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Nondemolition Continuous Measurement and the Quantum Stochastic Differential Equations

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1 Introduction

The problem [1], why the track of an injected particle in the cloud chamber keeps its width finite, will be investigated dynamically by means of the quantum stochastic calculus within the method of Non-Equilibrium Thermo Field Dynamics (NETFD) [2]-[4]. The measurement (a continuous non-demolition measurement) of the incident particle by ionizing the gas molecules in the cloud chamber is interpreted as a stochastic agitation due to the quantum Brownian motion [5]. It will be shown that the watch-dog effect, i.e., the continuous non-demolition measurement of the injected particle by the gas molecules, prevents the wave packet spreading out in contrast with the case of a free particle.

NETFD [2]-[4] is a *canonical formalism* of quantum systems in far-from-equilibrium state, which enables us to treat *dissipative* quantum systems by a method similar to the usual quantum mechanics and/or quantum field theory that accommodate the concept of the dual structure in the interpretation of nature, i.e., in terms of the *operator algebra* and the *representation space.* In NETFD, the time evolution of the unstable vacuum of a dissipative system is realized by a condensation of a kind of particle pairs ($\gamma^{\natural}\tilde{\gamma}^{\natural}$-pairs) into vacuum, and that the amount how many pairs are condensed is described by the one-particle distribution function whose time-dependence is given by the kinetic equation of the system. The framework of NETFD provides us with a unified viewpoint covering whole the aspects in non-equilibrium statistical mechanics, i.e., the Boltzmann equation, the Fokker-Planck equation, the Langevin equation, the stochastic Liouville equation [6] (Fig.1).

2 Characteristics of the Liouville Equation

Let us investigate first what is the most fundamental characteristics of the Liouville equation:

$$\frac{\partial}{\partial t}\rho(t) = -iL\rho(t). \tag{1}$$

Its general characteristics are summarized as

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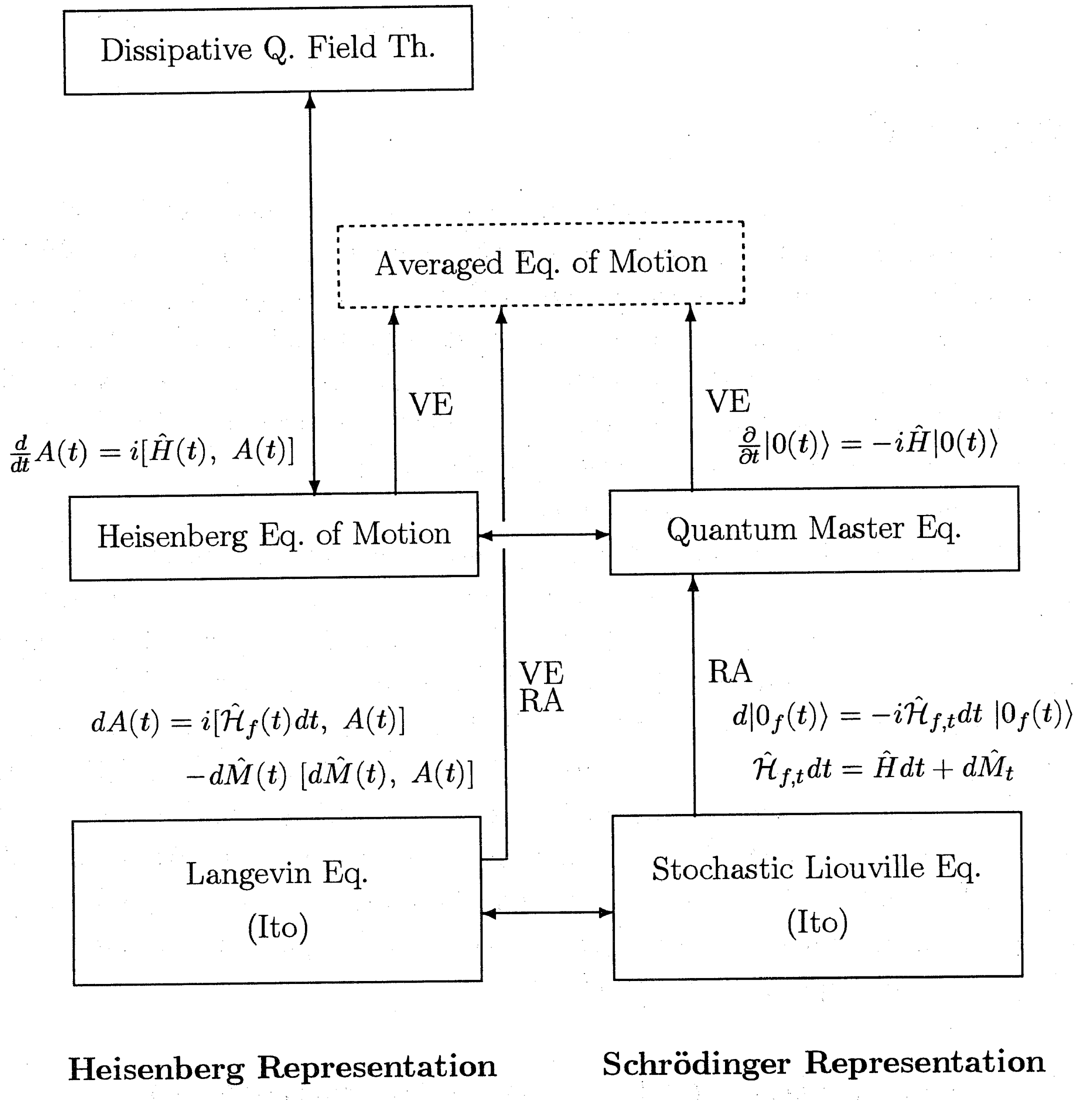

Figure 1: System of the Stochastic Differential Equations within Non-Equilibrium Thermo Field Dynamics. RA stands for the random average. VE stands for the vacuum expectation.

D1. The Hermiticy of the Liouville operator iL in the sense that

$$(iL\ \bullet)^{\dagger} = iL\ \bullet\ . \tag{2}$$

D2. The conservation of probability (tr $\rho = 1$):

$$\text{tr}\ L\ \bullet = 0. \tag{3}$$

D3. The Hermiticy of the density operator:

$$\rho^{\dagger}(t) = \rho(t). \tag{4}$$

The eigenvalues of $\rho(t)$ are positive.

In the trace formalism, the expectation value of an observable operator A is given by

$$\langle A \rangle_t = \text{tr}\ A\rho(t) = \text{tr}\ A\mathrm{e}^{-iLt}\rho(0) = \text{tr}\ \mathrm{e}^{iLt}A\mathrm{e}^{-iLt}\rho(0), \tag{5}$$

where we used the formal solution $\rho(t) = \mathrm{e}^{-iLt}\rho(0)$ of (1), and the property (3). Then, the Heisenberg operator $A(t)$ is defined by

$$A(t) = \mathrm{e}^{iLt}A\mathrm{e}^{-iLt}, \tag{6}$$

which satisfies the Heisenberg equation with the Liouville operator L:

$$\frac{dA(t)}{dt} = i[L,\ A(t)]. \tag{7}$$

3 Basics of NETFD

Referring to the general characteristics of the Liouville equation listed in the previous section, we will write down the general basics of NETFD.

3.1 Technical Basics of NETFD

In the first place, we list the technical basics of NETFD.

Tool 1. Any operator A in NETFD is accompanied by its partner (tilde) operator $\tilde{A}$. The tilde conjugation $\sim$ is defined by

$$\begin{aligned}
(A_1A_2)^{\sim} &= \tilde{A}_1\tilde{A}_2, && (8)\\
(c_1A_1 + c_2A_2)^{\sim} &= c_1^*\tilde{A}_1 + c_2^*\tilde{A}_2, && (9)\\
(\tilde{A})^{\sim} &= A, && (10)\\
(A^{\dagger})^{\sim} &= \tilde{A}^{\dagger}, && (11)
\end{aligned}$$

where c's are complex c-numbers.

Tool 2. Equal-time commutativity between the tilde and non-tilde operators:

$$[A,\ \tilde{B}] = 0. \tag{12}$$

Tool 3. Thermal state condition:

$$\langle 1|A^\dagger = \langle 1|\tilde{A}. \tag{13}$$

3.2 General Basics of NETFD

Within NETFD, the dynamical evolution of the system is described by the Schrödinger equation (the quantum master equation) ($\hbar = 1$):

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}\ |0(t)\rangle, \tag{14}$$

with the hat-Hamiltonian $\hat{H}$. The general characteristics of the Liouville equation are rephrased in NETFD as follows.

B1. The hat-Hamiltonian $\hat{H}$ is *tildian*, i.e.,

$$\left(i\hat{H}\right)^{\sim} = i\hat{H}. \tag{15}$$

Note that $\hat{H}$ is not necessarily Hermitian operator.

B2. The hat-Hamiltonian has zero eigenvalue for the thermal bra-vacuum:

$$\langle 1|\hat{H} = 0. \tag{16}$$

This is a manifestation of the conservation of probability, i.e. $\langle 1|0(t)\rangle = 1$.

B3. The thermal vacuums $\langle 1|$ and $|0\rangle$ are *tilde invariant*, i.e.,

$$\langle 1|^{\sim} = \langle 1|, \quad |0\rangle^{\sim} = |0\rangle, \tag{17}$$

and are normalized as $\langle 1|0\rangle = 1$.

In NETFD, the expectation value of an observable operator A is given by

$$\langle A\rangle_t = \langle 1|A|0(t)\rangle = \langle 1|A\mathrm{e}^{-i\hat{H}t}|0\rangle = \langle 1|\mathrm{e}^{i\hat{H}t}A\mathrm{e}^{-i\hat{H}t}|0\rangle, \tag{18}$$

where we used the formal solution $|0(t)\rangle = \mathrm{e}^{-i\hat{H}t}|0\rangle$ of (14), and the property (16). Then, the Heisenberg operator $A(t)$ is defined by

$$A(t) = \mathrm{e}^{i\hat{H}t}A\mathrm{e}^{-i\hat{H}t}, \tag{19}$$

which satisfies the Heisenberg equation with the hat-Hamiltonian $\hat{H}$:

$$\frac{d}{dt}A(t) = i[\hat{H},\ A(t)]. \tag{20}$$

4 Measurement in NETFD

4.1 Effective-Valued Measure

A measurement can be described by the bra-vector:

$$\langle M(B)| = \int_B \frac{d^2\beta}{\pi} \langle \beta,\, \tilde{\beta}|, \tag{21}$$

where

$$|\beta, \tilde{\beta}\rangle = \hat{D}(\beta)|0, \tilde{0}\rangle, \tag{22}$$

is the coherent state defined by

$$\hat{D}(\beta) = D(\beta)\tilde{D}(\beta), \qquad \text{with} \quad D(\beta) = \mathrm{e}^{\beta b^\dagger - \beta^* b}. \tag{23}$$

The vacuum $|0, \tilde{0}\rangle$ is introduced by

$$b|0, \tilde{0}\rangle = 0, \qquad \tilde{b}|0, \tilde{0}\rangle = 0. \tag{24}$$

We see that the coherent state is the eigen-state of b and $\tilde{b}$:

$$b|\beta, \tilde{\beta}\rangle = \beta|\beta, \tilde{\beta}\rangle, \qquad \tilde{b}|\beta, \tilde{\beta}\rangle = \beta^*|\beta, \tilde{\beta}\rangle, \tag{25}$$

satisfying the completeness

$$\int \frac{d^2\beta}{\pi} \int \frac{d^2\beta'}{\pi} |\beta, \tilde{\beta}'\rangle\langle\beta, \tilde{\beta}'| = 1. \tag{26}$$

The overlap among the coherent states is given by

$$\langle \beta_1, \tilde{\beta}_2|\beta', \tilde{\beta}'_2\rangle = \mathrm{e}^{-|\beta_1|^2/2 - |\beta'_1|^2/2 + \beta_1^*\beta'_1}\, \mathrm{e}^{-|\beta_2|^2/2 - |\beta'_2|^2/2 + \beta_2\beta_2'^*}. \tag{27}$$

The measurement procedure is described by the characteristic state defined by

$$\langle C(k)| = \int \frac{d^2\beta}{\pi}\, \mathrm{e}^{i(k\beta^* + k^*\beta)}\langle \beta, \tilde{\beta}| = \int \frac{d^2\beta}{\pi}\, \langle \beta, \tilde{\beta}|\mathrm{e}^{i\left(kb^\dagger + k^*\tilde{b}^\dagger\right)} = \langle 1|\mathrm{e}^{i\left(kb^\dagger + k^*\tilde{b}^\dagger\right)}. \tag{28}$$

4.2 Probability

The probability $P(B,t)$ of finding the measured value β within the region B, when the system is in the state $|0(t)\rangle$, is given by

$$P(B,t) = \int_B \frac{d^2\beta}{\pi}\, \langle \beta, \tilde{\beta}|0(t)\rangle. \tag{29}$$

Then, the characteristic probability is introduced by

$$C(k,t) = \int \frac{d^2\beta}{\pi}\, \mathrm{e}^{i(k\beta^* + k^*\beta)} \langle \beta, \tilde{\beta} | 0(t) \rangle = \langle 1 | \mathrm{e}^{i\left(kb^\dagger + k^* \tilde{b}^\dagger\right)} | 0(t) \rangle = \langle 1 | \hat{Y}(t) | 0 \rangle, \tag{30}$$

with the characteristic operator:

$$\hat{Y}(t) = \hat{V}^{-1}(t) \mathrm{e}^{i\left(kb^\dagger + k^* \tilde{b}^\dagger\right)} \hat{V}(t) = \mathrm{e}^{i\left(kb^\dagger(t) + k^* \tilde{b}^\dagger(t)\right)}, \tag{31}$$

where the Heisenberg operators $b^\dagger(t)$ and $\tilde{b}^\dagger(t)$ are defined by

$$b^\dagger(t) = \hat{V}^{-1}(t) b^\dagger \hat{V}(t), \qquad \tilde{b}^\dagger(t) = \hat{V}^{-1}(t) \tilde{b}^\dagger \hat{V}(t), \tag{32}$$

with the time-evolution generator $\hat{V}(t)$ satisfying

$$\frac{d}{dt}\hat{V}(t) = -i\hat{H}\hat{V}(t), \qquad \hat{V}(0) = 1. \tag{33}$$

4.3 An Example

Let us investigate, as an example, the system of a damped harmonic oscillator described by the quantum master equation

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{34}$$

with the hat-Hamiltonian

$$\hat{H} = \omega\left(b^\dagger b - \tilde{b}^\dagger \tilde{b}\right) - i\kappa\left[(1+2\bar{n})\left(b^\dagger b + \tilde{b}^\dagger \tilde{b}\right) - 2(1+\bar{n})\, b\tilde{b} - 2\bar{n} b^\dagger \tilde{b}^\dagger\right] - i2\kappa\bar{n}, \tag{35}$$

where $\bar{n}$ is the Planck distribution function defined by

$$\bar{n} = \left(\mathrm{e}^{\omega/T} - 1\right)^{-1}. \tag{36}$$

The operators b, $b^\dagger$ and their tilde conjugates satisfy the canonical commutation relation:

$$[b,\ b^\dagger] = 1, \quad [\tilde{b},\ \tilde{b}^\dagger] = 1. \tag{37}$$

The one-particle distribution function, $n(t) = \langle 1 | b^\dagger b | 0(t) \rangle$, satisfies the Boltzmann equation of the model:

$$\frac{d}{dt}n(t) = -2\kappa\left[n(t) - \bar{n}\right]. \tag{38}$$

The time-evolution of the vacuum is realized by a condensation of $\gamma^{\ddagger}\tilde{\gamma}^{\ddagger}$-pairs into vacuum, and that the amount how many pairs are condensed is described by the one-particle distribution function $n(t)$ whose time-dependence is given by the Boltzmann equation (38):

$$|0(t)\rangle = \exp\left\{\left[n(t) - n(0)\right]\gamma^{\ddagger}\tilde{\gamma}^{\ddagger}\right\}|0\rangle, \tag{39}$$

where

$$\gamma^{\natural} = b^{\dagger} - \tilde{b}, \qquad \tilde{\gamma}^{\natural} = \tilde{b}^{\dagger} - b. \tag{40}$$

These operators annihilate the thermal bra-vacuum:

$$\langle 1|\gamma^{\natural} = 0, \qquad \langle 1|\tilde{\gamma}^{\natural} = 0. \tag{41}$$

We see that

$$\langle \beta, \tilde{\beta}'|0(t)\rangle = \frac{1}{1+n(t)} \exp\left(-\frac{1}{2}|\beta|^2 - \frac{1}{2}|\beta'|^2 + \frac{n(t)}{1+n(t)}\beta^*\beta'\right). \tag{42}$$

Then, substituting

$$\langle \beta, \tilde{\beta}|0(t)\rangle = \frac{1}{1+n(t)} \exp\left(-\frac{1}{1+n(t)}|\beta|^2\right), \tag{43}$$

into (29) gives us the probability of measurement in the form:

$$P(r,t) = 1 - \exp\left[\frac{1}{1+n(t)}r^2\right] \tag{44}$$

where we chose for B the inside of the circle with a radius r in the β-space. Note that $P(\infty, t) = 1$, which represents the conservation of probability. The characteristic probability (30) of the model is given by

$$C(k,t) = \mathrm{e}^{-[1+n(t)]|k|^2}. \tag{45}$$

5 Quantum Brownian Motion

We will introduce the quantum Brownian motion.

Let us introduce the annihilation and creation operators b_t, $b_t^{\dagger}$ and their tilde conjugates satisfying the canonical commutation relation:

$$[b_t,\ b_{t'}^{\dagger}] = \delta(t-t'), \qquad [\tilde{b}_t,\ \tilde{b}_{t'}^{\dagger}] = \delta(t-t'). \tag{46}$$

The vacuums $\langle|$ and $|\rangle$ are defined by

$$b_t|\rangle = 0, \quad \langle|b_t^{\dagger} = 0, \quad \tilde{b}_t|\rangle = 0, \quad \langle|,\tilde{b}_t^{\dagger} = 0. \tag{47}$$

where the argument t represents time.

Introducing the operators

$$B_t = \int_0^{t-dt} dt'\ b_{t'}, \qquad B_t^{\dagger} = \int_0^{t-dt} dt'\ b_{t'}^{\dagger}, \tag{48}$$

and their tilde conjugates for $t \geq 0$, we see that they satisfy $B_{t=0} = 0$, $B^\dagger_{t=0} = 0$,

$$[B_s,\ B^\dagger_t] = \min(s,t), \tag{49}$$

and their tilde conjugates. These operators represent the ***quantum Brownian motion***, and annihilate the vacuums $|\rangle$ and $\langle|$:

$$dB_t|\rangle = 0, \quad d\tilde{B}_t|\rangle = 0, \quad \langle|dB^\dagger_t = 0, \quad \langle|d\tilde{B}^\dagger_t = 0. \tag{50}$$

Then, we have, for example,

$$\langle|dB^\dagger_t dB_t|\rangle = 0, \qquad \langle|dB_t dB^\dagger_t|\rangle = dt. \tag{51}$$

For finite temperature, within the weak relation, we have

$$dB^\dagger_t dB_t = \bar{n}dt, \quad dB_t dB^\dagger_t = (1+\bar{n})dt, \qquad d\tilde{B}_t dB_t = \bar{n}dt, \quad d\tilde{B}^\dagger_t dB^\dagger_t = (1+\bar{n})dt, \tag{52}$$

and their tilde conjugates.

6 A Model of the Tracks in the Cloud Chamber

6.1 Non-Demolition Continuous Measurement

The Heisenberg operator:

$$B(t) = \hat{V}_f^{-1}(t) B_t \hat{V}_f(t), \tag{53}$$

with the Brownian motion B_t satisfying (49) has the property

$$B(t') = \hat{V}_f^{-1}(t) B_{t'} \hat{V}_f(t), \qquad t \geq t'. \tag{54}$$

Here, the stochastic time-evolution generator $\hat{V}_f(t)$ is given by

$$d\hat{V}_f(t) = -i\hat{H}_{f,t}dt \circ \hat{V}_f(t), \tag{55}$$

($\hat{V}_f(0) = 1$), where

$$\hat{H}_{f,t}dt = H_S - \tilde{H}_S + d\hat{M}_t, \tag{56}$$

with the martingale operator

$$d\hat{M}_t = i\sqrt{2\kappa}\left(a^\dagger dB_t + \tilde{a}^\dagger d\tilde{B}_t - \text{h.c.}\right). \tag{57}$$

The symbol $\circ$ indicates to take the stochastic multiplication of the Stratonovich-type.

Note that (55) reduces to

$$d\hat{V}_f(t) = -i\hat{\mathcal{H}}_{f,t}dt\ \hat{V}_f(t), \tag{58}$$

with

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}dt + d\hat{M}_t, \qquad \hat{H} = \hat{H}_S + i\hat{\Pi}, \tag{59}$$

$$\hat{\Pi} = -\kappa\left[(1+2\bar{n})\left(a^\dagger a + \tilde{a}^\dagger \tilde{a}\right) - 2\,(1+\bar{n})\,a\tilde{a} - 2\bar{n}a^\dagger \tilde{a}^\dagger\right] - 2\kappa\bar{n}, \tag{60}$$

within the Ito stochastic multiplication.

Then, we can prove that

$$[B^{\#}(t'),\ A(t)] = \hat{V}_f^{-1}(t)[B_{t'}^{\#},\ A]\hat{V}_f(t) = 0, \quad t \geq t'. \tag{61}$$

with $\#$ being null, dagger and/or tilde. It leads us to

$$[Q(t'),\ A(t)] = 0, \quad t \geq t', \tag{62}$$

with

$$Q(t) = B(t) + B^\dagger(t). \tag{63}$$

Therefore, the measurement of $Q(t')$ does not disturb the states of the relevant system for $t \geq t'$. Note that the output Hermitian process $Q(t)$ is self-nondemolition:

$$[Q(t),\ Q(t')] = 0, \qquad \text{for } t,\ t' \geq 0. \tag{64}$$

Now, we introduce the characteristic operator representing the continuous observation of $Q(t)$ by

$$\hat{Y}(t) = \hat{V}_f^{-1}(t)\hat{Y}_t\hat{V}_f(t) \tag{65}$$

with

$$\hat{Y}_t = \exp\left[\frac{1}{2}\int_0^t \ell(s)\left(dQ_s + d\tilde{Q}_s\right)\right], \tag{66}$$

which satisfies

$$d\hat{Y}_t = \left[\frac{1}{2}\ell(t)\left(dQ_t + d\tilde{Q}_t\right) + \frac{1}{2}(1+2\bar{n})\ell^2(t)dt\right]\hat{Y}_t. \tag{67}$$

6.2 Introduction of the Effective Time-Evolution Generator

Let us now consider the characteristic function $\mathcal{A}(t)$ of the continuous measurement defined by

$$\mathcal{A}(t) = \langle\!\langle 1|\hat{Y}_t A|0_f(t)\rangle\!\rangle = \langle\!\langle 1|\hat{Y}_t A\hat{V}_f(t)|0\rangle\!\rangle, \tag{68}$$

which satisfies

$$\begin{aligned} d\mathcal{A}(t) \;=\; & -i\langle\!\langle 1|\hat{Y}_t[A,\ H_S]dt\hat{V}_f(t)|0\rangle\!\rangle \\ & -\kappa dt\langle\!\langle 1|\hat{Y}_t\left[(1+2\bar{n})\left(Aa^\dagger a + a^\dagger a A\right) - 2(1+\bar{n})a^\dagger A a - 2\bar{n}aAa^\dagger\right]\hat{V}_f(t)|0\rangle\!\rangle \\ & -\kappa dt\ell(t)\langle\!\langle 1|\hat{Y}_t\left[2\bar{n}\left(aA + Aa^\dagger\right) - 2(1+\bar{n})\left(Aa + a^\dagger A\right)\right]\hat{V}_f(t)|0\rangle\!\rangle \\ & +\kappa dt\left[2\bar{n}\left(\ell^2(t) - 1\right) + \ell^2(t)\right]\langle\!\langle 1|\hat{Y}_t A\hat{V}_f(t)|0\rangle\!\rangle. \end{aligned} \tag{69}$$

Here, we introduced the thermal vacuums $\langle\langle 1| = \langle|\langle 1|, \quad |0\rangle\rangle = |0\rangle|\rangle$.

The next step is to find $\hat{\mathcal{V}}_f(t)$ through the relation:

$$\langle\langle 1|\hat{Y}(t)A(t)|0\rangle\rangle = \langle\langle 1|\hat{Y}_t A\hat{V}_f(t)|0\rangle\rangle = \langle\langle 1|\hat{Y}_t A\hat{\mathcal{V}}_f(t)|0\rangle\rangle, \tag{70}$$

where $\hat{\mathcal{V}}_f(t)$ depends only on the input processes dQ_t and $d\tilde{Q}_t$, and satisfies the conservation of probability just within the relevant system.

Introducing the bare time-evolution generator $\hat{\mathcal{V}}_f(t)$ by

$$d\hat{\mathcal{V}}_f(t) = -\hat{\mathcal{H}}^B_{f,t}dt\hat{\mathcal{V}}_f(t), \tag{71}$$

with

$$\hat{\mathcal{H}}^B_{f,t}dt = \hat{H} + d\hat{\mathcal{M}}_t, \qquad d\hat{\mathcal{M}}_t = i\sqrt{2\kappa}\left(a\ dQ_t + \tilde{a}\ d\tilde{Q}_t\right), \tag{72}$$

we cast the normalization condition within the relevant system:

$$\langle 1|\hat{\mathcal{V}}^R_f(t)|0\rangle = N(t)\langle 1|\hat{\mathcal{V}}_f(t)|0\rangle = 1. \tag{73}$$

The condition (72) gives us

$$dN(t) = -\sqrt{2\kappa}\left(\langle a(t)\rangle + \langle a^\dagger(t)\right) N(t) + 2\kappa(1+2\bar{n})dt\left(\langle a(t)\rangle + \langle a^\dagger(t)\rangle\right)^2 N(t), \tag{74}$$

which leads

$$d\hat{\mathcal{V}}^R_f(t) = -\hat{\mathcal{H}}^R_{f,t}dt\hat{\mathcal{V}}^R_f(t), \tag{75}$$

with

$$\hat{\mathcal{H}}^R_{f,t}dt = \hat{H} + d\hat{\mathcal{M}}^R_t, \tag{76}$$

$$d\hat{\mathcal{M}}^R_t = i\sqrt{2\kappa}\left[\left(a - \langle a(t)\rangle\right)d\Delta Q_t + \left(\tilde{a} - \langle a^\dagger(t)\rangle\right)d\Delta\tilde{Q}_t\right]. \tag{77}$$

Here, we introduced

$$d\Delta Q_t = dQ_t - \sqrt{2\kappa}dt(1+2\bar{n})\left[\langle a(t)\rangle + \langle a^\dagger(t)\rangle\right]. \tag{78}$$

6.3 Track of the Injected Particle

Reinterpreting the operator a being the position of the injected particle, we introduce the position operator $\check{x}$ by

$$a = \sqrt{\lambda/2}\,\check{x}. \tag{79}$$

Then, the operators $\hat{\Pi}$ and $d\hat{\mathcal{M}}^R_t$ become, respectively,

$$\hat{\Pi} = -\kappa\frac{\lambda}{2}\left(\check{x} - \tilde{\check{x}}\right)^2, \tag{80}$$

$$d\hat{\mathcal{M}}^R_t = i\sqrt{\kappa\lambda}\left[\left(\check{x} - q(t)\right)\left(dQ_t - 2\sqrt{2\kappa}dtq(t)\right) + \left(\tilde{\check{x}} - q(t)\right)\left(d\tilde{Q}_t - 2\sqrt{2\kappa}dtq(t)\right)\right], \tag{81}$$

with

$$q(t) = \langle 1|\check{x}\hat{\mathcal{V}}_f^R(t)|0\rangle = \langle 1|\tilde{\check{x}}\hat{\mathcal{V}}_f^R(t)|0\rangle. \tag{82}$$

Here, we put the temperature of the cloud chamber be the absolute zero by comparing the energy of the injected particle: $\bar{n} = 0$.

The free Hamiltonian of the particle is given by

$$H_S = \frac{\check{p}^2}{2m}. \tag{83}$$

Note that the commutation relation is given by $[\check{x},\ \check{p}] = i$, and that the operators $\check{x}$ and $\tilde{\check{x}}$ are related each other by the condition:

$$\langle 1|\check{x} = \langle 1|\tilde{\check{x}}. \tag{84}$$

6.4 x-Representation

Let us introduce the stochastic distribution function $\rho_f^R(x, y; t)$ by

$$|0_f^R(t)\rangle = \int\int dx\ dy\ |x, y\rangle\langle x, y|0_f^R(t)\rangle, \tag{85}$$

with

$$\langle x, y|0_f^R(t)\rangle = \rho_f^R(x, y; t). \tag{86}$$

The states $|x, y\rangle$ represent the eigen-states of $\check{x}$ and $\tilde{\check{x}}$:

$$\check{x}|x, y\rangle = x|x, y\rangle, \qquad \tilde{\check{x}}|x, y\rangle = y|x, y\rangle, \tag{87}$$

satisfying the ortho-normality

$$\langle x, y|x', y'\rangle = \delta(x - x')\delta(y - y'), \tag{88}$$

and the completeness

$$\int\int dx\ dy\ |x, y\rangle\langle x, y| = 1. \tag{89}$$

In the x-representation, we have

$$\langle x, y|\check{p}|x', y'\rangle = \frac{1}{i}\frac{\partial}{\partial x}\delta(x - x')\delta(y - y'), \qquad \langle x, y|\tilde{\check{p}}|x', y'\rangle = \frac{1}{i}\frac{\partial}{\partial y'}\delta(x - x')\delta(y - y'). \tag{90}$$

We, then, have the stochastic Liouville equation for $\rho_f^R(x, y; t)$ in the form:

$$\begin{aligned} d\rho_f^R(x, y; t) &= \frac{i}{2m}\left(\frac{\partial^2}{\partial^2 x} - \frac{\partial^2}{\partial^2 y}\right)\rho_f^R(x, y; t) - \frac{\kappa\lambda}{2}(x - y)^2\rho_f^R(x, y; t) \\ &+\sqrt{\kappa\lambda}\left[(x - q(t))\left(dQ_t - 2\sqrt{2\kappa}dtq(t)\right) + (y - q(t))\left(d\tilde{Q}_t - 2\sqrt{2\kappa}dtq(t)\right)\right]\rho_f^R(x, y; t). \end{aligned} \tag{91}$$

6.5 Equation for the Width of the Wave Packet

Let us solve the Liouville equation in the form

$$\rho_f^R(x,y;t) = \sqrt{\frac{\omega'(t)}{\pi}}\, e^{-\omega(t)(x-q(t))^2/2-\omega(t)^*(y-q(t))^2/2+ip(t)(x-y)}, \tag{92}$$

with $\omega(t) = \omega'(t) + i\omega''(t)$, assuming that the particle was injected into the cloud-chamber at $t = 0$ with the momentum $p(0) = p$. The initial wave-packet of the particle is located at $q(0) = q$ with the width $\sigma_q^2 = (2\omega'(0))^{-1}$. We see that

$$\int\!\!\int dx\, dy\; x\; \rho_f^R(x,y;t) = q(t), \qquad \int\!\!\int dx\, dy\; \frac{1}{i}\frac{\partial}{\partial x}\; \rho_f^R(x,y;t) = p(t). \tag{93}$$

It is straightforward to derive the equation for $\omega(t)$:

$$\frac{d\omega(t)}{dt} + \frac{i}{m}\omega(t)^2 = 2\kappa\lambda, \tag{94}$$

which can be solved to give

$$\omega(t) = \alpha\, \frac{\omega(0) + \alpha \tanh\left((2\kappa\lambda/\alpha)t\right)}{\omega(0)\tanh\left((2\kappa\lambda/\alpha)t\right) + \alpha}, \tag{95}$$

with $\alpha = \sqrt{m\kappa}(1-i)$.

In the long time limit, $t \to \infty$, we observe that

$$\sigma_q^2 = \frac{1}{2\omega'(\infty)} = \frac{1}{2}\sqrt{\frac{\hbar}{m\kappa\lambda}}, \qquad \sigma_p^2 = \frac{\hbar^2|\omega(\infty)|^2}{2\omega'(\infty)} = \hbar\sqrt{\hbar m\kappa\lambda}. \tag{96}$$

This represents the watch-dog effect, i.e., the continuous nondemolition measurement of the injected particle by the ionization of the molecules in the cloud chamber prevents the wave packet spreading out in contrast with the free particle does.

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