The proof of this proposition is the same as that of [SW, Proposition I.1.4] except for obvious modifications. We should note that there is a error in its proof. The final equation should be read $t_1/t_0 = (s_0 - s) / (s_1 - s)$.

We prepare some notations to prove Theorem 2.2. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ be the external faces of $\mathcal{P}_n$, each of which is obtained by truncating $\mathcal{CP}_n$ at a vertex of $\mathcal{G}_n$, and $\mathcal{T}$ (resp. $\mathcal{B}$) the external face obtained by truncating $\mathcal{CP}_n$ at the top (resp. bottom) cone point. We denote by $\mathcal{L}_n$ the "cut locus of the external faces of $\mathcal{P}_n$ in $\mathcal{P}_n$", that is, $\mathcal{L}_n$ is the set of points in $\mathcal{P}_n$ each of which admits at least two distinct shortest paths to its external faces. We call the vertices of $\mathcal{L}_n$ in int$\mathcal{P}_n$ "the vertices of $\mathcal{L}_n$". To prove Theorem 2.2, we need the following lemma

Lemma 5.2 *The vertices of $\mathcal{L}_n$ are as follows:*

(1) If $n = 3$, then $\mathcal{L}_3$ has two vertices v_T and v_B on the z-axis. The vertex v_T (resp. v_B) has four shortest paths to four external faces $\mathcal{T}$ (resp. $\mathcal{B}$), $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ (see Figure 5).

(2) If $n = 4$, then $\mathcal{L}_4$ has one vertex v_O at the center of $\mathcal{P}_n$. The vertex v_O has the shortest paths to all external faces (see Figure 6).

(3) If $n \geq 5$, then $\mathcal{L}_n$ has n vertices $v_0, v_1, \ldots, v_{n-1}$ on the x-y plane. The vertex v_i has four shortest paths to four external faces $\mathcal{T}, \mathcal{B}, \mathcal{F}_i$ and $\mathcal{F}_{i+1}$ (mod n) *(see Figure 7).*

Proof of Lemma 5.2. By the definition of the cut locus $\mathcal{L}_n$, we can understand it as follows: let the external faces of $\mathcal{P}_n$ expand at a constant speed until they collide with themselves. Then the collision locus is equal to the cut locus $\mathcal{L}_n$. Using this interpretation, we now determine $\mathcal{L}_n$.

Case 1. $n = 3$. Since

$$r^2 - h_3^2(r) = \frac{r^2\,(r^2-1)\,(r^4+4)}{(r^2+2)^2} > 0\,,$$

we have $r > h_3(r)$. This means that $l_H(OG) < l_H(ON) = l_H(OS)$. By the symmetry of $\mathcal{P}_3$, when the external faces expand, first $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ arrive at the center O simultaneously. Hence, again by the symmetry of $\mathcal{P}_3$, a small neighborhood of O in the z-axis is contained in a collision locus by the external faces. Therefore $\mathcal{L}_3$ has precisely two vertices (Figure 5 shows $\mathcal{L}_3$ restricted to $\mathcal{Q}_3$). We have thus proved the case (1).

Case 2. $n = 4$. Since $h_4(r) = r$, we have $l_H(OG) = l_H(ON) = l_H(OS)$. Hence O is the unique vector of $\mathcal{L}_4$ (see Figure 6), and we have proved the case (2).

Case 3. $n \geq 5$. Since

$$h_n(r)^2 - r^2 = \frac{c_n\,r^2\,(r^2-1)\,(2-c_n\,r^2)}{(c_n\,r^2-1)^2} > 0\,.$$

we have $h_n(r) > r$. This means that $l_H(OG) > l_H(ON) = l_H(OS)$. This inequality requires that, when the external faces expand, $\mathcal{T}$ and $\mathcal{B}$ are the first to arrive at the center O. Thus $\mathcal{L}_n$ has precisely n-vertices on the x-y plane (see Figure 7), and we have proved the case (3).

We have thus completed the proof of Lemma 5.2. ■

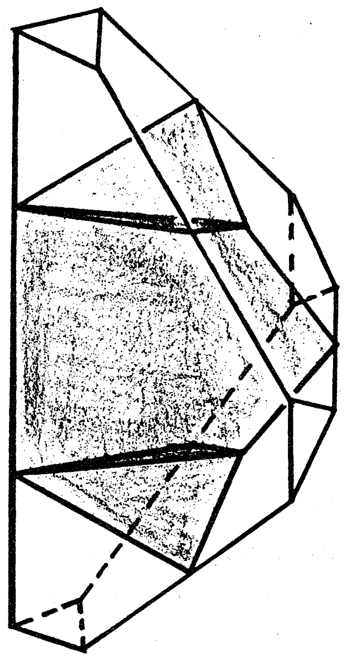

Figure 5: $\mathcal{L}_3$ restricted to $\mathcal{Q}_3$

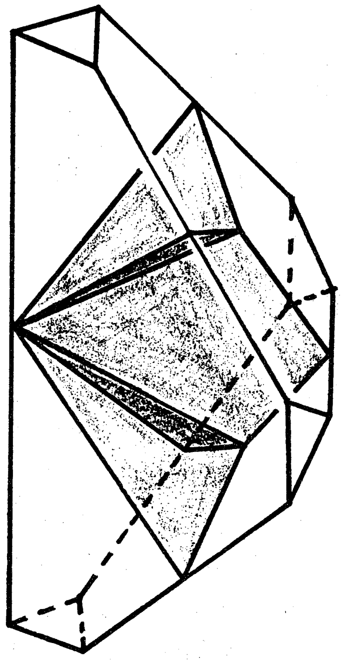

Figure 6: $\mathcal{L}_4$ restricted to $\mathcal{Q}_4$

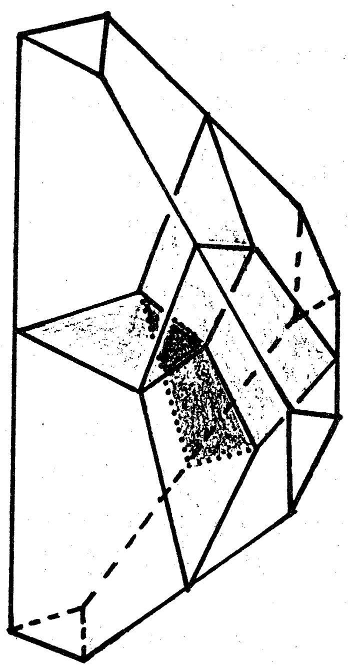

Figure 7: $\mathcal{L}_n$ restricted to $\mathcal{Q}_n$

Proof of Theorem 2.2. Proposition 5.1 and Lemma 5.2 (1) show that the truncated polyhedral decomposition obtained by cutting $\mathcal{P}_3$ along the x-y plane is the canonical decomposition $\mathcal{D}_{3,k}$. In case $n = 4$, Proposition 5.1 and Lemma 5.2 (2) shows that $\mathcal{D}_{4,k}$ consists of $\mathcal{P}_4$. In case $n \geq 5$, Proposition 5.1 and Lemma 5.2 (3) requires that we should slice $\mathcal{P}_n$ by half planes, each of which is bounded by the z-axis and includes a vertex of $\mathcal{G}_n$, to obtain $\mathcal{D}_{n,k}$.

We have thus proved Theorem 2.2. □

References

[Ad] Colin Adams, *Unknotting tunnels in hyperbolic 3-manifolds*, Mathematische Annalen **302** (1995), 177–195.

[ANS] Hirotaka Akiyoshi, Yoshiyuki Nakagawa, and Makoto Sakuma, *Shortest vertical geodesics of manifolds obtained by hyperbolic Dehn surgery on the Whitehead link*, Proceedings of Knots 96, edited by Shin'ichi Suzuki, World Scientific Publishing Co., 1997, 433–448.

[EP] D. B. A. Epstein and R. C. Penner, *Euclidean decompositions of noncompact hyperbolic manifolds*, Journal of Differential Geometry **27** (1988), 67–80.

[He] Daniel J. Heath, *On classifications of Heegaard splittings*, Osaka Journal of Mathematics **34** (1997), 497–523.

[Ki] Shin'ichi Kinoshita, *Alexander polynomials as isotopy invariants, I*, Osaka Mathematical Journal **10** (1958), 263–271.

[KG] Teruaki Kitano and Hiroshi Goda (editors), *Teijigen Toporojî Mondaishû (Japanese; Problems in Low-Dimensional Topology)*, 1996.

[Ko1] Sadayoshi Kojima, *Polyhedral decomposition of hyperbolic manifolds with boundary*, On the Geometric Structure of Manifolds, edited by Dong Pyo Chi, Proceedings of Workshops in Pure Mathematics, Volume 10, Part III (1990), 37–57.

[Ko2] Sadayoshi Kojima, *Polyhedral decomposition of hyperbolic 3-manifolds with totally geodesic boundary*, Aspects of Low Dimensional Manifolds, edited by Y. Matsumoto and S. Morita, Advanced Studies in Pure Mathematics **20** (1992), 93–112.

[KM] Sadayoshi Kojima and Yosuke Miyamoto, *The smallest hyperbolic 3-manifolds with totally geodesic boundary*, Journal of Differential Geometry **34** (1991), 175–192.

[Ma] Bernard Maskit, *Poincaré theorem for fundamental polygons*, Advances in Mathematics **7** (1971), 219–230.

[PZ] Luisa Paoluzzi and Bruno Zimmermann, *On a class of hyperbolic 3-manifolds and groups with one defining relation*, Geometriae Dedicata **60** (1996), 113–123.

[Ra] John G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Graduate Texts of Mathematics, **149**, Springer-Verlag, 1994.

[SW] Makoto Sakuma and Jeffrey Weeks, *Examples of canonical decompositions of hyperbolic link complements*, Japanese Journal of Mathematics **21** (1995), 393–439.

[Sc] Martin Scharlemann, *Some pictorial remarks on Suzuki's Brunnian graph*, Topology '90, edited by Boris Apanasov, Walter D. Neumann, Alan W. Reid and Laurent Siebenmann, Walter de Gruyter Verlag, Berlin and New York, 1992, 351–354.

[Su] Shin'ichi Suzuki, *Almost unknotted θ_n-curves in the 3-sphere*, Kobe Journal of Mathematics **1** (1984), 19–22.

[Th] William P. Thurston, *Three Dimensional Geometry and Topology*, Princeton Mathematical Series, **35**, Princeton University Press, 1997.

[Us] Akira Ushijima, *The canonical decompositions of some family of compact orientable hyperbolic 3-manifolds with totally geodesic boundary*, preprint.

[Wo] Keith Wolcott, *The knotting of theta curves and other graphs in S^3*, Geometry and Topology: Manifolds, Varieties, and Knots, edited by Clint McCrory and Theodore Shifrin, Lecture notes in pure and applied mathematics, Volume 105, Marcel Dekker, inc., New York and Basel, 1987, 325–346.

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, 1–1 MACHIKANEYAMA-CHO, TOYONAKA, OSAKA 560–0043, JAPAN

E-mail address: smv001ua@ex.ecip.osaka-u.ac.jp