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Optimality of $t-$policy for Imperfect Repair Problem

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Abstract

We consider the optimal imperfect repair problem of reliability system under the average cost criterion. It is formulated as a semi-Markov decision process for the optimal maintenance problem of reliability systems with two types of imperfect repairs which are probabilistically mixed maintenance activities of minimal-repair and replacement. One side of maintenance activities need not much cost but repairs the system likely new one in lesser degree, and the other side need much cost but repairs the system likely new one in greater degree. It is shown that there is an optimal $t-$policy for the system of which failure rate monotonically increases to infinity. A $t-$policy implies that a failure before age t is repaired by one of these activities, a failure after age t is repaired by the other.

1 Introduction

Imperfect repair considering in this paper is probabilistically mixed maintenance activities of minimal-repair and replacement. It is a extension of minimal-repair and replacement. A minimal-repair is the maintenance activity which recovers the function of the failed system without changing its age. A replacement restores the entire system into the new one. In the past three decades, vast literature has discussed various maintenance problems with the above maintenance activities. A pioneering work on the maintenance problem with minimal repair was done by Barlow and Hunter [1] in 1960. They considered minimal repair and preventive replacement as maintenance activities. Later, Phelps [6] discussed the maintenance problem with minimal repair and failure replacement under the average cost criterion. He formulated the problem as a semi-Markov decision process. Assuming that the failure time distribution has increasing failure rate (IFR) , he showed that the optimal policy of all allowable policies is a t-policy, that is, there exists a threshold age t such that failures before age t are minimally repaired, but the system is replaced at the first failure after age t. Segawa and Ohnishi [8] discuss the minimal-repair and replacement problem under the average cost criterion, assuming that the cost structure is dependent on age. They formulate the problem as a semi-Markov decision process, and show that an optimal policy of all allowable policies is in the class of t-policies under some assumptions which are weaker than those of Phelps [6]. We consider the optimal imperfect repair problem of reliability system under the average cost criterion. It is formulated as a semi-Markov decision process for the optimal maintenance problem of reliability systems with two types of imperfect repairs which are probabilistically mixed maintenance activities of minimal-repair and replacement. One side of maintenance activities need not much cost but repairs the system likely new one in lesser degree, and the other side need much cost but repairs the system likely new one in greater degree. It is shown that there is an optimal $t-$policy for the system of which failure rate monotonically increases to infinity. A $t-$policy implies that a failure before age t is repaired by one of these activities, a failure after age t is repaired by the other.

2 Model and Optimality Equation

We consider a reliability system described in following. Maintenance activities are 2-types imperfect repair R1 and R2. When the system failed at age x, imperfect repair R1 recovers its function with cost c_1 by probability p_1 like new and by probability $1-p_1$ without changing the age, and imperfect repair R2 recovers its function with cost c_2 by probability p_2 like new and by probability $1-p_2$ without changing the age. The time required for performing these activity is assumed to negligible. The cumulative distribution function of the failure time of the system is $F(x)$ and it has the continuous density function $f(x)$. We denote the reliability function as $\bar{F}(x) = 1 - F(x)$, and the failure rate function as $\lambda(x) = f(x)/\bar{F}(x)$. It is assumed that $\bar{F}(x)$ is positive and $\lambda(x)$ is continuous for all $x \in [0, \infty)$. Our problem is to find the optimal policy of the average cost, i.e., the sum of the expected maintenance costs par unit time averaged over the infinite time horizon.

Assumption 1

$$0 \le p_1 < p_2 \le 1. \tag{2.1}$$

Assumption 2

$$0 < c_1 < c_2. \tag{2.2}$$

Assumption 3 *The reliability function follows IFR, $\lambda(x)$ is continuous and $\lambda(\infty) = \infty$.*

Under Assumptions 1, 2, and 3, we discuss the problem that we can find a policy of threshold type. This problem can be formulated as a semi-Markov decision process in which the age of the system is chosen as the state, and decision on maintenance activities are made just after the epochs at which system failures occur. The following theorem is well-known for a semi-Markov decision process under the average cost criterion (see [3]).

Theorem 1 *If there exist a constant g and a bounded function v which satisfies the following equations, called the optimality equation, and a policy described by these equation is an optimal policy.*

$$v(x) = \min \left\{ \begin{array}{l} c_1 + \dfrac{1-p_1}{\bar{F}(x)} \left\{ \displaystyle\int_x^\infty v(s) f(s) ds - g \int_x^\infty \bar{F}(s) ds \right\} + \dfrac{p_1}{\bar{F}(x)} \left\{ \displaystyle\int_0^\infty v(s) f(s) ds - g \int_0^\infty \bar{F}(s) ds \right\} \\ c_2 + \dfrac{1-p_2}{\bar{F}(x)} \left\{ \displaystyle\int_x^\infty v(s) f(s) ds - g \int_x^\infty \bar{F}(s) ds \right\} + \dfrac{p_2}{\bar{F}(x)} \left\{ \displaystyle\int_0^\infty v(s) f(s) ds - g \int_0^\infty \bar{F}(s) ds \right\} \end{array} \right. \tag{2.3}$$

□

The function $v(\cdot)$ which appears in the above optimality equation is called the relative cost function. Because $v(\cdot)$ is determined unique without an adaptive constant, we can normalize, without any loss of generality, by adding an equation to the above optimality equations for simplification and obtain the following.

Theorem 2 *If there exist a constant g and a bounded function v which satisfy the following equations called the optimality equation, and a policy described by these equation is an optimal policy.*

$$v(x) = \min \left\{ \begin{array}{l} c_1 + \dfrac{1-p_1}{\bar{F}(x)} \left\{ \displaystyle\int_x^\infty v(s) f(s) ds - g \int_x^\infty \bar{F}(s) ds \right\}, \\ c_2 + \dfrac{1-p_2}{\bar{F}(x)} \left\{ \displaystyle\int_x^\infty v(s) f(s) ds - g \int_x^\infty \bar{F}(s) ds \right\}, \end{array} \right. \tag{2.4}$$

$$\int_0^\infty v(s) f(s) ds - g \int_0^\infty \bar{F}(s) ds = 0. \quad \square \tag{2.5}$$

We describe the optimality equation by following operator.

$$U_1[v](x) \equiv c_1 + \frac{1-p_1}{\bar{F}(x)} \left\{ \int_x^\infty v(s)f(s)ds - g\int_x^\infty \bar{F}(s)ds \right\}, \tag{2.6}$$

$$U_2[v](x) \equiv c_2 + \frac{1-p_2}{\bar{F}(x)} \left\{ \int_x^\infty v(s)f(s)ds - g\int_x^\infty \bar{F}(s)ds \right\}, \tag{2.7}$$

$$U[v](x) \equiv \min\{U_1[v](x), U_2[v](x)\}. \tag{2.8}$$

$$\begin{aligned} \overline{U}_1[v](x) &\equiv U_1[v](x) \\ &= c_1 + \frac{1-p_1}{\bar{F}(x)} \left\{ \int_x^\infty v(s)f(s)ds - g\int_x^\infty \bar{F}(s)ds \right\}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \overline{U}_2[v](x) &\equiv U_2[v](x) \\ &= c_2 + \frac{1-p_2}{\bar{F}(x)} \left\{ \int_x^\infty v(s)f(s)ds - g\int_x^\infty \bar{F}(s)ds \right\}, \end{aligned} \tag{2.10}$$

$$\underline{U}_1[v](x) \equiv c_1 - \frac{1-p_1}{\bar{F}(x)} \left\{ \int_0^x v(s)f(s)ds - g\int_0^x \bar{F}(s)ds \right\}, \tag{2.11}$$

$$\underline{U}_2[v](x) \equiv c_2 - \frac{1-p_2}{\bar{F}(x)} \left\{ \int_0^x v(s)f(s)ds - g\int_0^x \bar{F}(s)ds \right\}. \tag{2.12}$$

We can describe the optimality equation by the operator as following.

$$\begin{cases} v(x) = U[v](x), \\ \int_0^\infty v(s)f(s)ds - g\int_0^\infty \bar{F}(s)ds = 0. \end{cases} \tag{2.13}$$

Now we have the relations for these operators.

Lemma 1 *If*

$$\int_0^\infty v(s)f(s)ds - g\int_0^\infty \bar{F}(s)ds = 0, \tag{2.14}$$

then we can describe

$$\begin{aligned} \overline{U}_1[v](x) = U_1[v](x) = \underline{U}_1[v](x), \\ \overline{U}_2[v](x) = U_2[v](x) = \underline{U}_2[v](x). \quad \square \end{aligned} \tag{2.15}$$

We consider following integral equations.

$$v_1(x) = \underline{U}_1[v_1](x). \tag{2.16}$$

$$v_2(x) = \overline{U}_2[v_2](x). \tag{2.17}$$

They are

$$v_1(x) = c_1 - \frac{1-p_1}{\bar{F}(x)} \left\{ \int_0^x v_1(s)f(s)ds - g\int_0^x \bar{F}(s)ds \right\}, \tag{2.18}$$

$$v_2(x) = c_2 + \frac{1-p_2}{\bar{F}(x)} \left\{ \int_x^\infty v_2(s)f(s)ds - g\int_x^\infty \bar{F}(s)ds \right\}. \tag{2.19}$$

We multiply $\bar{F}(x)$ to (2.16) and differentiate by x, we get a differential equation.

$$v_1'(x) - p_1\lambda(x)v_1(x) + c_1\lambda(x) - (1-p_1)g = 0. \tag{2.20}$$

It has a general solution with integration constant C,

$$v_1(x) = \left[C - \int_0^x \{c_1\lambda(s) - (1-p_1)g\} \exp\left(-p_1\int_0^s \lambda(u)du\right) ds \right] \exp\left(p_1 \int_0^x \lambda(s)ds\right). \tag{2.21}$$

By definition of reliability function,

$$\bar{F}(x) = \exp\left(-\int_0^x \lambda(s)ds\right), \tag{2.22}$$

$$v_1(x) = \left[C - c_1\int_0^x \lambda(s)\bar{F}^{p_1}(s) + (1-p_1)g\int_0^x \bar{F}^{p_1}(s)ds\right]\bar{F}^{-p_1}(x). \tag{2.23}$$

Since $v_1(0) = c_1$, then $C = c_1$. We think separately for the case of $p_1 = 0$ and $0 < p_1 < 1$. Now we have a explicit solution for $p_1 = 0$,

$$\begin{aligned} v_1(x) &= \left[c_1 - c_1\int_0^x \lambda(s)ds + g\int_0^x ds\right] \\ &= c_1 - c_1\int_0^x \lambda(s)ds + gx \end{aligned} \tag{2.24}$$

$$v_1(x) = c_1 - c_1\int_0^x \lambda(s)ds + gx. \tag{2.25}$$

This is a same solution as Segawa and Ohnishi(1992). For $0 < p_1 < 1$, we can get integration by parts,

$$\int_0^x \lambda(s)\bar{F}^{p_1}(s)ds = \frac{1}{p_1} - \frac{\bar{F}^{p_1}(x)}{p_1}. \tag{2.26}$$

Then we can describe similarly

$$\begin{aligned} v_1(x) &= \left[C - c_1\left(\frac{1}{p_1} - \frac{\bar{F}^{p_1}(x)}{p_1}\right) + (1-p_1)g\int_0^x \bar{F}^{p_1}(s)ds\right]\bar{F}^{-p_1}(x) \\ &= \frac{c_1}{p_1} - \frac{1-p_1}{p_1}c_1\bar{F}^{-p_1}(x) + (1-p_1)g\bar{F}^{-p_1}(x)\int_0^x \bar{F}^{p_1}(s)ds. \end{aligned} \tag{2.27}$$

When $p_1 \to 0$,

$$\lim_{p_1 \to 0} v_1(x) = c_1 - c_1\int_0^x \lambda(s)ds + gx, \tag{2.28}$$

then, we can describe generally for $0 \le p_1 < 1$,

$$v_1(x) = \frac{c_1}{p_1} - \frac{1-p_1}{p_1}c_1\bar{F}^{-p_1}(x) + (1-p_1)g\bar{F}^{-p_1}(x)\int_0^x \bar{F}^{p_1}(s)ds. \tag{2.29}$$

Next, for $0 < p_2 < 1$,

$$v_2(x) = c_2 + \frac{1-p_2}{\bar{F}(x)}\left\{\int_x^\infty v_2(s)f(s)ds - g\int_x^\infty \bar{F}(s)ds\right\} \tag{2.30}$$

has a general solution similarly for v_1,

$$v_2(x) = \frac{c_2}{p_2} + (C - \frac{c_2}{p_2})\bar{F}^{-p_2}(x) + (1-p_2)g\bar{F}^{-p_2}(x)\int_0^x \bar{F}^{p_2}(s)ds. \tag{2.31}$$

Because $v_2(x)$ is bounded for x, then $v(\infty)$ is finite. We get initial condition for $x = \infty$,

$$\begin{aligned} v_2(\infty) &= c_2 + (1-p_2)\lim_{x\to\infty}\frac{1}{\bar{F}(x)}\left\{\int_x^\infty v_2(s)f(s)ds - g\int_x^\infty \bar{F}(s)ds\right\} \\ &= c_2 + (1-p_2)\lim_{x\to\infty}\left\{v_2(x) - \frac{g}{\lambda(x)}\right\} \\ &= c_2 + (1-p_2)v_2(\infty). \end{aligned} \tag{2.32}$$

Then we have
$$v_2(\infty) = \frac{c_2}{p_2}. \tag{2.33}$$

We get similarly,
$$v_2(\infty) = \frac{c_2}{p_2} + \lim_{x\to\infty} \frac{1}{\bar{F}^{p_2}(x)} \left\{ \left(C - \frac{c_2}{p_2} \right) + (1-p_2)g \int_0^x \bar{F}^{p_2}(s)ds \right\}. \tag{2.34}$$

Since the denominator tends to 0 at $x \to \infty$, then the numerator is 0 at $x \to \infty$.
$$C - \frac{c_2}{p_2} + (1-p_2)g \int_0^\infty \bar{F}^{p_2}(s)ds = 0. \tag{2.35}$$

Then we have
$$v_2(x) = \frac{c_2}{p_2} - (1-p_2)g\bar{F}^{-p_2}(x) \int_x^\infty \bar{F}^{p_2}(s)ds. \tag{2.36}$$

For the case of $p_2 = 1$, (2.30) is not integral equation, but $v_2(x) = c_1$ is the solution of (2.30). Then we can explicitly describe v_1, v_2 for $0 \le p_1 < p_2 \le 1$,
$$\begin{cases} v_1(x) = \dfrac{c_1}{p_1} - \dfrac{1-p_1}{p_1} c_1 \bar{F}^{-p_1}(x) + (1-p_1)g\bar{F}^{-p_1}(x) \displaystyle\int_0^x \bar{F}^{p_1}(s)ds, \\ v_2(x) = \dfrac{c_2}{p_2} - (1-p_2)g\bar{F}^{-p_2}(x) \displaystyle\int_x^\infty \bar{F}^{p_2}(s)ds. \end{cases} \tag{2.37}$$

We define
$$W_1(x) \equiv \underline{U}_2[v_1](x) - \underline{U}_1[v_1](x). \tag{2.38}$$

Then
$$\begin{aligned} W_1(x) &= c_2 - c_1 + \frac{p_2 - p_1}{\bar{F}(x)} \left\{ \int_0^x v_1(s)f(s)ds - g\int_0^x \bar{F}(s)ds \right\} \\ &= c_2 - c_1 + \frac{p_2 - p_1}{1-p_1} \{c_1 - v_1(x)\} \\ &= c_2 - c_1 + \frac{p_2 - p_1}{1-p_1} \left\{ c_1 - \frac{c_1}{p_1} + \frac{1-p_1}{p_1} c_1 \bar{F}^{-p_1}(x) - (1-p_1)g\bar{F}^{-p_1}(x) \int_0^x \bar{F}^{p_1}(s)ds - c_1 \right\} \\ &= \frac{p_1c_2 - p_2c_1}{p_1} + \frac{p_2 - p_1}{p_1} c_1 \bar{F}^{-p_1}(x) - (p_2 - p_1)g\bar{F}^{-p_1}(x) \int_0^x \bar{F}^{p_1}(s)ds \end{aligned} \tag{2.39}$$

We concern the signature of $W_1(x)$, then we multiple $\dfrac{\bar{F}^{p_1}(x)}{p_2 - p_1}$ for the both side,
$$w_1(x,g) \equiv \frac{\bar{F}^{p_1}(x)}{p_2 - p_1} W_1(x). \tag{2.40}$$

$$w_1(x,g) = \frac{p_1c_2 - p_2c_1}{p_1(p_2 - p_1)} \bar{F}^{p_1}(x) + \frac{c_1}{p_1} - g\int_0^x \bar{F}^{p_1}(s)ds. \tag{2.41}$$

In the same way,
$$W_2(x) \equiv \overline{U}_2[v_2](x) - \overline{U}_1[v_2](x). \tag{2.42}$$

$$\begin{aligned} W_2(x) &= c_2 - c_1 - \frac{p_2 - p_1}{\bar{F}(x)} \left\{ \int_x^\infty v_2(s)f(s)ds - g\int_x^\infty \bar{F}(s)ds \right\} \\ &= c_2 - c_1 - \frac{p_2 - p_1}{1-p_1} \{v_2(x) - c_2\} \\ &= \frac{p_1c_2 - p_2c_1}{p_2} + (p_2 - p_1)g\bar{F}^{-p_2}(x) \int_x^\infty \bar{F}^{p_2}(s)ds \end{aligned} \tag{2.43}$$

We concern the signature of $W_2(x)$, then we multiple $\dfrac{\bar{F}^{p_2}(x)}{p_2 - p_1}$ for the both side,

$$w_2(x,g) \equiv \frac{\bar{F}^{p_2}(x)}{p_2 - p_1} W_2(x) \tag{2.44}$$

$$w_2(x,g) = \frac{p_1 c_2 - p_2 c_1}{p_2(p_2 - p_1)} \bar{F}^{p_2}(x) + g \int_x^\infty \bar{F}^{p_2}(s) ds. \tag{2.45}$$

We have after all

$$\begin{cases} w_1(x,g) = \dfrac{p_1 c_2 - p_2 c_1}{p_1(p_2 - p_1)} \bar{F}^{p_1}(x) + \dfrac{c_1}{p_1} - g \displaystyle\int_0^x \bar{F}^{p_1}(s) ds, \\ w_2(x,g) = \dfrac{p_1 c_2 - p_2 c_1}{p_2(p_2 - p_1)} \bar{F}^{p_2}(x) + g \displaystyle\int_x^\infty \bar{F}^{p_2}(s) ds. \end{cases} \tag{2.46}$$

Lemma 2 *If g is an optimal average cost and v is an optimal relative function, then $w_1 \geq 0$ means* R1 *is a good selection, $w_1 < 0$ means* R2 *is a good selection. When $w_2 \geq 0, w_2 < 0$, it is similarly.* □

In order to check the signature of w_1 and w_2, we partially differentiate w_1 and w_2 by x. We get

$$\frac{\partial}{\partial x} w_1(x,g) = -\bar{F}^{p_1}(x) \left\{ \frac{p_1 c_2 - p_2 c_1}{p_2 - p_1} \lambda(x) + g \right\}, \tag{2.47}$$

$$\frac{\partial}{\partial x} w_2(x,g) = -\bar{F}^{p_2}(x) \left\{ \frac{p_1 c_2 - p_2 c_1}{p_2 - p_1} \lambda(x) + g \right\}. \tag{2.48}$$

There is the relation for them,

$$\frac{\partial}{\partial x} w_2(x,g) = \bar{F}^{p_2 - p_1}(x) \frac{\partial}{\partial x} w_1(x,g). \tag{2.49}$$

We define

$$D(x,g) \equiv -\frac{p_1 c_2 - p_2 c_1}{p_2 - p_1} \lambda(x) - g. \tag{2.50}$$

Lemma 3 *If $p_1 c_2 - p_2 c_1 < 0$, there exists at least one pair solution of simultaneous equation of*

$$\begin{cases} w_1(x,g) = 0, \\ w_2(x,g) = 0. \end{cases} \tag{2.51}$$

□

(Proof)
For $x \in (0, \infty)$, we define

$$\begin{cases} g_1(x) \equiv \dfrac{1}{\displaystyle\int_0^x \bar{F}^{p_1}(s) ds} \left\{ \dfrac{p_1 c_2 - p_2 c_1}{p_1(p_2 - p_1)} \bar{F}^{p_1}(x) + \dfrac{c_1}{p_1} \right\}, \\ g_2(x) \equiv -\dfrac{(p_1 c_2 - p_2 p_1) \bar{F}^{p_2}(x)}{p_2(p_2 - p_1) \displaystyle\int_x^\infty \bar{F}^{p_2}(s) ds}. \end{cases} \tag{2.52}$$

And

$$\begin{aligned} Z(x) &\equiv g_2(x) - g_1(x) \\ &= -\frac{(p_1 c_2 - p_2 c_1) \bar{F}^{p_2}(x)}{p_2(p_2 - p_1) \displaystyle\int_x^\infty \bar{F}^{p_2}(s) ds} \\ &\quad - \frac{1}{\displaystyle\int_0^x \bar{F}^{p_1}(s) ds} \left\{ \frac{p_1 c_2 - p_2 c_1}{p_1(p_2 - p_1)} \bar{F}^{p_1}(x) + \frac{c_1}{p_1} \right\}. \end{aligned} \tag{2.53}$$

From de l'Hôspital's rule,

$$\begin{aligned} Z(+0) &= \frac{p_1c_2-p_2c_1}{p_2(p_2-p_1)\displaystyle\int_0^\infty \bar{F}^{p_2}(s)ds} - \frac{\dfrac{p_1c_2-p_2c_1}{p_1(p_2-p_1)}+\dfrac{c_1}{p_1}}{\displaystyle\lim_{x\to+0}\int_0^x \bar{F}^{p_1}(s)ds} \\ &= \frac{p_1c_2-p_2c_1}{p_2(p_2-p_1)\displaystyle\int_0^\infty \bar{F}^{p_2}(s)ds} - \frac{\dfrac{c_2-c_1}{p_2-p_1}}{\displaystyle\lim_{x\to+0}\int_0^x \bar{F}^{p_1}(s)ds} \\ &= -\infty, \end{aligned} \tag{2.54}$$

and

$$\begin{aligned} Z(\infty) &= -\frac{\dfrac{c_1}{p_1}}{\displaystyle\int_0^\infty \bar{F}^{p_1}(s)ds} - \frac{p_1c_2-p_2c_1}{p_2(p_2-p_1)}\lim_{x\to\infty}\left\{\frac{\bar{F}^{p_2}(x)}{\displaystyle\int_0^x \bar{F}^{p_2}(s)ds}\right\} \\ &= -\frac{\dfrac{c_1}{p_1}}{\displaystyle\int_0^\infty \bar{F}^{p_1}(s)ds} - \frac{p_1c_2-p_2c_1}{p_2(p_2-p_1)}\lim_{x\to\infty} p_2\lambda(x) \\ &= \infty. \end{aligned} \tag{2.55}$$

Since $Z(x)$ is continuos for $x \in (0,\infty)$, there exists $t^* \in (0,\infty)$ such that $Z(t^*)=0$, i.e. $g_1(t^*)=g_2(t^*)$. We define g^* by

$$g^* = g_1(t^*) = g_2(t^*) > 0. \tag{2.56}$$

Then

$$g^* = \frac{1}{\displaystyle\int_0^{t^*} \bar{F}^{-p_1}(s)ds}\left\{\frac{p_1c_2-p_2c_1}{p_1(p_2-p_1)}\bar{F}^{p_1}(t^*)+\frac{c_1}{p_1}\right\} \tag{2.57}$$

and

$$g^* = -\left\{\frac{(p_1c_2-p_2c_1)\bar{F}^{p_2}(t^*)}{p_2(p_2-p_1)\displaystyle\int_{t^*}^\infty \bar{F}^{p_1}(s)ds}\right\}. \tag{2.58}$$

It means that

$$\frac{p_1c_2-p_2c_1}{p_1(p_2-p_1)}\bar{F}^{p_1}(t^*)+\frac{c_1}{p_1}-g^*\int_0^{t^*}\bar{F}^{p_1}(s)ds = 0 \tag{2.59}$$

and

$$\frac{p_1c_2-p_2c_1}{p_2(p_2-p_1)}\bar{F}^{p_2}(t^*)+g^*\int_{t^*}^\infty \bar{F}^{p_2}(s)ds = 0. \tag{2.60}$$

That is

$$w_1(t^*,g^*) = w_2(t^*,g^*) = 0. \quad \square \tag{2.61}$$

Definition 1 *For* (t^*,g^*) *which is a solution of* (2.51), *we define*

$$v^*(x) \equiv \begin{cases} \dfrac{c_1}{p_1} - \dfrac{1-p_1}{p_1}c_1\bar{F}^{-p_1}(x) + (1-p_1)g^*\bar{F}^{-p_1}(x)\displaystyle\int_0^x \bar{F}^{p_1}(s)ds, & x \in [0,t^*), \\ \dfrac{c_2}{p_2} - (1-p_2)g^*\bar{F}^{-p_2}(x)\displaystyle\int_x^\infty \bar{F}^{p_2}(s)ds, & x \in [t^*,\infty). \end{cases} \quad \square \tag{2.62}$$

Lemma 4 *g^*,v^* in (2.62) satisfy*

$$\underline{U}_2[v^*](x) \geq \underline{U}_1[v^*](x), \quad x \in [0, t^*), \tag{2.63}$$

and

$$\overline{U}_2[v^*](x) \leq \overline{U}_1[v^*](x), \quad x \in [t^*, \infty). \quad \square \tag{2.64}$$

(Proof) For g^*, v^*,

$$w_2(\infty, g^*) = 0. \tag{2.65}$$

$D(x, g^*)$ increases in x. There exists $x_D(g^*)$ such that $D(x_D(g^*), g^*) = 0$. $w_2(x, g)$is increasing in $[x_D(g^*), +\infty)$, then $w_2(x_D(g^*) < 0$. $w_1(x, g^*)$ and $w_2(x, g^*)$ is decreasing in $[0, x_D(g^*)]$. Because $w_1(t^*, g^*) = w_2(t^*, g^*) = 0$, then

$$w_1(x, g^*) \geq 0, \quad x \in [0, t^*) \tag{2.66}$$

and

$$w_2(x, g^*) \leq 0, \quad x \in [t^*, \infty). \tag{2.67}$$

That is

$$W_1(x) = \underline{U}_2[v^*](x) - \underline{U}_1[v^*](x) \geq 0 \tag{2.68}$$

and

$$W_2(x) = \overline{U}_2[v^*](x) - \overline{U}_1[v^*](x) \leq 0. \quad \square \tag{2.69}$$

We have following lemma.

Lemma 5

$$\int_0^\infty v^*(s)f(s)ds - g^* \int_0^\infty \bar{F}(s)ds = 0. \quad \square \tag{2.70}$$

From $w_1(t^*, g^*) = w_2(t^*, g^*) = 0$,

$$\textbf{(Proof)} \bar{F}^{-p_1}(t^*)g^* \int_0^{t^*} \bar{F}^{p_1}(s)ds = \frac{p_1c_2 - p_2c_1}{p_1(p_2 - p_1)} + \frac{c_1}{p_1}\bar{F}^{-p_1}(t^*), \tag{2.71}$$

$$\bar{F}^{-p_2}(t^*)g^* \int_{t^*}^\infty \bar{F}^{p_2}(s)ds = -\frac{p_1c_2 - p_2c_1}{p_2(p_2 - p_1)}, \tag{2.72}$$

then

$$\begin{aligned}
&\int_0^\infty v(s)f(s)ds - g^* \int_0^\infty \bar{F}(s)ds \\
&= \int_0^{t^*} v(s)f(s)ds - g^* \int_0^{t^*} \bar{F}(s)ds + \int_{t^*}^\infty v(s)f(s)ds - g^* \int_{t^*}^\infty \bar{F}(s)ds \\
&= \frac{\bar{F}(t^*)}{1 - p_1}\{c_1 - v_1(t^*)\} + \frac{\bar{F}(t^*)}{1 - p_2}\{v_2(t^*) - c_2\} \\
&= \frac{\bar{F}(t^*)}{1 - p_1}\left\{c_1 - \left\{\frac{c_1}{p_1} - \frac{1 - p_1}{p_1}c_1\bar{F}^{-p_1}(t^*) + (1 - p_1)g^*\bar{F}^{-p_1}(t^*)\int_0^{t^*} \bar{F}^{p_1}(s)ds\right\}\right\} \\
&\quad + \frac{\bar{F}(t^*)}{1 - p_2}\left\{\frac{c_2}{p_2} - (1 - p_2)g^*\bar{F}^{-p_2}(t^*)\int_{t^*}^\infty \bar{F}^{p_2}(s)ds - c_2\right\} \\
&= \bar{F}(t^*)\left\{-\frac{c_1}{p_1} + \frac{c_1}{p_1}\bar{F}^{-p_1}(t^*) - \left\{\frac{p_1c_2 - p_2c_1}{p_1(p_2 - p_1)} + \frac{c_1}{p_1}\bar{F}^{-p_1}(t^*)\right\} - \frac{c_2}{p_2} + \frac{p_1c_2 - p_2c_1}{p_2(p_2 - p_1)}\right\} \\
&= 0 \quad \square
\end{aligned}$$

Theorem 3 *If $p_1c_2 - p_2c_1 < 0$, then $t-$policy which takes the action* R1 *for $x \in [0, t^*)$ and* R2 *for $x \in [t^*, \infty)$ is optimal.* $\square$

(Proof) From Lemma 1 and 5, for (t^*, g^*), v^* satisfy

$$\begin{aligned} \underline{U}_2[v^*](x) &= U_2[v^*](x), \quad x \in [0, t^*), \\ \underline{U}_1[v^*](x) &= U_1[v^*](x), \end{aligned} \tag{2.73}$$

and

$$\begin{aligned} \overline{U}_1[v^*](x) &= U_1[v^*](x), \quad x \in [t^*, \infty), \\ \overline{U}_2[v^*](x) &= U_2[v^*](x). \end{aligned} \tag{2.74}$$

From Lemma 4, for $x \in [0, t^*)$,

$$U[v^*](x) = \min\{U_1[v^*](x), U_2[v^*](x)\} = U_1[v^*](x), \tag{2.75}$$

and for $x \in [t^*, \infty)$,

$$U[v^*](x) = \min\{U_1[v^*](x), U_2[v^*](x)\} = U_2[v^*](x). \tag{2.76}$$

Then

$$\begin{cases} v^*(x) = U[v^*](x), \\ \int_0^\infty v^*(s)f(s)ds - g^* \int_0^\infty \bar{F}(s)ds. \end{cases} \tag{2.77}$$

From Theorem 2, R1 is optimal for $x \in [0, t^*)$ and R2 is optimal for $x \in [t^*, \infty)$. It means that $t-$policy is optimal. □

3 The case that only R1 is optimal

Now, we consider the case of $p_1c_2 - p_2c_1 \geq 0$ in this chapter. Then, $p_1c_2 \geq p_2c_1 > 0$, that is $p_1 > 0$.

Lemma 6 *There exist a constant $\tilde{g}$ and a relative function $\tilde{v}$ which is bounded and continuously differentiable s.t.*

$$v(x) = c_1 + \frac{1-p_1}{\bar{F}(x)} \left\{ \int_x^\infty \tilde{v}(s)f(s)ds - \tilde{g} \int_x^\infty \bar{F}(s)ds \right\}, \tag{3.1}$$

and

$$\int_0^\infty \tilde{v}(s)f(s)ds - \tilde{g} \int_0^\infty \bar{F}(s)ds = 0. \quad \square \tag{3.2}$$

(Proof) If such function exists, it satisfies from Lemma 1

$$v(x) = \underline{U}_1[v](x). \tag{3.3}$$

We can describe its solution with a integration constant C for $p_1 > 0$

$$\tilde{v}(x) = \frac{c_1}{p_1} + \left(C - \frac{c_1}{p_1}\right) \bar{F}^{-p_1}(x) + (1-p_1)\tilde{g}\bar{F}^{-p_1}(x) \int_0^x \bar{F}^{p_1}(s)ds. \tag{3.4}$$

And $\tilde{v}(0) = c_1$, then $C = c_1$, it means

$$\tilde{v}(x) = \frac{c_1}{p_1} - \frac{1-p_1}{p_1} c_1 \bar{F}^{-p_1}(x) + (1-p_1)\tilde{g}\bar{F}^{-p_1}(x) \int_0^x \bar{F}^{p_1}(s)ds. \tag{3.5}$$

And we rewrite $\tilde{v}$,

$$\tilde{v}(x) = \frac{c_1}{p_1} - \frac{1-p_1}{\bar{F}^{p_1}(x)} \left\{ \frac{c_1}{p_1} - \tilde{g} \int_0^x \bar{F}^{p_1}(s)ds \right\}. \tag{3.6}$$

When $x \to \infty$, the denominator of its second part tends to 0, then it is necessarily

$$\frac{c_1}{p_1} - \tilde{g} \int_0^\infty \bar{F}^{p_1}(s)ds = 0. \tag{3.7}$$

That is

$$\tilde{g} = \frac{c_1}{p_1 \int_0^\infty \bar{F}^{p_1}(s)ds}. \tag{3.8}$$

We can consider that

$$\tilde{g}_1 \equiv \frac{c_1}{p_1 \int_0^\infty \bar{F}^{p_1}(s)ds} \tag{3.9}$$

and

$$\tilde{v}_1(x) = \frac{c_1}{p_1} - \frac{1-p_1}{p_1} c_1 \bar{F}^{-p_1}(x) + (1-p_1)\tilde{g}_1 \bar{F}^{-p_1}(x) \int_0^x \bar{F}^{p_1}(s)ds \tag{3.10}$$

satisfy (??),(3.2). □

Lemma 7 *If $p_1c_2 - p_2c_1 \geq 0$, then there exists $\tilde{g}_1$ and $\tilde{v}_1(x)$ such that*

$$\tilde{v}_1(x) = U_1[\tilde{v}_1](x) = U[\tilde{v}_1](x), \tag{3.11}$$

$$\int_0^\infty \tilde{v}_1(s)f(s)ds - \tilde{g}\int_0^\infty \bar{F}(s)ds = 0. \tag{3.12}$$

□

(Proof) We define

$$\overline{W}_1(x) \equiv \overline{U}_2[\tilde{v}_1](x) - \overline{U}_1[\tilde{v}_1](x), \tag{3.13}$$

then

$$\overline{W}_1(x) = \frac{p_1c_2 - p_2c_1}{p_1} + \frac{p_2 - p_1}{p_1} c_1 \bar{F}^{-p_1}(x) - (p_2 - p_1)\tilde{g}_1 \bar{F}^{-p_1}(x) \int_0^x \bar{F}^{p_1}(s)ds. \tag{3.14}$$

Now we describe

$$\tilde{w}_1(x, \tilde{g}_1) \equiv \frac{\bar{F}^{p_1}(x)}{p_2 - p_1} \overline{W}_1(x), \tag{3.15}$$

then

$$\tilde{w}_1(x) = \frac{p_1c_2 - p_2c_1}{p_1(p_2 - p_1)} \bar{F}^{p_1}(x) + \frac{c_1}{p_1} - \tilde{g}_1 \int_0^x \bar{F}^{p_1}(s)ds. \tag{3.16}$$

And

$$\begin{aligned} \frac{\partial}{\partial x} w_1(x, \tilde{g}_1) &= -\bar{F}^{p_1}(x) D(x, \tilde{g}_1) \\ &= -\bar{F}^{p_1}(x) \left\{ \frac{p_1c_2 - p_2c_1}{p_2 - p_1} \lambda(x) + \tilde{g}_1 \right\} \\ &< 0. \end{aligned} \tag{3.17}$$

Now $\tilde{w}_1(x, \tilde{g}_1)$ is decreasing in x,

$$\begin{aligned} \tilde{w}_1(\infty, \tilde{g}_1) &= \frac{p_1c_2 - p_2c_1}{p_1(p_2 - p_1)} \bar{F}^{p_1}(\infty) + \frac{c_1}{p_1} - \tilde{g}_1 \int_0^\infty \bar{F}^{p_1}(s)ds \\ &= \frac{c_1}{p_1} - \tilde{g}\int_0^\infty \bar{F}^{p_1}(s)ds \\ &= 0. \end{aligned} \tag{3.18}$$

Then for $\forall x \in [0, \infty), \tilde{w}_1(x, \tilde{g}) \geq 0$,that is

$$\overline{U}_2[\tilde{v}_1](x) \geq \overline{U}_1[\tilde{v}_1](x). \tag{3.19}$$

And $\overline{U}_1 = U_1, \overline{U}_2 = U_2$, then

$$\tilde{v}_1(x) = U[\tilde{v}_1](x). \tag{3.20}$$

□

Theorem 4 *If $p_1c_2 - p_2c_1 \geq 0$, then for all $x \in [0, +\infty)$,* R1 *is optimal action.* □

(Proof) From the Lemma 1, 2, $\tilde{g}_1, \tilde{v}_1$ satisfy

$$\tilde{v}_1 = U[\tilde{v}_1](x) \tag{3.21}$$

and

$$\int_0^\infty \tilde{v}_1(s)f(s)ds - \tilde{g}\int_0^\infty \bar{F}^{p_1}ds = 0. \tag{3.22}$$

By the theorem 2, for all $x \in [0, \infty)$ the activity R1 is optimal. □

4 Concluding Remarks

In this paper we discuss the optimal imperfect repair problem under the average cost criterion. We formulated the problem as semi-Markov decision process, and showed that an optimal policy for all policies is in the class of $t-$policies under some weak assumptions. Although preventive replacement is a very important maintenance activity, we could not consider it here. The study of the models which include this activity is left our future research.

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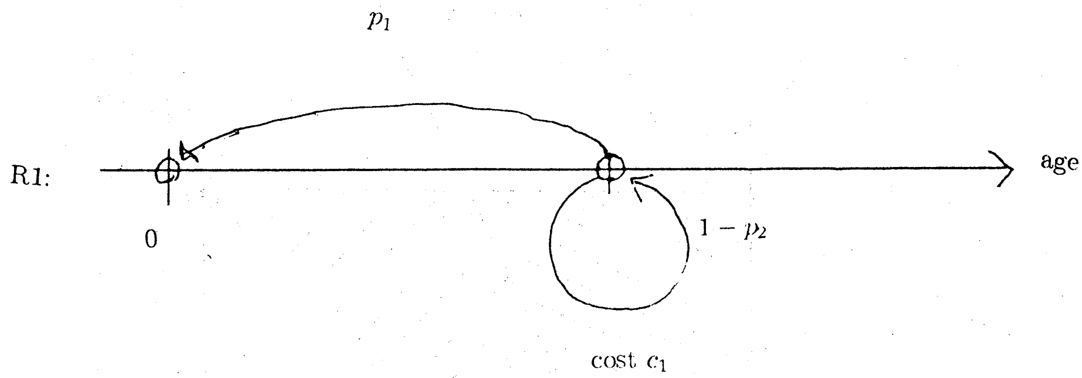
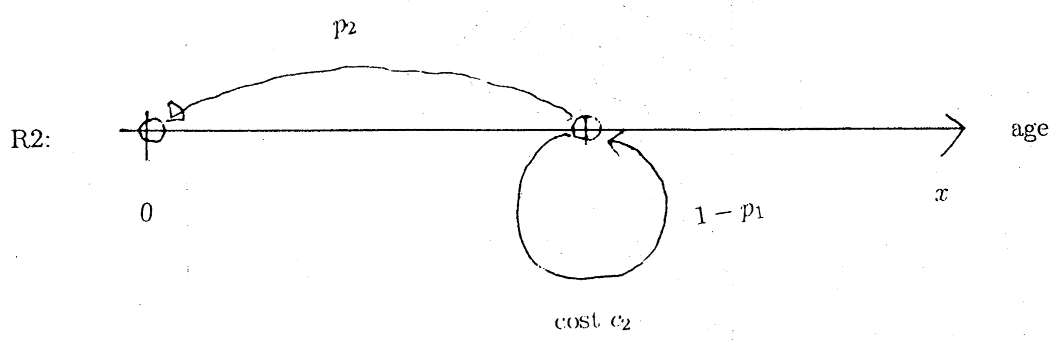
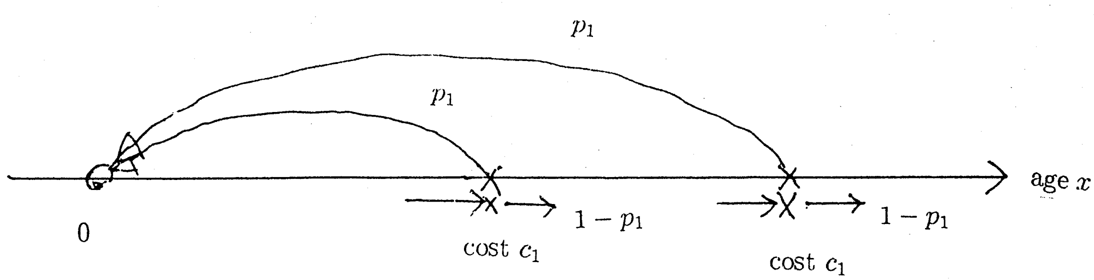
model (imperfect repair system)

maintenance activity





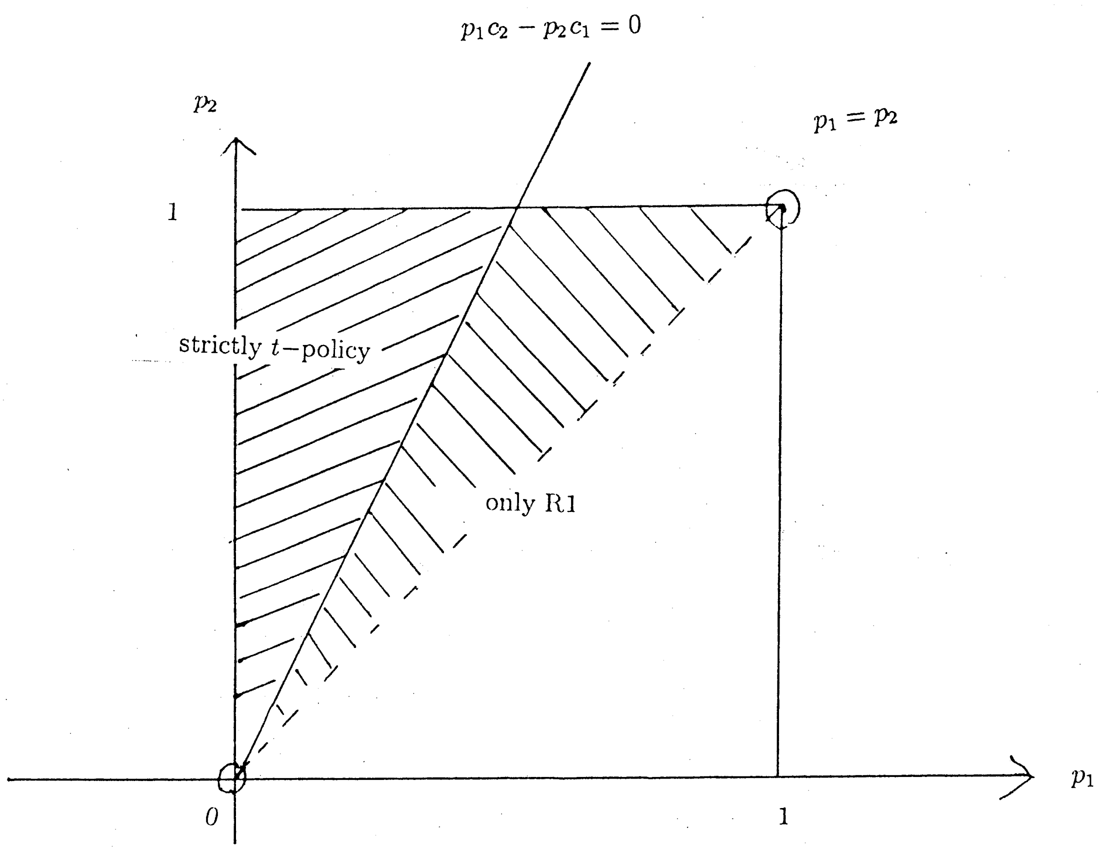
renewal process



x : failure age

The boundary $p_1c_2 - p_2c_1 = 0$ between strictly $t-$policy and only R1 policy is independent in the shape of reliability function.

(p_1, p_2)-plane



the structure of optimal policy