p-adic orbifolds and *p*-adic triangle groups.

by

Yves André

Summary. We study the *p*-adic linear differential equations which have the property that their pull-back on some finite étale covering of the base admits a full set of multivalued analytic solutions. Such equations admit a global monodromy group as in the complex case. We introduce the notion of *p*-adic orbifold fundamental group. Its algebraic structure depends on the relative *p*-adic position of the singularities. Its "discrete" representations correspond to the differential equations under study - for which it is then possible to describe the relationship between global and local monodromy.

Interesting examples occur in the context of *p*-adic period mappings, including some hypergeometric instances. This leads to a zoo of *p*-adic triangle (quadrangle...) groups. This embryo of *p*-adic crystallography will be developed in the Tôhoku part of the booklet [TCl[.

§1 Example: a hyperbolic quadrangle group and its pentadic counterpart.

In the complex upper half plane $\mathfrak{H}$, we consider the half-line L of slope $-1/2$ through the origin, the half-circle Σ (resp. Σ') centered at the origin of radius 1 (resp. $2 - \sqrt{3}$), and the half-circle Σ'' centered on the real axis, tangent to Σ and containing the point $D = L \cap \Sigma'$. We denote by B (resp. C) the point at the intersection of Σ (resp. Σ') with the imaginary axis, and by A the point at the intersection of Σ and Σ'' (see Fig.). Explicitly, $A = -\frac{1+2i}{\sqrt{5}}$.

Then $ABCD$ forms a hyperbolic quadrangle with angles $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{6})$. The symmetries around the edges of this polygon generate a group of Möbius transformations which contains a subgroup Γ of index 2 of conformal transformations.

This fuchsian group is conjugated in $PSL_2(\mathbf{R})$ to the group generated by the following unimodular matrices (up to sign):

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{\sqrt{5}+1}{2} \\ \frac{\sqrt{5}-1}{2} & 0 \end{pmatrix}, \begin{pmatrix} -\sqrt{3} & \sqrt{5}+1 \\ 1-\sqrt{5} & \sqrt{3} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}+\sqrt{15}}{2} & -2 \\ 2 & \frac{\sqrt{3}-\sqrt{15}}{2} \end{pmatrix},$$

of order 2, 2, 2, and 6 respectively; we notice that the product of these matrices (in that order) is the identity. This group was studied in different presentation by several people (J.F. Michon, M.F. Vignéras [Vi80] p.123, D. Krammer [K96] ...). A fundamental domain for Γ is given by $ABCD$ together with its reflection across AB. The quotient $\mathfrak{H}/\Gamma$ is isomorphic to the complex projective line. More precisely, as an orbifold, $\mathfrak{H}/\Gamma$ is $\mathscr{X}_\Gamma = (\mathbf{P}^1,(0;2),(1;2),(81;2),(\infty;6))$ (a branch point ζ with multiplicity n being denoted by $(\zeta;n)$), and its orbifold fundamental group is precisely Γ; the points A,B,C,D are mapped to $0,1,81,\infty$ respectively.

In order to figure out what could be a p-adic counterpart of the quadrangle group Γ, we look at a *uniformizing differential equation* attached to $\mathscr{X}_\Gamma$ [Y87]. Such a differential equation is of Lamé type, and has been displayed by Krammer:

$$(*) \quad 18Py'' + 9P'y' + (z-9)y = 0, \quad \text{where } ' = \frac{d}{dz} \text{ and } P = z(z-1)(z-81).$$

In particular, the projective monodromy group of $(*)$ is Γ.

Fig.
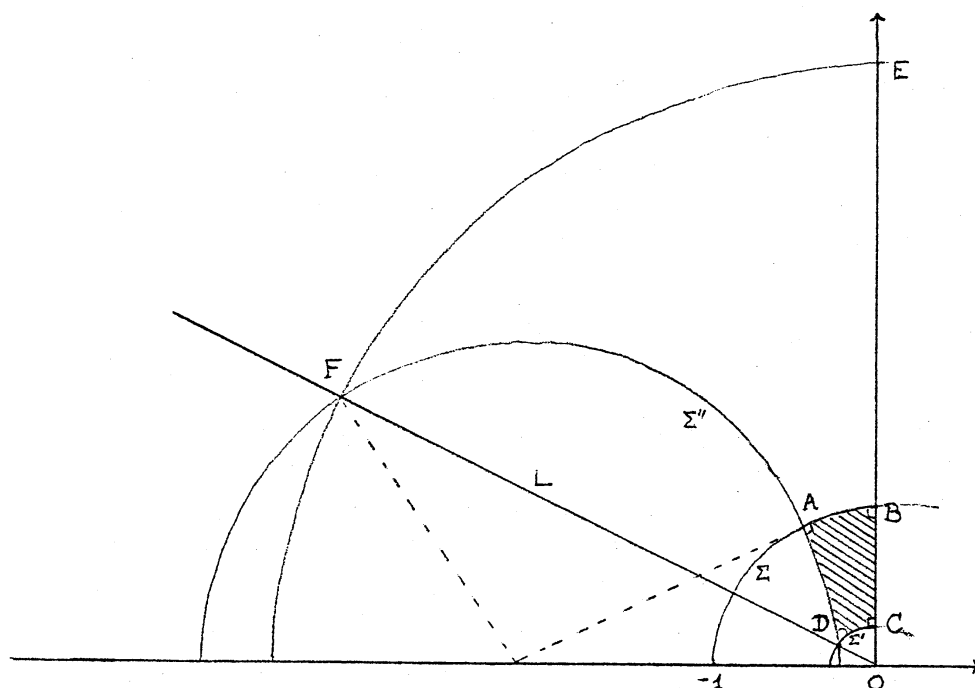
Let us now examine this differential equation from the p-adic viewpoint. For $p \geq 7$, one can show that the situation is the familiar one: one has a Frobenius structure, for which there are only finitely many supersingular disks; in the complement of these disks, the eigenvalues of Frobenius are of different magnitude, and this leads to a factorization of the differential operator $18Py'' + 9P'y' + (z - 9)y$ into two analytic operators of order one.

The case $p = 5$ is much more surprising. First, there is a *confluence* between the singularities 1 and 81 in characteristic 5. Next, it turns out that there is a Frobenius structure, for which *all residue classes are supersingular*. But the main feature for our purpose is given by the following

Theorem. *There is a Galois étale covering S of $\mathbf{P}^1 \setminus \{0, 1, 81, \infty\}$, such that the pull-back of* (*) *over S admits a full set of 5-adic multivalued analytic solutions. The associated projective monodromy group is a discrete subgroup Γ_5 of $PGL_2(\mathbf{Q}_5)$; there exist four elements of order 2, 2, 2, and 6 respectively, such that any three of them generate Γ_5.*

We call Γ_5 a *pentadic quadrangle group*. The situation is therefore very similar to the complex one. In fact, we shall obtain such generators of order 2, 2, 2, and 6 as local monodromy automorphisms in a suitable sense. We have not yet been able to compute such generators for Γ_5, but we can show that Γ_5 is generated in $PSL_2(\mathbf{Q}_5(\sqrt{3}, \sqrt{5}))$ by the following unimodular matrices:

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & -\sqrt{3} \\ \frac{\sqrt{3}}{3} & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 1+i & -3 \\ 1 & 1-i \end{pmatrix}, \quad \frac{1}{\sqrt{5}} \begin{pmatrix} 1+i & 3 \\ -1 & 1-i \end{pmatrix}, \quad \frac{1}{\sqrt{5}} \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix}.$$

We remark that if we consider these matrices in $PSL_2(\mathbf{C})$ instead of $PSL_2(\mathbf{Q}_5(\sqrt{3}, \sqrt{5})$ the group which they generate is no longer discrete.

We shall return to this example in §8, and give some explanation after having set up the framework for understanding such p-adic global monodromy phenomena.

§2 Topological coverings and étale coverings in the p-adic setting.

2.1. In complex geometry, there is no need to distinghish between topological coverings and étale coverings (finite or infinite). Complex manifolds are locally contractible, and have universal coverings.

In p-adic rigid geometry, the situation is more complicated. It is natural to call topological covering any morphism $f : Y \to X$ such that there is an admissible cover (X_i) of X and an admissible cover (Y_{ij}) of $f^{-1}(X_i)$ with disjoint Y_{ij} isomorphic to (X_i) via f. Indeed, such topological coverings correspond to locally constant sheaves of sets on X. It is still true that topological coverings are étale, but the converse is wrong, even if one restricts to finite surjective

morphisms. Indeed, the Kummer covering $z \to z^n$ of the punctured disk is an étale covering, but not a topological covering (see e.g. [vP83]).

2.2. It is more convenient to deal with these questions in the framework of V. Berkovich p-adic geometry [B90], due to the nice topological properties of Berkovich's analytic spaces. We consider a field k, complete under a p-adic valuation ($k \subset \mathbf{C}_p$), and work with smooth (Hausdorff) strictly k-analytic spaces, which we call *p-adic manifolds*, for simplicity. These spaces are locally compact and locally arcwise connected, and Berkovich has recently showed that they are locally contractible, hence have universal coverings [B97[.

Topological coverings of a p-adic manifold X are defined in the usual way; they correspond to locally constant sheaves of sets on X. They coincide with topological coverings of the rigid analytic variety associated to X at least if X is paracompact (e.g. in the one-dimensional case, cf. [LivP95]).

2.3. Berkovich has defined, and J. De Jong has studied [dJ95], *etale coverings* in this context: a morphism $f : Y \to X$ of p-adic manifolds is an étale covering (map) if for all $x \in X$, there exists an open neighborhood $U_x \subset X$ of x such that $f^{-1}(U_x)$ is a disjoint union of spaces, each mapping finite étale to U_x. In the case of a finite morphism, this just means that f is étale; if k is algebraically closed, this also means that f induces an isomorphism on the completed local rings of the associated rigid varieties. A typical example of an infinite étale covering map is the logarithm $log : D(1, 1^-) \longrightarrow k$ (with Galois group μ_{p^∞}).

2.4. It is probably not true that the composite of two étale covering maps remains an étale covering map. However:

Lemma. *Any morphism composed from an étale covering map followed or preceded by a finite étale morphism is an étale covering map. Moreover, a morphism f is an étale covering map if its composition $g \circ f$ with some finite etale map g is an étale covering map.*

Proof. This is clear if the étale covering map follows the finite étale morphism. Let us now consider the case of an étale covering map $f : Y \to X$ followed by a finite étale morphism $g : X \to X'$. Let x' be a point of X'. Then for any point x in the finite set $g^{-1}(x')$, there exists an open neighborhood $U_x \subset X$ of x such that $f^{-1}(U_x)$ is a disjoint union of spaces $U_{x,i}$, each mapping finite étale to U_x. We may assume that the U_x are pairwise disjoint. Since g is finite, the underlying topological map is closed, hence $g^{-1}(x')$ admits a basis of open neighborhoods of the form $g^{-1}(V)$. In particular, there is an open neighborhood $V_{x'}$ of x' such that $\amalg_{x \in g^{-1}(x')} U_x$ contains $g^{-1}(V_{x'})$. We may replace each U_x by its intersection with $g^{-1}(V_{x'})$ (which is a union of connected components of $g^{-1}(V_{x'})$). It is then clear that $g \circ f$ induces a finite étale morphism from $U_{x,i}$ to $V_{x'}$. Hence $g \circ f$ is an étale covering map.

2.5. Heuristically, one may say that there are more topological coverings in complex geometry than in p-adic geometry, but less étale coverings. Indeed, by Riemann's uniformization theorem, there are only three simply-connected

one-dimensional complex manifolds up to isomorphism; but a one-dimensional p-adic manifold is simply-connected if and only if the graph of its semi-stable reduction is a tree [DJ95]5.3; in particular, *p-adic algebraic curves with good reduction, p-adic punctured disks and annuli are simply-connected.* On the other hand, the complex projective line has no non-trivial étale coverings, while the p-adic projective line has many infinite connected étale coverings. An explicit example, for $p \equiv 3$ mod. 4, is given by $f : D(\frac{1}{2}, 1^-) \longrightarrow \mathbf{P}^1$, with

$$f = z \cdot {}_2F_1(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}, z)/({}_2F_1(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, z) + \frac{z}{2} \cdot {}_2F_1(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}, z)),$$

cf. [TCl[.

2.6. Let X be a *connected* p-adic manifold. Let $\bar{x}$ be a geometric point of X with value in some complete algebraically closed extension Ω of k. Let us consider the functor $F_{\bar{x}} : \{etale\ coverings\ of\ X\} \longrightarrow \{Sets\}$, which associates to a covering Y/X the set of Ω-valued geometric points of Y lying above $\bar{x}$. In [dJ95], De Jong defines the *etale fundamental group* $\pi_1{}^{et}(X, \bar{x})$ pointed at $\bar{x}$ as the automorphism group of $F_{\bar{x}}$, and equip it with a canonical topology of Hausdorff pro-discrete group. It is independent of $\bar{x}$ up to isomorphism, and has the property that the category of $\pi_1{}^{et}(X, \bar{x})$-sets is naturally equivalent to the category of disjoint unions of connected étale covering spaces of X (equivalence depending on $\bar{x}$). It is related to the usual discrete topological fundamental group $\pi_1{}^{top}(X, \bar{x})$ and to the (pro)finite-étale fundamental group $\pi_1{}^{alg}(X, \bar{x})$ by homomorphisms:

$$\pi_1{}^{et}(X, \bar{x}) \to \pi_1{}^{top}(X, \bar{x}) \text{ and } \pi_1{}^{et}(X, \bar{x}) \to \pi_1{}^{alg}(X, \bar{x}).$$

The first map is surjective, while the second one has only dense image (and is not strict) in general.

Lemma. *Let Y/X be a connected finite Galois étale covering with group G, and let $\bar{y}$ be a geometric point of Y above $\bar{x}$. Then the kernel of the map $\pi_1{}^{et}(X, \bar{x}) \to G$ is the closure of the image of $\pi_1{}^{et}(Y, \bar{y}) \to \pi_1{}^{et}(X, \bar{x})$.*

This means that for any étale covering X'/X which splits over Y, the corresponding action of $\pi_1{}^{et}(X, \bar{x})$ on $F_{\bar{x}}(X')$ factors through G, which is clear.

2.7. In this paper, we shall be concerned only with those étale covering maps which are obtained from an infinite topological covering map followed by a finite étale morphism. To study such simple covering maps, it is convenient to introduce the *reduced etale fundamental group*, which seems to be a reasonable analogue of the complex fundamental group:

$$\pi_1{}^{red}(X, \bar{x}) := Coim(\pi_1{}^{et}(X, \bar{x}) \longrightarrow (\pi_1{}^{top}(X, \bar{x}) \times \pi_1{}^{alg}(X, \bar{x})))$$

(*Coim* being isomorphic to *Im* only as an abstract group).

2.8. The topological groups $\pi_1{}^{et}(X, \bar{x}), \pi_1{}^{top}(X, \bar{x}), \pi_1{}^{alg}(X, \bar{x}), \pi_1{}^{red}(X, \bar{x})$ are functorial in $(X, \bar{x})$.

2.9. Let us give examples. Assume that X is an elliptic curve, and $k = \mathbf{C}_p$; there are two cases:

if X has good reduction, then $\pi_1{}^{top}(X, \bar{x}) = 0$ and $\pi_1{}^{red}(X, \bar{x}) \cong \pi_1{}^{alg}(X, \bar{x}) \cong \hat{\mathbf{Z}}^2$,

if X has bad (multiplicative) reduction, then $\pi_1{}^{top}(X, \bar{x}) \cong \mathbf{Z}$, $\pi_1{}^{alg}(X, \bar{x}) \cong \hat{\mathbf{Z}}^2$, and $\pi_1{}^{red}(X, \bar{x}) \cong \mathbf{Z} \times \hat{\mathbf{Z}}$.

(In contrast, one can show that the étale fundamental group contains a "huge" non-commutative subgroup, irrelevant for our study). It is a general principle that *bad reduction* reflects into the presence of *infinite discrete quotients for the reduced étale fundamental group*.

In the case of multiplicative reduction, X is a Tate curve: $X \cong \mathbf{C}_p^{\times}/q^{\mathbf{Z}}$ ($|q| < 1$), $\tilde{X} \cong \mathbf{C}_p^{\times}$. Let G denote group of order 2 generated by the inversion on $\tilde{X}$ or X. The morphism

$$(\mathbf{C}_p^{\times} \setminus \pm\sqrt{q}^{\mathbf{Z}})/G \longrightarrow (X \setminus X[2])/G$$

is an interesting example of an étale covering which is not a topological covering, but which becomes a topological covering by finite étale base-change $X \setminus X[2] \to (X \setminus X[2])/G$.

2.10. For any one-dimensional p-adic manifold X, the topological fundamental group $\pi_1{}^{top}(X, \bar{x})$ is a discrete free group isomorphic to the fundamental group of the dual graph Δ of the semistable reduction of X ([dJ95]5.3, [LivP95]). When $b_1(\Delta) < \infty$, it follows that $\pi_1{}^{top}(X, \bar{x})$ is residually finite, i.e. embeds into its profinite completion, which is a quotient of $\pi_1{}^{alg}(X, \bar{x})$; therefore, $\pi_1{}^{red}(X, \bar{x}) \cong Coim(\pi_1{}^{et}(X, \bar{x}) \longrightarrow \pi_1{}^{alg}(X, \bar{x}))$ in this case. The $\pi_1{}^{red}(X, \bar{x})$-sets then correspond to disjoint unions of étale coverings $Y \to X$ which can be "approximated" by finite étale sub-coverings $Y_\alpha \to X$ (Y dense in $\varprojlim Y_\alpha$).

§3 p-adic connections with locally constant sheaves of solutions.

3.1. Let us briefly recall the complex situation. Let S be a complex manifold, $(\mathcal{M}, \nabla)$ a vector bundle of rank r with integrable connection on S. The classical Cauchy theorem shows that for any $s \in S$, the solution space $(\mathcal{M} \otimes \mathcal{O}_{S,s})^{\nabla}$ at s has dimension r. Analytic continuation along paths gives rise to a homomorphism $\pi_1{}^{top}(S, s) \to Aut_{\mathbf{C}}((\mathcal{M} \otimes \mathcal{O}_{S,s})^{\nabla})$ (the *monodromy*). The sheaf of germs of solutions $\mathcal{M}^{\nabla}$ is locally constant on S: its pull-back over the universal covering $\tilde{S}$ of S is constant. Conversely, any complex representation V of $\pi_1{}^{top}(S, s)$ of dimension r gives rise naturally to a vector bundle $\mathcal{M}$ a vector bundle of rank r with integrable connection ∇ ($\mathcal{M} = (V \times \tilde{S})/\pi_1{}^{top}(S, s)$, $\nabla(V) = 0$).

This sets up an equivalence of categories:

$$(finite-dim.\ repr.\ of\ \pi_1{}^{top}(S, s)) \simeq (S-vector\ bundles\ with\ integrable\ connection).$$

3.2. Let S now be a p-adic manifold, and let $\bar{s}$ be a geometric point with image $s \in S$. It is still true that any k-linear representation V of $\pi_1{}^{top}(S, \bar{s})$ of dimension r gives rise naturally to a vector bundle $\mathcal{M}$ a vector bundle of rank r with integrable connection ∇ (same formula). The functor

$$(finite-dim.\ repr.\ of\ \pi_1{}^{top}(S, \bar{s})) \rightarrow (S-vector\ bundles\ with\ integrable\ connection)$$

is still fully faithful, but *no longer surjective*; its essential image consists of those connections whose sheaf of solutions is locally constant (i.e. becomes constant over $\tilde{S}$).

In fact, the classical "Cauchy theorem" according to which the solution space $(\mathcal{M} \otimes \mathcal{O}_{S,s})^{\nabla}$ at s has dimension r is true for every classical point of S - which corresponds to a point of the associated rigid variety -, but does not hold for non-classical points s of the Berkovich space S in general (it can be saved however by performing a suitable extension of scalars which makes s classical, as in Dwork's technique of generic points). When "Cauchy's theorem" holds at every point of S, one can continue the local solutions along paths as in the complex situation. This nice category of connections has not attracted much attention from p-adic analysts until now.

3.3. Let us consider the case when S is a Tate elliptic curve: $S = k^{\times}/q^{\mathbf{Z}}$, with $\bar{s} = s =$ its origin. Then $\pi_1{}^{top}(S, \bar{s}) = q^{\mathbf{Z}}$, and the connections on S which arise from representations of $q^{\mathbf{Z}}$ are those which become trivial over $\tilde{S} = k^{\times}$. It turns out that they correspond to certain q-*difference equations* with analytic coefficients on $k^{\times}$.

The simplest example is given by $\mathcal{M} = \mathcal{O}_S$, $\nabla(1) = \omega_{can}$ (the canonical differential induced by dt/t); this amounts to the differential equation

$$(**) \quad dy = y.\omega_{can} \ ,$$

for which an analytic multivalued generator of the space of solutions is given by the coordinate t on $k^{\times}$. The monodromy group is $q^{\mathbf{Z}}$ itself.

If we choose instead $\nabla(1) = \frac{1}{2}\omega_{can}$ associated with the representation $q \rightarrow \sqrt{q}$ of $q^{\mathbf{Z}}$ (assuming that q is a square in k), we encounter a seeming *paradox*: the basic solution seems to be $\sqrt{t}$, which is not analytic multivalued on S (i.e. not an analytic function on $k^{\times}$). The associated q-difference equation here is $y(qt) = \sqrt{q}.t$. If we choose $\sqrt{t}$ as basic solution, as did G. Birkhoff in his theory of q-difference equations, we encounter the paradox which was pointed out and analyzed by M. van der Put and M. Singer in the last chapter of their book [SvP97]. The solution of the paradox is that the vector bundle $\mathcal{M}$ associated to the representation $q \rightarrow \sqrt{q}$ of $q^{\mathbf{Z}}$ (or to the q-difference equation $y(qt) = \sqrt{q}.t$) is in fact a non-trivial vector bundle of rank one, and the basic solution is not $\sqrt{t}$, but $\frac{\theta(t/\sqrt{q})}{\theta(t)}$, where $\theta(t) = \Pi_{n>0}(1 - q^n t)\Pi_{n \leq 0}(1 - q^n/t)$.

3.4. We next turn to the more general case of a p-adic manifold S which "is" an algebraic geometrically irreducible k-curve. Let $\bar{S}$ be its projective completion. It follows from the Van Kampen theorem, together with the fact that

punctured disks are simply-connected, that $\pi_1{}^{top}(S, \bar{s}) \to \pi_1{}^{top}(\bar{S}, \bar{s})$ is an isomorphism (this argument also works in higher dimension, for p-adic manifolds deprived from a divisor with strict normal crossings, using Kiehl's existence theorem of a tubular neighborhoods [Ki67]). It follows that the vector bundles with connection attached to representations of $\pi_1{}^{top}(S, \bar{s})$ automatically extend to $\bar{S}$. Hence we may assume without loss of generality that S is compact.

By GAGA, vector bundles with connection on S are algebrizable, and one can use C. Simpson's construction [Si94] to define the *moduli space of connections* of rank r over S, denoted by $M_{dR}(S, r)$. On the other hand, we have seen that the topological fundamental group $\pi_1{}^{top}(S, \bar{s})$ is free on $b_1(\Delta)$ generators, being isomorphic to the fundamental group of the dual graph Δ of the semistable reduction of S. Simpson has also studied the *moduli space of representations of dimension r* of such a group. We denote it by $M_B(S, r)$.

3.5. Let us assume that S is of genus $g \geq 2$. Simpson shows that $M_{dR}(S, r)$ is algebraic irreducible of dimension $2(r^2(g-1)+1)$. On the other hand, $M_B(S, r)$ is an algebraic irreducible affine variety of dimension $(r^2(b_1(\Delta)-1)+1)$. We note that this dimension is half the dimension of $M_{dR}(S, r)$ in case S is a Mumford curve.

Proposition [TCl][(not used in the sequel). *The functor which associates a vector bundle with connection to any representation of the topological fundamental group induces an injective analytic map of moduli spaces* $M_B(S, r) \longrightarrow M_{dR}(S, r)$.

We do not know whether this is a closed immersion. In the complex situation, the corresponding map $M_B(S, r) \longrightarrow M_{dR}(S, r)$ turns out to be an analytic isomorphism (Riemann-Hilbert-Simpson).

3.6. Let us go back to the case $g = 1$, and to the differential equation (**) on the Tate elliptic curve $S = \mathbf{C}_p{}^\times / q^{\mathbf{Z}}$. We assume $p \neq 2$. Let us write a Legendre equation for $S \setminus \{s\}$: $y^2 = z(z-1)(z-\lambda)$, such that the points $-1/\sqrt{q}$, $1/\sqrt{q}$, -1, 1 of $\mathbf{C}_p{}^\times$ map to 0, λ, 1, ∞ in $\mathbf{P}^1$ respectively ($|\lambda| = |\sqrt{q}| < 1$). The direct image of (**) on $\mathbf{P}^1 \setminus \{0, \lambda, 1, \infty\}$ is a differential equation of the form

$$(*\,*\,*) \quad Qy'' + cQ'y' + 4c^2 y = 0, \quad \text{where} \quad ' = \frac{d}{dz} \text{ and } Q = z(z-1)(z-\lambda).$$

A basis of solutions is given by t, t^{-1}. We would like to attach the monodromy group $\left\{ \begin{pmatrix} 0 & q^n \\ q^{-n} & 0 \end{pmatrix} \right\} \bigcup \left\{ \begin{pmatrix} q^n & 0 \\ 0 & q^{-n} \end{pmatrix} \right\}$ to this differential equation. However, this cannot be done in terms of paths since $\mathbf{P}^1 \setminus \{0, \lambda, 1, \infty\}$ is simply-connected. In order to cope with such a situation which intermingles topological coverings and étale finite coverings, we introduce orbifold fundamental groups.

§4 Punctured disks.

4.1. For our study of local monodromy, it is crucial to investigate the reduced étale fundamental group of a punctured disk. Let D be a closed disk with center $\zeta \in \mathbf{C}_p$, and let $\bar{x}$ be a geometric point of the punctured disk $D^* = D \setminus \{\zeta\}$.

4.2. Corresponding to the full subcategory $\{$ *finite Kummer coverings of* $D^*\}$ of $\{$ *etale coverings of* $D^*\}$, there are arrows

$$\pi_1{}^{et}(D^*, \bar{x}) \longrightarrow \pi_1{}^{red}(D^*, \bar{x}) \longrightarrow \pi_1{}^{alg}(D^*, \bar{x}) \longrightarrow \hat{\mathbf{Z}} = \Pi_\ell \, \mathbf{Z}_\ell,$$

and the compact group $\pi_1{}^{alg}(D^*, \bar{x})$ maps onto $\hat{\mathbf{Z}}$.

Proposition. *i)* $\pi_1{}^{red}(D, \bar{x}) \cong \pi_1{}^{alg}(D, \bar{x})$ *; any finite quotient of this profinite group is generated by its p-Sylow subgroups.*
ii) The maps $\pi_1{}^{red}(D^*, \bar{x}) \to \pi_1{}^{alg}(D^*, \bar{x}) \to \pi_1{}^{alg}(D, \bar{x}) \times \hat{\mathbf{Z}}$ *are (topological) isomorphisms.*

Proof. The second assertion of *i)* means that any finite Galois étale covering of degree prime to p is trivial, which is proven in [B93]6.3.3 and [L93]2.11. We now show that $\pi_1{}^{red}(D, \bar{x}) \to \pi_1{}^{alg}(D, \bar{x})$ and $\pi_1{}^{red}(D^*, \bar{x}) \to \pi_1{}^{alg}(D^*, \bar{x})$ are topological isomorphisms. It suffices to show that $\pi_1{}^{red}(D, \bar{x})$ (resp.$\pi_1{}^{red}(D^*, \bar{x})$) is compact and maps onto $\pi_1{}^{alg}(D, \bar{x})$ (resp. $\pi_1{}^{alg}(D^*, \bar{x})$). This property does not depend on the geometric point $\bar{x}$. For $\bar{x}$ mapping to the maximal point of D (corresponding to the sup-norm of $\mathbf{C}_p\langle t \rangle$), the map $\pi_1{}^{et}(\mathcal{M}(\mathbf{C}_p\langle t \rangle), \bar{x}) \cong \mathrm{Gal}(\bar{\mathbf{C}}_p\langle t \rangle / \mathbf{C}_p\langle t \rangle) \to \pi_1{}^{alg}(D, \bar{x})$ is surjective ([dJ95]7.5.). Since this map factors through the map $\pi_1{}^{red}(D^*, \bar{x}) \to \pi_1{}^{red}(D, \bar{x})$, it is easy to conclude. Besides, this map also factors through $\pi_1{}^{alg}(D^*, \bar{x})$, which shows that $\pi_1{}^{alg}(D^*, \bar{x}) \to \pi_1{}^{alg}(D, \bar{x})$ is a (topological) epimorphism.

It remains to show that the continuous homomorphism of profinite groups $\pi_1{}^{alg}(D^*, \bar{x}) \to \pi_1{}^{alg}(D, \bar{x}) \times \hat{\mathbf{Z}}$ is bijective, hence an isomorphism. We have already noticed that both projections p_1, p_2 are surjective. On the other hand, any finite quotient of $\pi_1{}^{alg}(D^*, \bar{x})/\mathrm{Ker}\, p_2$. $\mathrm{Ker}\, p_1$ corresponds to Kummer covering of D^* which extends to a covering of D; it is necessarily trivial.This implies that $\pi_1{}^{alg}(D^*, \bar{x}) = \mathrm{Ker}\, p_2$. $\mathrm{Ker}\, p_1$. We deduce that $\mathrm{Ker}\, p_2 \to \pi_1{}^{alg}(D, \bar{x})$ and $\mathrm{Ker}\, p_1 \to \hat{\mathbf{Z}}$ are surjective, and so is $\pi_1{}^{alg}(D^*, \bar{x}) \xrightarrow{u} \pi_1{}^{alg}(D, \bar{x}) \times \hat{\mathbf{Z}}$.
In order to show that the latter map u is injective, we rely on a fundamental result of Gabber-Lütkebohmert [L93], according to which *any connected finite étale covering of D^* restricts to a Kummer covering over some smaller punctured disk D'^* with same center*. The quotient of the radii depends on p and on the degree d of the covering (it may be chosen to be 1 if d is prime to p and the covering is Galois). The injectivity of u can be checked at the level of finite quotients of $\pi_1{}^{alg}(D^*, \bar{x})$. We fix such a finite quotient $\pi_1{}^{alg}(D^*, \bar{x})/U$, and denote by $Y \to D^*$ the associated Galois covering of D^*, and by $\bar{u} : \pi_1{}^{alg}(D^*, \bar{x})/U \to (\pi_1{}^{alg}(D, \bar{x}) \times \hat{\mathbf{Z}})/u(U)$ the induced map. Let $D' \overset{\iota}{\hookrightarrow} D$ be a smaller disk centered at ζ over which the associated Galois étale covering $Y \to D^*$ becomes Kummer.

The injectivity of $\bar{u}$ does not depend on the choice of $\bar{x}$, hence we may assume that $\bar{x}$ defines a geometric point of D'^*. We consider the commutative diagram

$$
\begin{array}{ccc}
\pi_1{}^{alg}(D'^*,\bar{x}) & \xrightarrow{\ u'\ } & \pi_1{}^{alg}(D',\bar{x}) \times \hat{\mathbf{Z}} \\
\iota_* \downarrow & & \downarrow \iota_* \times id \\
\pi_1{}^{alg}(D^*,\bar{x}) & \xrightarrow{\ u\ } & \pi_1{}^{alg}(D,\bar{x}) \times \hat{\mathbf{Z}} \\
\downarrow & & \downarrow \\
\pi_1{}^{alg}(D^*,\bar{x})/U & \xrightarrow{\ \bar{u}=\bar{p}_1 \times \bar{p}_2\ } & (\pi_1{}^{alg}(D,\bar{x}) \times \hat{\mathbf{Z}})/u(U).
\end{array}
$$

Since $Y \times_{D^*} D'^*$ is a Kummer, in particular *connected*, covering of D'^*, its fibre over $\bar{x}$ identifies with $\pi_1{}^{alg}(D'^*,\bar{x})/\iota_*{}^{-1}U$, and the composite left vertical map is surjective. On the other hand, the composite map $\pi_1{}^{alg}(D'^*,\bar{x}) \longrightarrow (\pi_1{}^{alg}(D,\bar{x}) \times \hat{\mathbf{Z}})/u(U)$ factors through the second factor of u'. Thus we see that the preimage of $\mathrm{Ker}\,\bar{u}$ in $\pi_1{}^{alg}(D'^*,\bar{x})$ maps trivially to $\pi_1{}^{alg}(D^*,\bar{x})/U$, and we conclude that $\mathrm{Ker}\,\bar{u}$ is trivial.

4.3. *Examples.* The Artin-Schreier covering $D(0,1^+) \xrightarrow{z \mapsto z^p - z} D(0,1^+)$ is an example of a non-trivial finite Galois étale covering of the unit closed disk with group $\mathbf{Z}/p\mathbf{Z}$, which splits over any smaller disk. A less standard example, for $p = 3$, is given by $(D(1,1^+) \setminus D(0,1^-)) \xrightarrow{z \mapsto z^3 - z^{-2}} D(0,1^+)$. Its Galois closure is a non-trivial Galois étale covering of the unit closed disk with group A_5 (one notices that the discriminant of $z^5 - xz^2 - 1$ is $5^5 + 2^2.3^3.x^5$, a square in $\mathcal{O}(D(0,1^+))$), which induces in characteristic 3 a Galois étale covering of the affine line with group A_5 (cf. [Se91]3.3). In particular, we see that $\pi_1{}^{alg}(D,\bar{x})$ is not a pro-p-group.

In fact, it follows from Raynaud's solution of the Abhyankar conjecture [R94] that any finite group generated by its p-Sylow subgroups is a quotient of $\pi_1{}^{alg}(D,\bar{x})$.

§5 p-adic orbifold fundamental groups.

5.1. The notions of complex orbifolds has several avatars: Thurston's orbifolds, Grothendieck's stacks, Satake's V-manifolds. The latter viewpoint may be the most convenient in the p-adic setting. For simplicity, however, we shall restrict ourselves to dimension one, and by *p-adic orbifold*, we shall mean here the data $\mathcal{X} = (X,(\zeta_i; n_i))$ of a one-dimensional p-adic manifold X and finitely many distinct classical points $\zeta_i \in X, i = 1, \ldots, \nu$, equipped with a multiplicity $n_i \in \mathbf{Z}_{>0}$. We also assume that $k = \mathbf{C}_p$.
We set $Z = \{\zeta_1, \ldots, \zeta_\nu\}$ and fix a geometric point $\bar{x}$ of $X \setminus Z$. The map $\pi_1{}^{top}(X \setminus Z, \bar{x}) \to \pi_1{}^{top}(X, \bar{x})$ is an isomorphism (cf 3.4).

5.2. For each i, we choose a small closed disk $D_i \subset X$ centered at ζ_i, in such a way that the D_i are pairwise disjoint. In particular the punctured disks D_i^* lie on $X \setminus Z$. For each i, let us also choose a geometric point $\bar{x}_i$ of D_i^*, and a topological generator $\bar{\gamma}_i$ of the factor $\hat{\mathbf{Z}}$ of $\pi_1{}^{red}(D_i^*,\bar{x}_i)$ (cf. 4.2).

5.3. Let us further choose an étale path α_i between $\bar{x}_i$ and $\bar{x}$ in $X \setminus Z$, i.e. ([dJ95]2.9) an isomorphism between the fiber functors $F_{\bar{x}_i}$ and $F_{\bar{x}}$. This induces a composite homomorphism $\pi_1{}^{red}(D_i^*, \bar{x}_i) \to \pi_1{}^{red}(X \setminus Z, \bar{x}_i) \overset{ad(\alpha_i)}{\longrightarrow} \pi_1{}^{red}(X \setminus Z, \bar{x})$. We denote by γ_i the image of $\tilde{\gamma}_i$ in $\pi_1{}^{red}(X \setminus Z, \bar{x})$.

Proposition. *The closure $\langle \gamma_i \rangle_i$ of the subgroup generated by the γ_i is the kernel of the homomorphism $\pi_1{}^{red}(X \setminus Z, \bar{x}) \longrightarrow \pi_1{}^{red}(X, \bar{x})$.*

Proof. We have to show that any étale covering map $Y^b \to X \setminus Z$ which corresponds to a $\pi_1{}^{red}(X \setminus Z, \bar{x})$-set extends to an étale covering map $Y \to X$. Due to the previous proposition, we know that the restriction of $Y^b/(X \setminus Z)$ to each D_i^* extends (uniquely) to an étale covering map Y_i/D_i. We then obtain Y/X by patching Y^b and the Y_i together.

Corollary. *If $X = \mathbf{A}^1$, $\pi_1{}^{red}(X \setminus Z, \bar{x})$ is topologically generated by the γ_i.*

Indeed, $\pi_1{}^{top}(\mathbf{A}^1)$ is trivial; $\pi_1{}^{alg}(\mathbf{A}^1)$ is also trivial, due to the p-adic version of Riemann's existence theorem ([L93]). Therefore $\pi_1{}^{red}(\mathbf{A}^1)$ is trivial, whence the result.
This result implies that when $X = \mathbf{P}^1$, $\pi_1{}^{red}(X \setminus Z, \bar{x})$ is topologically generated by any $\nu - 1$ elements among the γ_i.

5.4. We can now define the *orbifold fundamental group of $\mathcal{X}$ pointed at $\bar{x}$* to be the quotient $\pi_1{}^{orb}(\mathcal{X}, \bar{x})$ of $\pi_1{}^{red}(X \setminus Z, \bar{x})$ by the closure of the normal subgroup generated by the elements $(\gamma_i)^{n_i}$.
This is a Hausdorff pro-discrete topological group. It is easy to see that this definition does not depend on the choice of D_i, x_i, α_i, and $\tilde{\gamma}_i$.
We have not been able to interpret the $\pi_1{}^{orb}(\mathcal{X}, \bar{x})$-sets in terms of a satisfactory notion of étale coverings of p-adic orbifolds - but see 5.8 below.

Corollary. *If $X = \mathbf{A}^1$ or $\mathbf{P}^1$, the images $\bar{\gamma}_i$ of the γ_i generate $\pi_1{}^{orb}(\mathcal{X}, \bar{x})$ topologically.*

5.5. *Example.* Let us consider the orbifold $\mathcal{X}=(\mathbf{P}^1,(0;2),(\lambda;2),(1;2),(\infty;2))$. Applying 2.7 to the Legendre elliptic curve covering $\mathcal{X}$, it is not difficult to see that $\pi_1{}^{orb}(\mathcal{X}, \bar{x})$ is
　　a split extension of $\mathbf{Z}/2\mathbf{Z}$ by $\hat{\mathbf{Z}} \times \hat{\mathbf{Z}}$ if $|\lambda(\lambda - 1)| = 1$,
　　a split extension of $\mathbf{Z}/2\mathbf{Z}$ by $\mathbf{Z} \times \hat{\mathbf{Z}}$ if $|\lambda(\lambda - 1)| \neq 1$.
5.6. It is a general principle that, *unlike what happens in the complex case, the structure of the p-adic orbifold fundamental group depends on the* position *of the points ζ_i.* Especially, the existence of infinite discrete quotients depends on the position of the ζ_i.

Lemma 5.7. *Any continuous surjective homomorphism $\pi_1{}^{orb}(\mathcal{X}, \bar{x}) \longrightarrow \Gamma$ to a torsion-free discrete group Γ arises from a topological covering of X; in particular Γ is free (cf 2.10).*

Proof. Such a homomorphism corresponds to an étale Galois covering map $Y^b \to X \setminus Z$ with group Γ. Since Γ is torsion-free, the $\bar{\gamma}_i$ map to $1 \in \Gamma$. According

to proposition 5.2, this implies that $Y^b \to X \setminus Z$ extends to an étale Galois covering map $Y \to X$ with group Γ (corresponding to a surjective continuous map $\pi_1^{et}(X \setminus Z, \bar{x}) \longrightarrow \Gamma$). On the other hand, $\mathrm{Ker}(\pi_1^{et}(X \setminus Z, \bar{x}) \to \pi_1^{top}(X \setminus Z, \bar{x}))$ is topologically generated by compact subgroups (cf. [dJ95]3.9.ii). Since Γ is torsion-free and discrete, this implies that this kernel maps trivially to Γ. Hence $\pi_1^{et}(X \backslash Z, \bar{x}) \to \Gamma$ factors through $\pi_1^{top}(X \backslash Z, \bar{x})$, i.e. Y/X is a topological covering.

5.8. We say that an abstract group is virtually torsion-free if it has a normal subgroup of finite index which is torsion-free.

Proposition. *Let* $\varphi : \pi_1^{orb}(\mathcal{X}, \bar{x}) \longrightarrow \Gamma$ *be a continuous surjective homomorphism to a virtually torsion-free discrete group* Γ. *Then there exists*
i) *a connected p-adic manifold* S *of dimension one, and a finite morphism* $S \to X$ *ramified exactly above the points* ζ_i, *with ramification index dividing* n_i,
ii) *a connected topological covering* $S' \to S$,
such that the restriction of the composite morphism $S' \to S \to X$ *above* $X \setminus Z$ *is the étale covering map corresponding to* φ.

Conversely, for any $S \to X$ *and* $S' \to S$ *as in* $i), ii)$, *the restriction of the composite morphism* $S' \to S \to X$ *above* $X \setminus Z$ *is a Galois étale covering map with Galois group* Γ, *and the associated homomorphism* $\pi_1^{et}(X \setminus Z, \bar{x}) \longrightarrow \Gamma$ *factors through* $\pi_1^{orb}(\mathcal{X}, \bar{x})$.

Proof. Let $Y^b/(X \setminus Z)$ be the connected étale covering corresponding to φ. Let $\Gamma' \subset \Gamma$ be a torsion-free normal subgroup of finite index. The composite morphism $\bar{\varphi} : \pi_1^{orb}(\mathcal{X}, \bar{x}) \to \Gamma \to \Gamma/\Gamma'$ corresponds to a finite morphism $h : S \to X$ as in i). Its restriction above $X \setminus Z$ is a Galois étale covering $S^b/(X \setminus Z)$ which is a subcovering of $Y^b/(X \setminus Z)$. The pull-back of $Y^b/(X \setminus Z)$ over S^b splits: $Y^b = \coprod_{g \in \Gamma/\Gamma'} Y^b_g$, and any component Y^b_g is Galois étale over S^b with group Γ'. Let us consider the orbifold $\mathcal{S} = (S, (\xi_{ij} ; n_i))$, where the ξ_{ij} are the points lying above ζ_i, and let $\bar{s}$ be a geometric point of S above $\bar{x}$. It is clear that the homomorphism $\pi_1^{et}(S^b, \bar{s}) \longrightarrow \Gamma'$ corresponding to a given Y^b_g factors through a continuous surjective homomorphism $\pi_1^{orb}(\mathcal{S}, \bar{s}) \longrightarrow \Gamma'$. We then find a topological covering $S' \to S$ as in ii) by applying (5.7). By (2.4), $S' \times_X (X \backslash Z)$ is a étale covering of $X \setminus Z$. It is then easy to see that $S' \times_X (X \setminus Z) \cong Y^b$ using lemma 2.6.

Let us turn to the converse statement. We know by (2.4) that the restriction $S'^b \longrightarrow (X \setminus Z)$ of the composite morphism $S' \to S \to X$ is an étale covering map; it is clearly Galois with group Γ. Since Γ is in fact virtually free (cf. 5.7), it is residually finite. This implies that $\pi_1^{et}(X \setminus Z, \bar{x}) \longrightarrow \Gamma$ factors through $\pi_1^{red}(X \setminus Z, \bar{x})$. By the ramification property of S/X and using the already quoted result of Gabber-Lütkebohmert, we see that the restriction of S/X to any sufficiently small punctured disk centered at ζ_i is a disjoint union of Kummer coverings of order dividing n_i. The same is true for the restriction of S'/X to any

sufficiently small punctured disk centered at ζ_i. This implies that the image of $(\gamma_i)^{n_i} \in \pi_1{}^{red}(X \setminus Z, \bar{x})$ in Γ is trivial. This achieves the proof of the proposition.

§6 Global versus local p-adic monodromy.

6.1. We now consider *discrete* representations of the orbifold fundamental group $\pi_1{}^{orb}(\mathcal{X}, \bar{x})$. By "discrete representation", we mean a continuous homomorphism $\rho : \pi_1{}^{orb}(\mathcal{X}, \bar{x}) \longrightarrow GL_r(\mathbf{C}_p)$ which factors through a discrete group, i.e. such that the coimage $Coim\ \rho$ is discrete.

We denote by $\Gamma \subset GL_r(\mathbf{C}_p)$ the image of ρ. Of course, $\Gamma \cong Coim\ \rho$ as an abstract group, but Γ need not be discrete (this subtlety is already familiar in the complex situation, where monodromy groups are not always discrete).

6.2. Let us assume moreover that Γ is finitely generated (this occurs in particular if $X = \mathbf{A}^1$ or $\mathbf{P}^1$ according to corollary 5.4). By Selberg's lemma, Γ is virtually torsion-free; thus proposition 5.8 applies, and we get an associated representation $\sigma : \pi_1{}^{red}(S, \bar{s}) \longrightarrow \Gamma' \subset GL_r(\mathbf{C}_p)$, hence a vector bundle $(\mathcal{M}_\sigma, \nabla_\sigma)$ of rank r with connection on S. We set $S^b = S \times_X (X \setminus Z)$ and $G = \Gamma/\Gamma'$. Because the formation of $\sigma \mapsto (\mathcal{M}_\sigma, \nabla_\sigma)$ is compatible with base change on S, we see that $(\mathcal{M}_\sigma, \nabla_\sigma)$ admits a G-action compatible with the G-action on S. We can then define a vector bundle of rank r with connection on $S^b/G = X \setminus Z$ by setting: $(\mathcal{M}_\rho, \nabla_\rho) := (\mathcal{M}_{\sigma|S^b}, \nabla_{\sigma|S^b})/G$.

It is clear that the pull-back of $(\mathcal{M}_\rho, \nabla_\rho)$ over S^b identifies with $(\mathcal{M}_{\sigma|S^b}, \nabla_{\sigma|S^b})$.

6.3. By construction, $(\mathcal{M}_\rho, \nabla_\rho)$ has the property that *its pull-back over S^b extends to S and admits a full set of multivalued analytic solutions on S*. This property actually *characterizes the connections* $(\mathcal{M}, \nabla)$ *which arise from a discrete representation of* $\pi_1{}^{orb}(\mathcal{X}, \bar{x})$ (we call them *"connections with global monodromy"* for short).

Indeed, one can reconstruct the representation ρ in the following way. The representation space $\mathbf{C}_p^r$ of ρ is identified with the solution space $(\mathcal{M} \otimes \mathcal{O}_{S,s})^\nabla$. Let $\sigma : \pi_1{}^{red}(S, \bar{s}) \longrightarrow \pi_1{}^{top}(S, \bar{s}) \longrightarrow GL_r(\mathbf{C}_p)$ be the (topological) monodromy representation of the (unique) vector bundle with connection $(\mathcal{M}_\sigma, \nabla_\sigma)$ on S which extends $(\mathcal{M}_\rho, \nabla_\rho)_{|S^b}$. Let S'/S be the topological covering which corresponds to $Ker\ \sigma$. Due to the converse part of proposition 5.8, ρ is defined by the restriction of S'/X above $(X \setminus Z)$.

If $(\mathcal{M}, \nabla) = (\mathcal{M}_\rho, \nabla_\rho)$, the representation we just found is clearly the original ρ.

6.4. By abuse, we shall say that ρ is the (non-topological) *monodromy representation* attached to $(\mathcal{M}, \nabla)$, and that Γ is the associated *global monodromy group*. We already saw a non-trivial example in 3.6 (the differential equation $(* * *)$).

The formation of $\rho \mapsto (\mathcal{M}_\rho, \nabla_\rho)$ is clearly functorial in ρ, and commutes with base change of $(X, \bar{x})$. Moreover, it is independent of $\bar{x}$ in the sense that if α is an étale path from $\bar{x}'$ to $\bar{x}$, the corresponding representation $\rho' : \pi_1{}^{orb}(\mathcal{X}, \bar{x}') \longrightarrow GL_r(\mathbf{C}_p)$ defined by $\rho \circ ad(\alpha)$ leads to the same vector bundle with connection.

6.5. Let us now assume that $X = \mathbf{P}^1$, so that any discrete quotient of $\pi_1{}^{orb}((\mathbf{P}^1, (\zeta_i; n_i)), \bar{x})$ is finitely generated (by the images of the γ_i). For convenience, we also assume that $\zeta_\nu = \infty$. Any vector bundle on $\mathbf{P}^1 \setminus Z$ is trivial; the

choice of a basis (resp. cyclic basis, if any) identifies connections with ordinary linear differential systems of order one (resp. differential equations).

Theorem. *The construction $\rho \mapsto (\mathcal{M}_\rho, \nabla_\rho)$ defines a fully faithful functor*

$\{$*Discrete representations of* $\pi_1{}^{orb}((\mathbf{P}^1, (\zeta_i; n_i)), \bar{x})\} \to \{$*Algebraic regular connections on* $\mathbf{P}^1 \setminus Z$ *such that the local monodromy at each* ζ_i *is of finite order dividing* $n_i\}$.
The essential image of this functor consists of the connections with global monodromy in the sense of 6.3. The monodromy group of $(\mathcal{M}_\rho, \nabla_\rho)$ *is generated by any subset of* $\nu - 1$ *elements among the* $\rho(\gamma_i)$, $i = 1, \ldots \nu$.

Note that the condition "the local monodromy at ζ_i is of finite order dividing n_i" is purely algebraic: it means that for some (hence for every) logarithmic extension of ∇_ρ across ζ_i, the residue of ∇_ρ is semi-simple, and that its eigenvalues (the exponents) are rational with denominator dividing n_i.

Proof. Indeed, we have constructed finite covering $S/\mathbf{P}^1$ which restricts to a finite etale covering $S^b/(\mathbf{P}^1 \setminus Z)$, and such that the pull-back of $(\mathcal{M}_\rho, \nabla_\rho)$ on S^b extends to a vector bundle with connection $(\mathcal{M}_\sigma, \nabla_\sigma)$ on S. By the p-adic version of Riemann's existence theorem [L93], S is algebraic compact. Hence, by GAGA, $(\mathcal{M}_\sigma, \nabla_\sigma)$ is algebraic, and so is $(\mathcal{M}_\rho, \nabla_\rho)$ - being a factor of the direct image of $(\mathcal{M}_\sigma, \nabla_\sigma)_{|S^b}$. The result follows easily from this and the previous discussion.

6.6. In the situation of 6.5, let ∇_ρ be a connection "with global monodromy". The monodromy representation ρ defines a Galois étale covering $S'^b \to (\mathbf{P}^1 \setminus Z)$ (with Galois group $Im\ \rho$), which factors through some topological covering $S'^b \to S^b$ followed by a finite Galois étale covering $h^b : S^b \to (\mathbf{P}^1 \setminus Z)$ (with Galois group denoted by G).

Proposition. *Let us assume that p does not divide $|G|$. Then:*
i) for any open or closed disk $D \subset (\mathbf{P}^1 \setminus Z)$, the restriction of ∇_ρ to D is solvable in $\mathcal{O}(D)$;
ii) for any open or closed annulus $\mathcal{A} \subset (\mathbf{P}^1 \setminus Z)$ centered at some point $\zeta \in \mathbf{A}^1$, the restriction of ∇_ρ to $\mathcal{A}$ is solvable in $\mathcal{O}(\mathcal{A})[(z - \zeta)^{1/|G|}]$.

Proof. We may consider only closed disks and annuli. We know that ∇_ρ is solvable in $\mathcal{O}(S'^b)$. But the inverse image of D in S'^b is a topological covering of a finite Galois étale covering of D; since p does not divide $|G|$, we conclude by [B93]6.3.3 or [L93]2.11 that this inverse image of D is isomorphic to a disjoint sum of copies of D, whence i). Similarly, the inverse image of $\mathcal{A}$ in S'^b is a topological covering of a finite Galois étale covering $\mathcal{A}_1$ of $\mathcal{A}$ with group G. By *loc. cit.*, $\mathcal{A}_1$ is a disjoint sum of Kummer coverings of degree dividing $|G|$. Because annuli are simply-connected, the inverse image of $\mathcal{A}$ in S'^b itself is such a disjoint (infinite) sum, whence ii).

6.7. *Remark.* Point i) implies that ∇_ρ is solvable in the generic disk in the sense of B. Dwork (or overconvergent in the sense of P. Berthelot). Point

ii) shows that ∇_ρ has rational p-adic exponents on any annulus, in the sense of G.Christol and Z. Mebkhout [ChM97]. In particular, we see that the theory of p-adic exponents cannot predict the existence of infinite global monodromy. In the special case of connections with global monodromy, the theory of p-adic orbifolds provides a *geometric interpretation of p-adic exponents*, and a *link between local monodromies at different points (which cannot be obtained by considering annuli surrounding these points)*.

Example. Let us consider again our differential equation $(***)$ on $\mathbf{P}^1 \setminus \{0, \lambda, 1, \infty\}$, for $p \neq 2$. Here S is our Tate curve $\mathbf{C}_p^\times / q^{\mathbf{Z}}$,

$$\Gamma = \{\begin{pmatrix} 0 & q^n \\ q^{-n} & 0 \end{pmatrix}\} \bigcup \{\begin{pmatrix} q^n & 0 \\ 0 & q^{-n} \end{pmatrix}\} \subset GL_2(\mathbf{C}_p),$$

$G \cong \mathbf{Z}/2\mathbf{Z}$ is the image of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Actually, one can choose γ_i to be

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & q^{-1} \\ q & 0 \end{pmatrix}$$

for $\zeta_i = 0, \lambda, 1, \infty$ respectively. Let now $\mathcal{A}$ be an annulus surrounding 0 and λ alone. We know by point *ii*) above that the p-adic exponents on $\mathcal{A}$ are 0 or $1/2$. They are in fact 0: the argument of point *ii*) shows that the pull-back of equation $(***)$ on $\mathcal{A}_1$ (the inverse image of $\mathcal{A}$ in S) is solvable in $\mathcal{O}(\mathcal{A}_1)$. It then suffices to show that the covering $\mathcal{A}_1/\mathcal{A}$ splits. This follows from the fact that $\mathcal{O}(\mathcal{A}_1) = \mathcal{O}(\mathcal{A})[y]/(y^2 - z(z-1)(z-\lambda))$ and that $z-1$ and $z(z-\lambda)$ are squares in $\mathcal{O}(\mathcal{A})$.

In the complex situation, a similar picture holds, but for a different reason: the monodromy along $\mathcal{A}$ can be computed in terms of local monodromies around 0 and around λ, and there is a cancellation.

In order to obtain more interesting connections with global monodromy, we shall use the Cherednik-Drinfeld-Boutot-Zink uniformization of Shimura curves.

§7 Uniformization of Shimura curves.

7.1. Let B be a quaternion division algebra over a totally real number field F, which is ramified at every place at infinity except one ∞_0. Let v be a finite place of F such that B_v is a division algebra. Let p be the residue characteristic of v. Let Γ_∞ be a congruence subgroup of B^*/F^* (viewed as an algebraic group over $\mathbf{Q}$). We assume that Γ_∞ is maximal at p (i.e. the p-part of the associated adelic group in B_v is the maximal compact subgroup. Then $\mathfrak{H}/\Gamma_\infty$ is a projective algebraic curve, which has a canonical model $Sh = Sh_{\Gamma_\infty}$ (Shimura curve) over some class-field of F unramified at v. This Shimura curve has bad reduction at any prime above v.

7.2. Let F_v^{nr} be the completion of the maximal unramified extension of F_v. I. Cherednik [Č76] has represented $Sh(F_v^{nr})$ as a Mumford curve, as follows. Let Ω_v

be the Drinfeld upper half-space $\mathbf{P}^1(F_v^{nr}) \setminus \mathbf{P}^1(F_v)$ (viewed as a p-adic manifold over F_v^{nr}). Let $\bar{B}$ be the quaternion algebra obtained from B by changing the local invariants exactly at ∞_0 and v; in particular, $\bar{B}$ is totally definite. Let Σ be the set of places at infinity together with v. Then there exists a Σ-congruence subgroup Γ_v of $\bar{B}^*/F^*$ (viewed as an algebraic group over $\mathbf{Q}$) such that $\Omega_v/\Gamma_v \cong Sh(F_v^{nr})$ as p-adic manifolds.

7.3. Note that Γ_v is a discrete subgroup of $(\bar{B}^*/F^*)(F_v) \cong PGL_2(F_v)$. This subgroup can be made explicit from Γ_∞. For instance, let $\mathcal{B}$ be a maximal $\mathcal{O}_F$-order in B and let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$. Let $\mathcal{O}_F^{(v)}$ be the subring of F of elements integral at every finite place except v, and let $\bar{\mathcal{B}}^{(v)}$ be a maximal $\mathcal{O}_F^{(v)}$-order in $\bar{B}$. If Γ_∞ is the image of $(1+\mathfrak{n}\mathcal{B})^*$ in B^*/F^*, then Γ_v is the image of $(1+\mathfrak{n}\bar{\mathcal{B}}^{(v)})^*$ in $\bar{B}^*/F^*$ [Č76].

Another instance: let $N(\mathcal{B})$ be the normalizer of $\mathcal{B}$ in B^*. Let $\Gamma_\infty^+ \subset \Gamma_\infty^* \subset B^*/F^*$ be the images of the subgroups of elements with totally positive reduced norm in $\mathcal{B}^*$ and $N(\mathcal{B})$ respectively. Let $N(\bar{\mathcal{B}}^{(v)})$ be the normalizer of $\bar{\mathcal{B}}^{(v)}$ in $\bar{B}^*$. Let $\Gamma_v^+ \subset \Gamma_v^* \subset \bar{B}^*/F^*$ be the image of the subgroup of elements of $(\bar{\mathcal{B}}^{(v)})^*$ with reduced norm equal to a unit in $\mathcal{O}_F^{(v)}$ times a square in F, and the image of $N(\bar{\mathcal{B}}^{(v)})$, respectively. Assume that the narrow class number $h^+(F)$ is 1. Then there is a natural isomorphism $\Gamma_\infty^*/\Gamma_\infty^+ \cong \Gamma_v^*/\Gamma_v^+$. For any intermediate $\Gamma_\infty^+ \subset \Gamma_\infty \subset \Gamma_\infty^*$, $\Gamma_v^+ \subset \Gamma_v \subset \Gamma_v^*$ is the corresponding intermediate subgroup (cf. [Ku79]5.1).

7.4. When $F = \mathbf{Q}$ and when Γ_∞ is small enough, Sh is the solution of a moduli problem for polarized abelian surfaces with action of $\mathcal{B}$. Their p-divisible groups are certain formal groups of height 4 and dimension 2, called special $\mathcal{B}_v$-formal groups. V. Drinfeld has shown that Cherednik's uniformization holds at the level of formal groups over $\mathcal{O}_{F_v^{nr}}$ by interpreting the formal model $\hat{\Omega}_v$ of Ω_v as a moduli space for special $\mathcal{B}_v$-formal groups (cf. [BoC91],[RZ96]); Γ_v appears as a group of $\mathcal{B}^{(v)}$-isogenies of such formal groups in characteristic p.

When $F \neq \mathbf{Q}$, Sh does not admit such a direct modular interpretation; but a certain "twisted form" $Sh^\bullet$ of Sh does, as follows (cf. [D71]7,[BoZ95[). Let K be a quadratic totally imaginary extension of F, such that every place v_i of F above p splits in K. We denote by w_i, $\bar{w}_i$ the two places of K above v_i (with $v_0 = v$). The quaternion algebra $B^\bullet = B \otimes_F K$ over K is ramified at $w := w_0$. We fix an extension $\infty_0' : K \to \mathbf{C}$ of the real embedding ∞_0 of F. We fix a double embedding $\mathbf{C} \leftarrow \bar{\mathbf{Q}} \to \bar{\mathbf{Q}}_\mathbf{p}$ such that the embedding $\tau_0 : K \to \bar{\mathbf{Q}}_p$ corresponding to ∞_0' lies above v. The embeddings which factor through w_i, not $\bar{w}_i$ form a CM type Φ. Let $\Gamma_\infty^\bullet$ be a congruence subgroup of $B^{\bullet*}/K^*$ (viewed as an algebraic group over $\mathbf{Q}$). The symmetric domain associated to $B^{\bullet*}/K^*$ is isomorphic to $\mathfrak{H}$. The quotient $\mathfrak{H}/\Gamma_\infty^\bullet$ is a projective algebraic curve, which has a canonical model $Sh^\bullet = Sh_{\Gamma_\infty^\bullet}^\bullet$ over some class-field of K unramified at w. For $\Gamma_\infty^\bullet$ small enough, this Shimura curve is a geometric component of a fine moduli space $\mathcal{S}h^\bullet$ defined over $\mathcal{O}_K$, for polarized abelian varieties of dimension $g = 4[F : \mathbf{Q}]$ with action of $\mathcal{B}^\bullet = \mathcal{B} \otimes_{\mathcal{O}_F} \mathcal{O}_K$, Shimura type Φ and level structure (loc. cit.). Moreover,

for any Γ_∞ small enough, one can attach a $\Gamma_\infty^\bullet$ (isomorphic to Γ_∞) such that Sh_{Γ_∞} and $Sh_{\Gamma_\infty^\bullet}^\bullet$ become isomorphic after a finite extension of the base field. The connection between Sh and $Sh^\bullet$ is much more precise in the adelic context.

7.5. Via the theory of p-adic period spaces of Drinfeld-Rapoport-Zink, Boutot and Zink obtain in [BoZ95[a modular proof of Cherednik's uniformization [BoZ95[, which we sketch very roughly as follows. The non-connected Shimura variety of unitary type $Sh^\bullet$ admits a model $Sh_v^\bullet$ over $\mathcal{O}_{F_v^{nr}}$. Let $A \longrightarrow Sh^\bullet$ be the universal abelian scheme with $\mathcal{B}^\bullet$-action. It also admits a model $A_v \longrightarrow Sh_v^\bullet$. Due to the action of $\mathcal{B}^\bullet$, the p-divisible group of A_v splits over the formal completion of $Sh^\bullet \otimes_{\mathcal{O}_K} \mathcal{O}_{F_v^{nr}}$: $A_v[p^\infty] \cong \Pi_i \mathcal{G}_{w_i} \times \mathcal{G}_{\bar{w}_i}$. The factor $\mathcal{G}_{w_0}$ is a special $\mathcal{B}_v$-formal group (note that $\mathcal{B}_w^\bullet \cong \mathcal{B}_v$), which comes by descent from the universal special $\mathcal{B}_v$-formal group $\tilde{\mathcal{G}}_{w_0}$ over $\hat{\Omega}_v$ - the latter being quasi-isogenous to a fixed special $\mathcal{B}_v$-formal group $\mathbf{G}$ over $\bar{\mathbf{F}}_p$; moreover $\tilde{\mathcal{G}}_{w_0}$ is in duality with the corresponding $\tilde{\mathcal{G}}_{\bar{w}_0}$, in a way compatible, up to a factor in F_v^*, with the quasi-isogenies to $\mathbf{G}$ and $\check{\mathbf{G}}$ (loc. cit., 1).

7.6. Due to the action of $\mathcal{O}_K$ on A, the *Gauss-Manin connection* of $A \to Sh^\bullet$ splits: $(H^1_{dR}(A/Sh^\bullet), \aleph) = \bigoplus_{\tau: K \to \bar{\mathbf{Q}}_p} (H^1_{dR}(A/Sh^\bullet)_\tau, \aleph_\tau)$. Moreover, after an extension scalars to a splitting field K'/K for $B^\bullet$, each factor $(H^1_{dR}(A/Sh^\bullet)_\tau, \aleph_\tau)$ splits itself into two isomorphic factors of rank 2, according to the action of $\mathcal{O}_{K'}$. We take one of these factors of rank two of $(H^1_{dR}(A/Sh^\bullet)_{\tau_0}, \aleph_{\tau_0})$, and consider it as a connection of rank 2 on Sh (after finite extension of the base number field, which will play no role). We denote this connection by ∇_{Γ_∞}.

This construction works even without the assumption that Γ_∞ is small enough, away from the branch locus of $\mathfrak{H} \to \mathfrak{H}/\Gamma_\infty$ (identified with the branch locus of $\Omega_v \to \Omega_v/\Gamma_v$). Indeed, the restriction of A descends to an abelian scheme with action of $\mathcal{B}^\bullet$ outside this branch locus (transcendentally, this is $(B^\bullet \otimes_{\mathbf{Q}} \mathbf{R}) \times (\mathfrak{H} \backslash Fix(\tilde{\Gamma}_\infty^\bullet)) / B^\bullet .. \tilde{\Gamma}_\infty^\bullet$, where $\tilde{\Gamma}_\infty^\bullet$ is a lifting of $\Gamma_\infty^\bullet$ in $N(B^\bullet)^*$).

Any auxiliary generic cyclic vector allows to consider it as a differential equation, and it is clearly a *uniformizing differential equation* for the orbifold $\mathfrak{H}/\Gamma_\infty$ in the sense of [Y87] (cf. also [K96]4): the associated projective monodromy group is Γ_∞. In the sequel, we shall be especially interested in the case where $\mathfrak{H}/\Gamma_\infty = (\mathbf{P}^1, (\zeta_i; n_i))$ as an orbifold.

Theorem. *Viewed as a p-adic connection, ∇_{Γ_∞} is a connection with global monodromy in the sense of 6.3. The associated projective monodromy group is the discrete group $\Gamma_v \subset PGL_2(F_v)$.*

Proof. We may assume that Γ_∞ is small enough (but maximal at p, cf. 7.1). We shall then show that ∇_{Γ_∞}, as a p-adic analytic connection, becomes trivial on the universal covering Ω_v of $Sh_{F_v^{nr}}$. Let us consider the factor $(H^1_{dR}(A/Sh^\bullet)_{w_0}, \aleph_{w_0})$ of $(H^1_{dR}(A/Sh^\bullet), \aleph)$, defined to be the direct sum of the $(H^1_{dR}(A/Sh^\bullet)_\tau, \aleph_\tau)$ corresponding to those embeddings τ which factor through w_0. It is enough to show that the pull-back of $(H^1_{dR}(A/Sh^\bullet)_{w_0}, \aleph_{w_0})$ on Ω_v is a trivial analytic connection. $H^1_{dR}(A/Sh^\bullet)_{w_0}$ is locally free ([BBM82]2.5.2), and we shall actually deal

with the dual connection, which may be identified with $(H^1_{dR}(\check{A}/\mathcal{S}h^\bullet)_{\bar{w}_0}, \aleph_{\bar{w}_0})$ ($\check{A}$: dual abelian scheme). We note that $(H^1_{dR}(\check{A}/\mathcal{S}h^\bullet)_{\bar{w}_0}, \aleph_{\bar{w}_0}) \otimes F^{nr}_v$ descends to a factor $(H^1_{dR}(\check{A}_v/\mathcal{S}h^\bullet_v)_{\bar{w}_0}, \aleph_{\bar{w}_0})$ of $(H^1_{dR}(\check{A}_v/\mathcal{S}h^\bullet_v), \aleph)$ over $\mathcal{O}_{F^{nr}_v}$. We look at the pull-back of $(H^1_{dR}(\check{A}_v/\mathcal{S}h^\bullet_v)_{\bar{w}_0}, \aleph_{\bar{w}_0})$ on $\hat{\Omega}_v$. It is is canonically isomorphic to the Lie algebra of the universal vectorial extension of the p-divisible group $A_v[p^\infty]_{\hat{\Omega}_v}$ (cf.[MM74]). By functoriality, we get a canonical isomorphism between $H^1_{dR}(\check{A}_v/\mathcal{S}h^\bullet_v)_{\bar{w}_0} \otimes \mathcal{O}_{\hat{\Omega}_v}$ and the Lie algebra $LE(\tilde{\mathcal{G}}_{w_0})$ of the universal vectorial extension of $\tilde{\mathcal{G}}_{w_0}$. Actually, the universal extension $E(\tilde{\mathcal{G}}_{w_0})$ itself carries a connection (Grothendieck's $\natural$-structure, *loc. cit.*), and the induced connection on its Lie algebra identifies with the Gauss-Manin connection on $H^1_{dR}(\check{A}_v/\mathcal{S}h^\bullet_v)_{\bar{w}_0} \otimes \mathcal{O}_{\hat{\Omega}_v}$. Coming back to the rigid-analytic context, we conclude by the following variant of [RZ]5.15:

Proposition. *Let $\mathcal{M}$ be a formal scheme formally locally of finite type over $Spf\mathcal{O}_{F^{nr}_v}$, let $\mathcal{M}^{rig}$ be the rigid variety associated to $\mathcal{M}$ by the Raynaud-Berthelot construction ([RZ]5.5), and let $\mathcal{M}_\pi$ be the $\bar{\mathbf{F}}_p$-scheme defined by an ideal of definition of $\mathcal{M}$ containing a uniformizer π of $\mathcal{O}_{F^{nr}_v}$. We assume that $\mathcal{M}^{rig}$ is smooth. Let $\mathcal{G}$ be a p-divisible group over $\mathcal{M}$, $\mathbf{G}$ a p-divisible group over $\bar{\mathbf{F}}_p$, and $q: \mathbf{G}_{\mathcal{M}_\pi} \longrightarrow \mathcal{G}_{\mathcal{M}_\pi}$ a quasi-isogeny. Let us denote by $LE(\mathcal{G})$ the Lie algebra of the universal vectorial extension of $\mathcal{G}$, and by $\mathbf{D}(\mathbf{G})$ the Dieudonné module of $\mathbf{G}$. Then q induces a canonical functorial isomorphism of vector bundles with connection over $\mathcal{M}^{rig}$: $q_{\mathcal{M}^{rig}} : \mathbf{D}(\mathbf{G}) \otimes_{W(\bar{\mathbf{F}}_p)} \mathcal{O}_{\mathcal{M}^{rig}} \cong LE(\mathcal{G})^{rig}$ compatible with base change.*

The only point which is not in [RZ]5.15 concerns the connections. In order to establish it, we follow the reasoning of *loc. cit.* One reduces by gueing to the case where $\mathcal{M}$ is affine π-adic, $\mathcal{M}_\pi$ being defined by the image of π. Then $\mathcal{M}$ embeds into a formal scheme $\mathcal{P}$ formally smooth of finite type over $\mathbf{Z}_p$. For any $n > 0$, let $\mathcal{M}_{p^n} \subset \mathcal{M}$ be defined by the image of p^n. Let $\tilde{\mathbf{G}}$ be any lifting of $\mathbf{G}$ to $\mathcal{O}_{F_v^{nr}}$. Then q extends in a unique way into a quasi-isogeny of p-divisible groups $q_{p^n} : \tilde{\mathbf{G}}_{\mathcal{M}_{p^n}} \longrightarrow \mathcal{G}_{\mathcal{M}_{p^n}}$ (rigidity of quasi-isogenies [Dr76]). In particular, let $N > 0$ be such that $p^N q_p$ is an isogeny. Let us consider as in [RZ]5.15 the canonical homomorphism associated to q_p by [M72]IV,2.2, $(p^N q_p)^*$: $LE(\tilde{\mathbf{G}}_{\mathcal{M}}) \longrightarrow LE(\mathcal{G})$. This homomorphism need not preserve the structure of extension, but it certainly induces a morphism of crystals on $(\mathcal{M}_{p^n}, (p))$, hence by [BBM82]1.2.3, a morphism of $\mathcal{O}_{\mathcal{P}_{p^n}}$-modules with connection. But $q_{\mathcal{M}^{rig}}$ is given by $p^{-N}(p^N q_p)^*$. Therefore, it is compatible with the connections (taking into account the fact that $\mathcal{M}^{rig}$ is a smooth subvariety of $\mathcal{P}^{rig}$).

7.7. *Remark.* One can drop the assumption that Γ_∞ is maximal at p on replacing Ω_v by a finite etale covering, cf. [BoZ95[.

§8 Explanation of the first example.

8.1. According to [Vi80] IV 3.B,C, the fuchsian group Γ considered in this example is the group Γ^*_∞ (denoted by $\bar{G}$ in *loc. cit.*) attached to a maximal order $\mathcal{B}$ in the quaternion algebra $B/\mathbf{Q}$ with discriminant 15. On the other hand, it turns out that Γ is conjugated in $PSL_2(\mathbf{R})$ to the group denoted by W^+ in [K96], which is generated by the matrices displayed in §1. The point is that the order R of B considered in [K96]10 is maximal: indeed, R is spanned as an additive group by the matrices

$$u_1 = id, u_2 = \begin{pmatrix} \frac{\sqrt{5}+1}{2} & 0 \\ 0 & \frac{-\sqrt{5}+1}{2} \end{pmatrix}, \ u_3 = \begin{pmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}, \ u_4 = \begin{pmatrix} 0 & -\frac{\sqrt{3}+\sqrt{15}}{2} \\ \frac{\sqrt{3}-\sqrt{15}}{2} & 0 \end{pmatrix},$$

one computes that the matrix built from the reduced traces $t(u_i u_j)$ has determinant $-(3.5)^2$, and one concludes by the criterion [Vi80] III 5.3. By loc. cit.III.5.10, R is right principal, and it follows that R is conjugated to $\mathcal{B}$ in B. By [Vi80]IV.3.B or [K96], *loc. cit.*, we have $\mathfrak{H}/\Gamma = \mathcal{X}_\Gamma = (\mathbf{P}^1,(0;2),(1;2),(81;2),(\infty;6))$, and [K96]9 exhibits $(*)$ as a uniformizing differential equation.
Another uniformizing differential equation is given by the piece of Gauss-Manin connection $\nabla_{\Gamma^*_\infty}$ considered in 7.6. It follows from [K96]4.5 that the two connections $\nabla_{\Gamma^*_\infty}, \nabla_{(*)}$ are related to each other by torsion by a rank-one isotrivial connection.

8.2. For $p = 3$ or $p = 5$, it follows from theorem 7.6 that $\nabla_{\Gamma^*_\infty}$, viewed as a p-adic connection, is a connection with global monodromy; moreover (cf. 7.3), the projective monodromy is the discrete subgroup $\Gamma^*_p \subset PGL_2(\mathbf{Q}_p)$, which appears as a p-adic quadrangle group $\Diamond_p(2,2,2,6)$. It follows that $(*)$ has the same properties.

8.3. When $p = 5$, one can take $\bar{\mathcal{B}} = \mathbf{Z}[1, i, \frac{i+j}{2}, \frac{1+ij}{2}]$, with $i^2 = -1$, $j^2 = -3$, $ij = -ji$. Finding generators for Γ^*_5 amounts to a tedious calculation similar to those carried out in [GvP80]9.1 for the Hurwitz quaternions.

8.4. The fact that there is a Frobenius structure for which every residue class is supersingular may be drawn from the fact that any abelian surface with $\mathcal{B}$-action has potentially good, supersingular, reduction. It would be interesting to determine whether the p-adic exponents on any annulus surrounding 1 and 81 alone are 0.

§9 p-adic triangle groups.

9.1. In this last section, we consider the case of the Gauss hypergeometric differential equation. We are interested in finding out for which parameters in $\mathbf{C}_p$ the hypergeometric equation has global monodromy group in the sense of 6.3, 6.5, and in that case, in describing the projective monodromy group. According to theorem 6.5, the exponents of the hypergeometric equation - hence the parameters - are then rational. When this situation occurs, the projective

global monodromy group is a p-adic counterpart of the Schwarz (projective) triangle group $\triangle(e_1, e_2, e_3)$ (cf [Ma74]); we call it a p-adic triangle group and denote it by $\triangle_p(e_1, e_2, e_3)$.

The corresponding p-adic hypergeometric function $y = {}_2F_1(a, b, c; z)$ *has the following property: there is a finite covering* $h : S \to \mathbf{P}^1$ *ramified above* $0, 1, \infty$, *such that* $y \circ h$ *extends to a global multivalued meromorphic function on* S^{an}, *with poles lying above* $0, 1, \infty$. Poles may occur because the vector bundle on S, underlying the extension to S of the pull-back of the hypergeometric connection (cf. 6.3), is not necessarily free.

We place ourselves in the hyperbolic case $1/e_1 + 1/e_2 + 1/e_3 < 1$ (e_1, e_2, e_3 positive integers).

9.2. *Arithmetic p-adic triangle groups.* These are p-adic triangle groups corresponding to the arithmetic triangle groups $\triangle(e_1, e_2, e_3)$ classified by K. Takeuchi [T77]. Recall that $\triangle(e_1, e_2, e_3)$ is called arithmetic if it is commensurable in $PSL_2(\mathbf{R})$ to some quaternionic arithmetic group Γ_∞^* as in 7.3 above (corresponding to a quaternion division algebra over a totally real number field, which is ramified at every place at infinity except one). More specifically, we shall $\triangle_p(e_1, e_2, e_3)$ an arithmetic p-adic triangle group only when some place above p divides the discriminant of the quaternion algebra (it is very likely that $\triangle_p(e_1, e_2, e_3)$ does not exist for other p, but we have not proven it).

Theorem. *i)* *There are 45 arithmetic diadic triangle groups:* $\triangle_2(2, 4, 6)$, $\triangle_2(2, 6, 6)$, $\triangle_2(3, 4, 4)$, $\triangle_2(3, 6, 6)$, $\triangle_2(2, 3, 8)$, $\triangle_2(2, 4, 8)$, $\triangle_2(2, 6, 8)$, $\triangle_2(2, 8, 8)$, $\triangle_2(3, 3, 4)$, $\triangle_2(3, 8, 8)$, $\triangle_2(4, 4, 4)$, $\triangle_2(4, 6, 6)$, $\triangle_2(4, 8, 8)$, $\triangle_2(2, 3, 12)$, $\triangle_2(2, 6, 12)$, $\triangle_2(3, 3, 6)$, $\triangle_2(3, 4, 12)$, $\triangle_2(3, 12, 12)$, $\triangle_2(6, 6, 6)$, $\triangle_2(2, 4, 5)$, $\triangle_2(2, 4, 10)$, $\triangle_2(2, 5, 5)$, $\triangle_2(2, 10, 10)$ $\triangle_2(4, 4, 5)$, $\triangle_2(5, 10, 10)$, $\triangle_2(3, 4, 6)$, $\triangle_2(2, 4, 18)$, $\triangle_2(2, 18, 18)$, $\triangle_2(4, 4, 9)$, $\triangle_2(9, 18, 18$ $\triangle_2(2, 3, 16)$, $\triangle_2(2, 8, 16)$, $\triangle_2(3, 3, 8)$, $\triangle_2(4, 16, 16)$, $\triangle_2(8, 8, 8)$, $\triangle_2(2, 5, 20)$, $\triangle_2(5, 5, 10)$ $\triangle_2(2, 3, 24)$, $\triangle_2(2, 12, 24)$, $\triangle_2(3, 3, 12)$, $\triangle_2(3, 8, 24)$, $\triangle_2(6, 24, 24)$, $\triangle_2(12, 12, 12)$, $\triangle_2(2, 5, 8)$, $\triangle_2(4, 5, 5)$.

ii) *There are 16 arithmetic triadic triangle groups:* $\triangle_3(2, 4, 6)$, $\triangle_3(2, 6, 6)$, $\triangle_3(3, 4, 4)$, $\triangle_3(3, 6, 6)$, $\triangle_3(2, 4, 12)$, $\triangle_3(2, 12, 12)$, $\triangle_3(4, 4, 6)$, $\triangle_3(6, 12, 12)$, $\triangle_3(2, 5, 6)$, $\triangle_3(3, 5, 5)$, $\triangle_3(2, 4, 18)$, $\triangle_3(2, 18, 18)$, $\triangle_3(4, 4, 9)$, $\triangle_3(9, 18, 18)$, $\triangle_3(2, 5, 30)$, $\triangle_3(5, 5, 15)$.

iii) *There are 9 arithmetic pentadic triangle groups:* $\triangle_5(2, 3, 10)$, $\triangle_5(2, 5, 10)$, $\triangle_5(3, 3, 5)$, $\triangle_5(5, 5, 5)$, $\triangle_5(2, 3, 30)$, $\triangle_5(2, 15, 30)$, $\triangle_5(3, 3, 15)$, $\triangle_5(3, 10, 30)$, $\triangle_5(15, 15, 15)$.

For $p > 5$, there is no arithmetic p-adic triangle group.

Proof. Let Γ_1 and Γ_2 be Fuchsian commensurable subgroups of $PSL_2(\mathbf{R})$ such that $\mathfrak{H}/(\Gamma_1 \cap \Gamma_2) \cong \mathbf{P}^1$. For j=1, 2, let ∇_j be a regular connection with rational exponents on $\mathfrak{H}/\Gamma_j \backslash$ branch locus, with projective monodromy group Γ_j; then over $\mathfrak{H}/(\Gamma_1 \cap \Gamma_2)$, $\nabla_1 \cong \nabla_1 \otimes \nabla'$, where ∇' is an isotrivial connection of rank one (cf. [K96]4.5). Let B/F be the quaternion algebra attached to $\triangle(e_1, e_2, e_3)$. We use the fact that the commensurable subgroups $\triangle(e_1, e_2, e_3)$, Γ_∞^* of $PSL_2(\mathbf{R})$ can be joined by a finite chain of commensurable subgroups $\Gamma_1, \Gamma_2, \ldots$ as before

(cf. [T77]4). The intersection of $\triangle(e_1, e_2, e_3)$ and Γ_∞^* corresponds to a projective smooth curve S', which is defined over some number field, equipped with two rational functions $h_\triangle, h_{\Gamma_\infty^*} : S' \longrightarrow \mathbf{P}^1$. It follows that over the complement $S'^b \subset S'$ of the ramification loci of these rational functions, the hypergeometric connection ∇_{hyp} becomes isomorphic to $\nabla_{\Gamma_\infty^*} \otimes \nabla''$, where ∇'' is an isotrivial connection of rank one.

Let now p be a prime such that some place $v|p$ of F divides the discriminant $d(B)$. According to [T77]2, v is uniquely determined by p. According to theorem 7.6, $\nabla_{\Gamma_\infty^*}$ is a connection with global monodromy (with discrete projective monodromy group Γ_v^*): there is a finite etale covering of S''^b/S'^b such that the p-adic connection $\nabla_{\Gamma_\infty^*}$ becomes trivial on the p-adic universal covering of S''^b. Replacing S''^b by an etale covering which trivializes ∇'', we conclude that ∇_{hyp} is a connection with global monodromy. *Therefore, for fixed quadruple (e_1, e_2, e_3, p) such that $\triangle(e_1, e_2, e_3)$ is arithmetic and p divides a power of the discriminant of the associated quaternion algebra, there is exactly one arithmetic $\triangle_p(e_1, e_2, e_3)$.* One concludes by inspection of [T77], table 1.

9.3. Remarks. *i)* We observe that for any arithmetic triangle group $\triangle_p(e_1, e_2, e_3)$, p always divides one of the e_i. Therefore, the exponents of the corresponding hypergeometric equations are not p-integral. This situation was always ruled out in the literature on p-adic hypergeometric equations.

ii) It follows from our structure theorem that $\triangle_p(e_1, e_2, e_3)$ contains distinct elements g_1, g_2, g_3 of respective order e_1, e_2, e_3 such that $\triangle_p(e_1, e_2, e_3)$ is generated by any two of them. However, we do not know whether $\triangle_p(e_1, e_2, e_3)$ is a homomorphic image of $\triangle(e_1, e_2, e_3)$.

iii) According to [T77] p.207, one has $h^+(F) = 1$ for every F occuring in the tables, which allows to apply 7.3. Together with the information about commensurability provided by [T77]3,4, this allows to describe precisely $\triangle_p(e_1, e_2, e_3)$ in a number of cases. For instance, let us consider the case $F = \mathbf{Q}$, corresponding to a unique B in the list, with discriminant 6. Let $p = 2$ or 3. Then $\triangle_p(2, 4, 6)$ is Γ_p^*, while Γ_p^+ is a p-adic quadrangle group $\Diamond_p(2, 2, 3, 3)$ (of index 4 in the former); $\triangle_p(2, 6, 6)$ and $\triangle_p(3, 4, 4)$ are intermediate subgroups, and $\triangle_p(3, 6, 6)$ is of index 2 in $\triangle_p(2, 6, 6)$. For $p = 3$, groups of this type were studied in detail in [GvP80]9.1, since $\mathcal{B}$ is then the standard ring of Hurwitz quaternions.

iv) In the latter examples, one can use the p-adic Betti lattices of [A95]5 in order to express special values of the multivalued extension of the relevant pull-back of p-adic hypergeometric function (e.g. $_2F_1(\frac{1}{12}, \frac{1}{6}, \frac{1}{2}; z)$ for $\triangle_3(2, 6, 6)$) in term of rational values of the p-adic gamma function [TCI[.

9.4. Non-arithmetic p-adic triangle groups? Even when $\triangle(e_1, e_2, e_3)$ is non-arithmetic, there is attached a quaternion algebra B over an abelian totally real number field F, but B splits at several places at infinity $\infty_0, \ldots \infty_m$. Let p be a prime number, and assume that

(H) there are at least $m + 1$ distinct places $v_0, \ldots v_m$ above p which divide

the discriminant of B.

In [BoZ95[(1.16, 3.4), Boutot and Zink extend their p-adic uniformization theory to this situation. We have an arithmetic group Γ_∞ acting on $(\mathfrak{H})^{m+1}$, a subgroup $\triangle'$ of $\triangle(e_1, e_2, e_3)$ of finite index, and a finite morphism $\mathfrak{H}/\triangle' \longrightarrow (\mathfrak{H})^{m+1}/\Gamma_\infty$. It would be possible to generalize our arguments and deduce that $\triangle_p(e_1, e_2, e_3)$ is a p-adic triangle group. However, we have not yet found any example of a quadruple (e_1, e_2, e_3, p) satisfying (H).

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Yves André
Institut de Mathématiques
4 place Jussieu
Tour 46-56, 5$^{\text{ème}}$ étage, case 247
F-75252 PARIS CEDEX 05 - FRANCE
e-mail: andre@math.jussieu.fr