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On stochastic dynamical systems leaving fields of geometric objects invariant: revisited

(幾何学的対象の場を不変量にもつ確率的力学系について:再訪問)

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1 Introduction

Given a stochastic dynamical system on a manifold described by a stochastic differential equation (cf. [5]), we obtain a condition for the stochastic dynamical system to have a "field of geometric objects" (not necessary a tensor field) as an invariant. We also study a stochastic dynamical system leaving a G-structure of degree r invariant. For this end, we use a generalized Itô's formula applicable to fields of geometric objects ([1]).

In §2, we recall the notion of fields of geometric objects of order r, as well as the important notion of Lie differentiation of a field of geometric objects with respect to a vector field in the sense of Salvioli ([10]). In §3, using a generalized Itô's formula, we obtain a condition for the stochastic flow of diffeomorphisms generated by a stochastic differential equation on a manifold to leave a field of geometric objects invariant. In particular, we also obtain a condition for such a stochastic dynamical system to leave a G-structure of degree r invariant. Several examples are given in §4.

Some results in this note are based on [1] and [3].

2 Fields of geometric objects

Let M be a σ-compact, n-dimensional C^∞ manifold, and let $P^r(M)$ be the bundle of frames of r-th order contact over M with structure group $G^r(n)$ and projection π. Here, each point of $P^r(M)$ is the r-jet $j_0^r(f)$ at the origin $0 \in \mathbb{R}^n$ given by a diffeomorphism f of an open neighborhood of $0 \in \mathbb{R}^n$ onto an open set of M, and

$\pi(j_0^r(f)) := f(0)$. Also,

$$G^r(n) = \{j_0^r(\psi) : \psi \text{ is a diffeomorphism of an open neighborhood of } \\ 0 \in \mathbb{R}^n \text{ onto an open neighborhood of } 0 \in \mathbb{R}^n \text{ such that } \psi(0) = 0\}$$

acts on $P^r(M)$ on the right by the usual composition law of jets ([6]).

By a **field of geometric objects** (of order r), we shall mean a C^∞ section of a fiber bundle E associated with $P^r(M)$. (For simplicity, we assume that E admits a global C^∞ section and that the domain of the definition of a field of geometric objects is M.) Note that the (C^∞) vector fields, differential forms, (usual) tensor fields, pseudo-tensor fields, oriented tensor fields, and tensor densities are examples of fields of geometric objects of order one, and the affine connections without torsion and the projective structures (over M) are examples of fields of geometric objects of order two.

In particular, when $G^r(n)$ acts transitively on the standard fiber E_0 of E on the left, that is, E_0 is a homogeneous space $G^r(n)/G$ with G a closed subgroup of $G^r(n)$, then $E = P^r(M)/G = P^r(M) \times_{G^r(n)} (G^r(n)/G)$, and the **$G$-structures** of degree (or order) r (that is, the G-subbundles of $P^r(M)$) are in one-to-one correspondence with the fields of geometric objects $\sigma : M \to P^r(M)/G$ ([6]).

Now let $\pi_{T(M)} : T(M) \to M$ denote the tangent bundle over M and $T_x(M)$ the tangent space of M at $x \in M$. Let $\tilde{\varphi}$ be the transformation of $P^r(M)$ induced naturally from a C^∞ transformation φ of M, that is,

$$\tilde{\varphi}(j_0^r(f)) = j_0^r(\varphi \circ f), \quad j_0^r(f) \in P^r(M).$$

Then $\tilde{\varphi}$ induces naturally a transformation $\bar{\varphi}$ of E such that $\pi_E \circ \bar{\varphi} = \varphi \circ \pi_E$, where $\pi_E : E \to M$ is the projection. We define a section $\varphi^\sharp\sigma : M \to E$ by $\varphi^\sharp\sigma = \bar{\varphi}^{-1} \circ \sigma \circ \varphi$.

Correspondingly, a C^∞ vector field $X : M \ni x \mapsto X(x) \in T_x(M)$ induces a C^∞ vector field $\tilde{X}$ on $P^r(M)$ and a C^∞ vector field $\bar{X}$ on E, respectively, in a natural manner. In other words, X generates a local one-parameter group of local transformations φ_t of M, and φ_t induces a local one-parameter group of local transformations $\tilde{\varphi}_t$ (resp. $\bar{\varphi}_t$) of $P^r(M)$ (resp. E). Then $\tilde{X}$ (resp.$\bar{X}$) is the vector field generating $\tilde{\varphi}_t$ (resp. $\bar{\varphi}_t$). We set

$$\varphi_t^\sharp\sigma = (\bar{\varphi}_t)^{-1} \circ \sigma \circ \varphi_t.$$

The vector field $\tilde{X}$ (resp. $\bar{X}$) is called the **natural lift** of X to $P^r(M)$ (resp. E) (see [8] (for the case $r = 1$) and [6]).

Define the **Lie derivative**

$$\hat{L}_X\sigma : M \longrightarrow T(E)$$

of σ with respect to X **in the sense of Salvioli** by ([10])

$$(\hat{L}_X\sigma)(x) := \frac{d}{dt}(\varphi_t^\sharp\sigma)(x)\Big|_{t=0}$$
$$= \sigma_*(X(x)) - \bar{X}(\sigma(x)) \in T_{\sigma(x)}(E), \quad x \in M,$$

where σ_* denotes the differential of the map $\sigma : M \to E$. Note that $(\hat{L}_X\sigma)(x)$ is tangent to the fiber of E through $\sigma(x)$.

Let G be a closed subgroup of $G^r(n)$ and let P be a G-structure of degree r on M. A C^∞ transformation φ of M is called an **automorphism** of P if the induced transformation $\tilde{\varphi}$ of $P^r(M)$ maps P onto P. We prepare the following lemma (see, e.g., [3]).

Lemma 2.1 *Let G be a closed subgroup of $G^r(n)$. Let P be a G-structure of degree r on M, and $\sigma : M \to P^r(M)/G$ the field of geometric objects corresponding to P. Then:*

(1) For a C^∞ transformation φ of M, the G-structure of degree r corresponding to $\varphi^\sharp\sigma$ is given by $\tilde{\varphi}^{-1}(P)$.
(2) A C^∞ transformation φ of M is an automorphism of $P \iff \varphi^\sharp\sigma = \sigma$.
(3) X is an infinitesimal automorphism of $P \iff \hat{L}_X\sigma = 0$.

3 Stochastic dynamical systems leaving fields of geometric objects invariant

Let M and G be as in §2. Let $X_0, X_1, \ldots, X_k$ be C^∞ vector fields on M. For each $\lambda = 0, 1, \ldots, k$, let $\tilde{X}_\lambda$ be the natural lift of X_λ to $P^r(M)$, and consider the following stochastic differential equation in the Stratonovich form:

$$dp_t = \sum_{\lambda=0}^{k} \tilde{X}_\lambda(p_t) \circ dw_t^\lambda. \tag{3.1}$$

Here, $w_t^0 \equiv t$, and $w_t = (w_t^1, \ldots, w_t^k)$ is the k-dimensional Wiener process realized canonically on the k-dimensional standard Wiener space. The solution with the initial condition $p_s = p \in P^r(M)$ is denoted by $p_{s,t}(p) = (p_{s,t}(p,w))$, $0 \le s \le t$. Then $p_{s,t}$ is a (stochastic) map $p_{s,t} : P^r(M) \to P^r(M)$. Assume that $p_{s,t}$ defines a stochastic flow of (C^∞) diffeomorphisms of $P^r(M)$. Then $p_{s,t}$ induces a stochastic flow $\xi_{s,t}$ of diffeomorphisms of M. Note that $\xi_{s,t}$ is also generated by the following stochastic differential equation:

$$d\xi_t = \sum_{\lambda=0}^{k} X_\lambda(\xi_t) \circ dw_t^\lambda.$$

Note also that, for almost all w,

$$p_{s,t}(j_0^r(f)) = j_0^r(\xi_{s,t} \circ f) = \tilde{\xi}_{s,t}(j_0^r(f)), \quad j_0^r(f) \in P^r(M), \quad 0 \leq s \leq t.$$

Moreover, $\xi_{s,t}$ generates a stochastic flow $\eta_{s,t}(= \bar{\xi}_{s,t})$ of diffeomorphisms of E; $\eta_{s,t}$ is also generated by the following stochastic differential equation:

$$d\eta_t = \sum_{\lambda=0}^{k} \bar{X}_\lambda(\eta_t) \circ dw_t^\lambda,$$

where $\bar{X}_\lambda$ is the natural lift of X_λ to E. We define the **stochastic deformation** of σ by

$$\xi_{s,t}^\sharp \sigma = \eta_{s,t}^{-1} \circ \sigma \circ \xi_{s,t}.$$

We shall say that a field of geometric objects $\sigma : M \to E$ is an **invariant** of $\xi_{s,t}$ if

$$\xi_{s,t}^\sharp \sigma = \sigma \text{ (a.s.)}.$$

Theorem 3.1 *Let $\sigma : M \to E$ be a field of geometric objects. Suppose the equation (3.1) generates a stochastic flow $p_{s,t}$ of (C^∞) diffeomorphisms of $P^r(M)$ (with probability 1). Then for the stochastic flow $\xi_{s,t}$ of diffeomorphisms of M induced from $p_{s,t}$, it holds that*

$$\sigma \text{ is an invariant of } \xi_{s,t} \iff \hat{L}_{X_\lambda} \sigma = 0 \quad (\lambda = 0, 1, \ldots, k).$$

Theorem 3.2 *Let G be a closed subgroup of $G^r(n)$, and let P be a G-structure of degree r on M. Let $\sigma : M \to P^r(M)/G$ be the field of geometric objects corresponding to P. Assume that the equation (3.1) generates a stochastic flow $p_{s,t}$ of diffeomorphisms of $P^r(M)$ (with probability 1). Then for the stochastic flow $\xi_{s,t}$ of diffeomorphisms of M induced from $p_{s,t}$, it holds that*

$$\xi_{s,t} \text{ is a stochastic flow of automorphisms of } P \iff \hat{L}_{X_\lambda} \sigma = 0 \quad (\lambda = 0, 1, \ldots, k).$$

These theorems are proved by using the following theorem ([1]). (For a tangent vector or a vector field Y on N, we denote by $Y[H]$ the operation of Y on a C^∞ function $H : N \to \mathbb{R}$.)

Theorem 3.3 (Generalized Itô's formula for $\xi_{s,t}^\sharp \sigma$) *For a C^∞ function $F : E \to \mathbb{R}$, it holds that*

$$\begin{aligned} F \circ (\xi_{s,t}^\sharp \sigma)(x) &- F \circ \sigma(x) \\ &= \sum_{\lambda=0}^{k} \Phi_{s,t}^\lambda(x, F) + \frac{1}{2} \sum_{\alpha=1}^{k} \int_s^t (X_\alpha(\xi_{s,u}(x))[((\hat{L}_{X_\alpha}\sigma)(\cdot))[F \circ \eta_{s,u}^{-1}]] \\ &\qquad - ((\hat{L}_{X_\alpha}\sigma) \circ \xi_{s,u}(x))[\bar{X}_\alpha[F \circ \eta_{s,u}^{-1}]]) \cdot du, \quad (x \in M), \end{aligned}$$

where

$$\Phi^{\lambda}_{s,t}(x,F) := \int_s^t (\eta_{s,u})_*^{-1}((\hat{L}_{X_\lambda}\sigma) \circ \xi_{s,u}(x))[F] \cdot dw_u^{\lambda},$$

and $\cdot dw_u^{\lambda}$ *denotes the Itô stochastic differential.*

In the case where E is a vector bundle, we have the following.

Theorem 3.4 ([1]) *Let E be a vector bundle associated with $P^r(M)$. Then, for a field of geometric objects $\sigma : M \to E$,*

$$\xi^{\sharp}_{s,t}\sigma - \sigma = \sum_{\alpha=1}^{k} \int_s^t \xi^{\sharp}_{s,u}\mathcal{L}_{X_\alpha}\sigma \cdot dw_u^{\alpha} + \int_s^t \xi^{\sharp}_{s,u}\left[\mathcal{L}_{X_0} + \frac{1}{2}\sum_{\alpha=1}^{k}(\mathcal{L}_{X_\alpha})^2\right]\sigma\, du.$$

Here

$$\mathcal{L}_X\sigma = \lim_{t\to 0}\frac{1}{t}(\bar{\varphi}_t^{-1} \circ \sigma \circ \varphi_t - \sigma)$$

under the notations in §2.

Remark 3.1 In particular, in the case where σ is a tensor field, the corresponding result is also given in, e.g., [4] and [9].

4 Examples

Example 4.1 *Stochastic flow of projective transformations.*

Let P be a projective structure on M with $\dim M = n \geq 2$. (Therefore, $r = 2$ and $G = H^2(n)$ in the sense of [6] and [7].) Then $\xi_{s,t}$ is a stochastic flow of projective transformations of M with respect to P if and only if each X_λ is an infinitesimal projective transformation ([3]).

Example 4.2 *Stochastic dynamical system having an ℓ-dimensional C^∞ distribution on M as an invariant.*

Let $L(M)(= P^1(M))$ be the bundle of linear frames over M ($\dim M \geq 2$). Let $\ell \in \mathbb{N}$ be such that $1 \leq \ell < n = \dim M$, and let $G(n,\ell)$ be the Grassmann manifold formed of ℓ-dimensional subspaces of $\mathbb{R}^n$ ([8]). The general linear group $G(n,\mathbb{R})$ acts on $G(n,\ell)$ on the left. Then we have a fiber bundle E (with standard fiber $G(n,\ell)$ and structure group $GL(n,\mathbb{R})$) associated with $L(M)$. This E is called the (unoriented) **Grassmann bundle** of ℓ-planes over M in the literature. The C^∞ sections of E are in one-to-one correspondence with the ℓ-dimensional C^∞ distributions on M.

Let $\mathcal{D}$ be an ℓ-dimensional C^∞ distribution on M, and let $\sigma_{\mathcal{D}} : M \to E$ be the field of geometric objects corresponding to $\mathcal{D}$. Then we can define an ℓ-dimensional stochastic distribution $\mathcal{D}_{s,t}$ (the stochastic deformation of $\mathcal{D}$) as the (stochastic) distribution corresponding to $\xi^{\sharp}_{s,t}\sigma_{\mathcal{D}}$, and it holds that

$$\mathcal{D} \text{ is an invariant of } \xi_{s,t} \iff \xi^{\sharp}_{s,t}\sigma_{\mathcal{D}} = \sigma_{\mathcal{D}} \text{ (a.s.)} \iff \hat{L}_{X_\lambda}\sigma_{\mathcal{D}} = 0 \ (\lambda = 0, 1, \ldots, k).$$

Example 4.3 *Stochastic dynamical system leaving a second order linear (partial) differential operator on C^∞ functions on M invariant.*

Let $\mathbb{F} = (\mathbb{R}^n \circledcirc \mathbb{R}^n) \oplus \mathbb{R}^n \oplus \mathbb{R} (\cong \mathbb{R}^m, m = n(n+1)/2 + n + 1)$, where the symbol $\circledcirc$ stands for the symmetric tensor product. Define the action of $G^2(n)$ on $\mathbb{F}$ on the left as follows:

$$(s^i_j; s^i_{jm})(a^{ij}; b^i; c) = \left(\sum_{m,\ell=1}^{n} s^i_m s^j_\ell a^{m\ell}; \sum_{j,m=1}^{n} s^i_{jm} a^{jm} + \sum_{j=1}^{n} s^i_j b^j; c \right),$$

where $(s^i_j; s^i_{jm})$ $(s^i_{jm} = s^i_{mj})$ and $(a^{ij}; b^i; c)$ $(a^{ij} = a^{ji})$, $i, j, m = 1, \ldots, n$, are natural coordinates of $G^2(n)$ and $\mathbb{F}$, respectively. Then we obtain a vector bundle $E(M, \mathbb{F}, G^2(n), P^2(M))$ with standard fiber $\mathbb{F}$ and structure group $G^2(n)$, associated with $P^2(M)$. Each C^∞ section σ of E corresponds to a second order (possibly degenerate) linear (partial) differential operator $\mathcal{A}_\sigma$ on $\mathbb{R}$-valued C^∞ functions on M (cf. [2]). Each field of geometric objects $\sigma : M \to E$ is expressed locally as

$$\left(x^i; \left[\left(\frac{\partial^2}{\partial x^i \partial x^j}; \frac{\partial}{\partial x^i} \right), (a^{ij}(x); b^i(x); c(x)) \right] \right), (i, j = 1, \ldots, n),$$

and $\mathcal{A}_\sigma$ is expressed locally as

$$\mathcal{A}_\sigma = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x^i} + c(x).$$

It holds that

$$\mathcal{A}_\sigma \text{ is an invariant of } \xi_{s,t} \iff \mathcal{L}_{X_\lambda}\sigma = 0 \quad (\lambda = 0, 1, \ldots, k).$$

Example 4.4 *Mean invariants.*

Let E be a vector bundle associated with $P^r(M)$. Then for a field of geometric objects $\sigma : M \to E$, we have, by Theorem 3.4,

$$\sigma \text{ is a } \textbf{mean invariant} \text{ of } \xi_{s,t} \overset{\text{def}}{\iff} \text{(the expectation) } \mathbb{E}[\xi^{\sharp}_{s,t}\sigma] = \sigma$$
$$\iff \left(\frac{1}{2} \sum_{\alpha=1}^{k} (\mathcal{L}_{X_\alpha})^2 + \mathcal{L}_{X_0} \right) \sigma = 0.$$

We also give the following example, although the setting is slightly different from that of §3.

Example 4.5 *Random acceleration on a Riemannian manifold.*

Let (M, g) be a C^∞ Riemannian manifold, and let Φ be the geodesic spray. Let $\hat{X}_\lambda$ be the vertical lift of X_λ to $T(M)$, $\lambda = 0, 1, \ldots, k$; that is,

$$(\hat{X}_\lambda)_v = \frac{d}{dt}(v + t(X_\lambda)_{\pi_{T(M)}(v)})\bigg|_{t=0} \in T_v(T(M)).$$

Consider the following stochastic differential equation on $T(M)$ (an equation of random acceleration):

$$dV_t = (\Phi + \hat{X}_0)(V_t)dt + \sum_{\alpha=1}^{k} \hat{X}_\alpha(V_t) \circ dw_t^\alpha.$$

Let θ be a C^∞ differential 1-form on M, and define a C^∞ function F_θ on $T(M)$ by

$$F_\theta(v) = \theta(v), \quad v \in T(M).$$

Then

$$d(\theta(V_t)) = dF_\theta(V_t) = (\Phi + \hat{X}_0)[F_\theta](V_t)dt + \sum_{\alpha=1}^{k} \hat{X}_\alpha[F_\theta](V_t) \circ dw_t^\alpha.$$

For $v \in T(M)$, we have

$$\begin{aligned} \hat{X}_\lambda[F_\theta](v) &= \frac{d}{dt}(v + t(X_\lambda)_{\pi_{T(M)}(v)})[F_\theta]\bigg|_{t=0} \\ &= \frac{d}{dt}\theta(v + t(X_\lambda)_{\pi_{T(M)}(v)})\bigg|_{t=0} = \theta(X_\lambda)(\pi_{T(M)}(v)). \end{aligned}$$

Also, $\Phi[F_\theta](v)$ is expressed locally as follows:

$$\begin{aligned} &\left(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} - \sum_{i=1}^{n} \sum_{j,m=1}^{n} \Gamma_{jm}^i v^j v^m \frac{\partial}{\partial v^i}\right) \left[\sum_{\ell=1}^{n} \theta_\ell v^\ell\right] \\ &= \sum_{i,j=1}^{n} \left(\frac{\partial \theta_j}{\partial x^i} - \sum_{m=1}^{n} \Gamma_{ij}^m \theta_m\right) v^i v^j = \frac{1}{2} \sum_{i,j=1}^{n} (\nabla_i \theta_j + \nabla_j \theta_i) v^i v^j, \end{aligned}$$

where ∇ stands for covariant differentiation with respect to the Levi-Civita connection and Γ_{jm}^i the connection coefficients. Therefore, if

$$\nabla_i \theta_j + \nabla_j \theta_i = 0 \quad \text{(Killing equation)} \quad \text{and} \quad \theta(X_\lambda) = 0$$

for $i, j = 1, \ldots, n$ and $\lambda = 0, 1, \ldots, k$, that is, if the vector field X_θ corresponding to θ through g (namely, $g(X_\theta, \cdot) = \theta$) is a Killing vector field and is orthogonal to each X_λ in the sense that $g(X_\theta, X_\lambda) = 0$, then F_θ is an invariant of the solution of the above equation of random acceleration.

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