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Lecture I : Liftings of Galois covers of smooth curves

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These two lectures are a report on the lifting problem of Galois covers of smooth curves from char. $p>0$ to char. 0. The references are listed as [G-M 1], [G-M 2], [M].

The first lecture will focus on lifting problems, the main results are a local global principle and the positive answer to the lifting problem for p^2-cyclic covers generalizing a former result in p-cyclic case due to F.Oort, T.Sekiguchi, N.Suwa.

The second lecture will focuss on the geometry of order p-automorphisms of an open disc over a p-adic field.

I would like to thank Professors T.Sekiguchi and N.Suwa for inviting me to Japan and for organizing this symposium which gives me the opportunity for this report. I am very grateful to T.Ito and M.Yato for writing the TeX version of my notes of the lectures.

Finally I would like to dedicate this work to Michel Raynaud who has so much influenced us.

Notations:

k is an algebraicaly closed field of char. $p>0$.
R is a complete DVR finite over $W(k)$.
π is a uniformising parameter for R.
$\mathrm{Fr}R =: K \subset K^{\mathrm{alg}}$ is endowed with the unique polongation of the valuation v.

0. Introduction

Let C/k be a smooth irreducible complete curve of genus g.

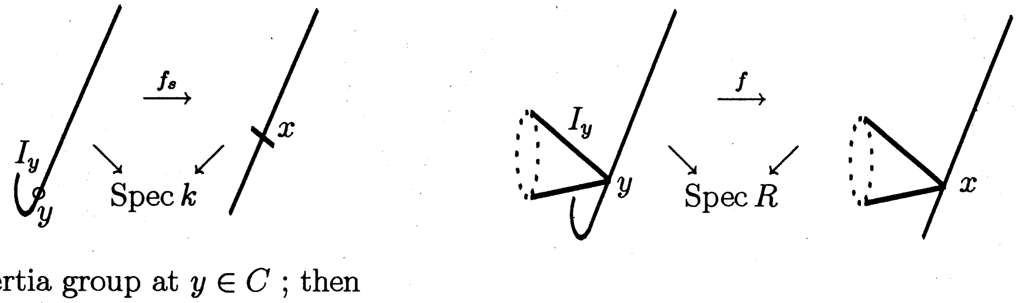
We are interested in the following:

Global lifting problem. *Let* $G \underset{\text{finite}}{\subset} \mathrm{Aut}_k C$. *Is it possible to find an* R *as above and* $\mathcal{C}/R$ *a relative smooth* R*-curve such that* $G \subset \mathrm{Aut}_R \mathcal{C}$ *and* $(\mathcal{C}, G)$ *gives* $(C, G) \mod \pi$*; i.e. we have a commutative diagramm:*

$$\begin{tikzcd} \mathrm{Aut}_k C & \mathrm{Aut}_R \mathcal{C} \arrow[l] \\ G \arrow[u, hook] \arrow[ur, dashed] & \end{tikzcd}$$

It is possible to formulate this in terms of G-covers. Let

$$\begin{array}{ccc} C & \xrightarrow{f_s} & C/G = D \\ & \searrow\swarrow & \\ & \operatorname{Spec} k & \end{array} \quad \underset{\text{lift}}{\rightsquigarrow} \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{C}/G = \mathcal{D} \\ & \searrow\swarrow & \\ & \operatorname{Spec} R & \end{array}$$



$I_y =$ Inertia group at $y \in C$; then
$I_y \subset \mathrm{Aut}_k \hat{\mathcal{O}}_{C,y} \simeq \mathrm{Aut}_k k[[z]]$
In the lifting process then I_y is lifted as $I_y \subset \mathrm{Aut}_R \hat{\mathcal{O}}_{\mathcal{C},y} \simeq \mathrm{Aut}_R R[[Z]]$.
Results.
· If $(|G|, p) = 1$ the answer is yes by Grothendieck (SGA I). In fact he proves that the answer is yes under the condition that the ramification in $C \xrightarrow{f_s} C/G$ is tame; i.e. I_y has prime to p order for any $y \in C$.

So the main problem occurs when the ramification is wild. Recall that any p-group can occur as an inertia group moreover in infinitely many ways.
· If $|G| > 84(g(C) - 1)$ the answer is no due to a trivial contradiction using Hurwitz bound for automorphism group in char 0.
· If G is cyclic of order pe(resp. p^2e) with $(e,p) = 1$, the answer is yes for R large enough namely $W(k)[\zeta_{(1)}]$(resp. $W(k)[\zeta_{(2)}]$, where $\zeta_{(n)} \in K^{\mathrm{alg}}$ is a primitive p^n-th root of 1. See [O-S-S] resp. [G-M1].

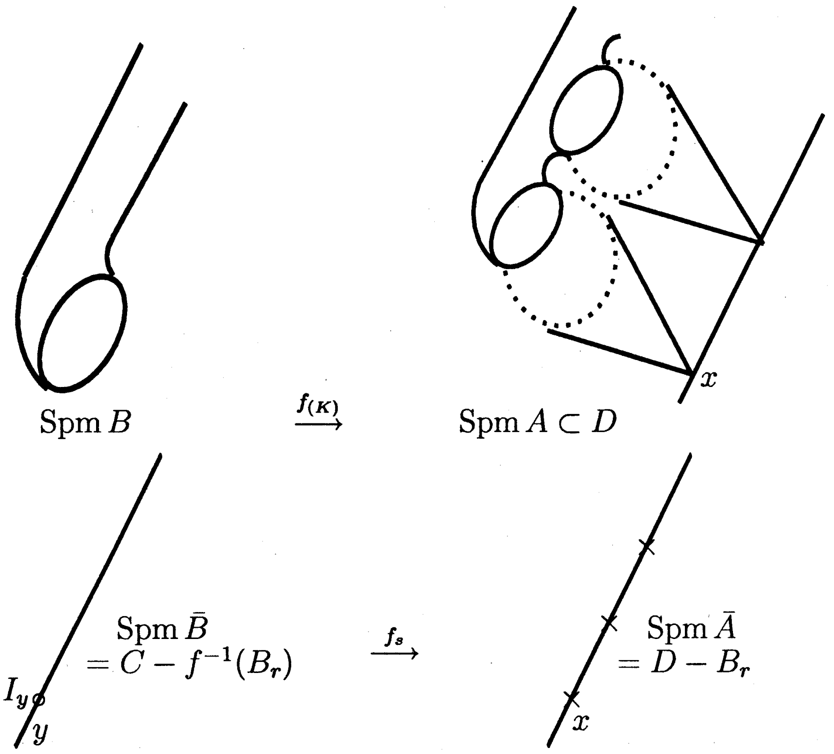
We have seen that the global lifting problem induces the:
Local lifting problem. *Let* $G \underset{\text{finite}}{\subset} \mathrm{Aut}_k k[[z]]$; *can we find* R *as above and a commutative diagram*

$$\begin{array}{ccc} \mathrm{Aut}_k k[[z]] & \longleftarrow & \mathrm{Aut}_R R[[Z]] \\ \uparrow & \nearrow & \\ G & & \end{array}$$

I. Local global principle

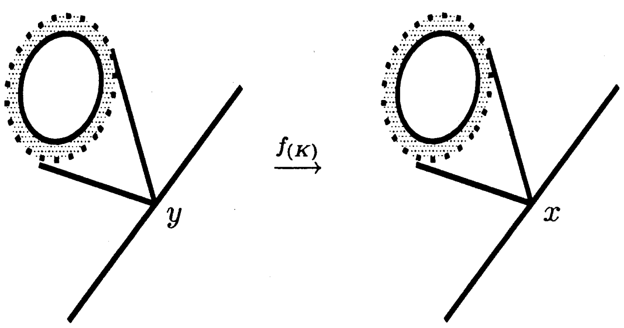
We prove the following
Theorem([G-M1]). *The global lifting problem over* R *for any* (C, G) *is equivalent to the local lifting problem over* R *for any* $(\hat{\mathcal{O}}_{C,y}, I_y \subset \mathrm{Aut}_k \hat{\mathcal{O}}_{C,y})$ *where* y *runs the branch locus of* $C \xrightarrow{f_s} C/G$.

Proof. Sketch
We use rigid analytic geometry
First Step: Lift D/k as $\mathcal{D}/R$ (SGA I). Let $B_r :=$ the branch locus for f_s.

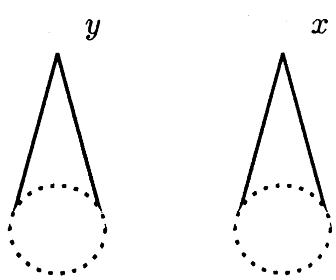


lift $\bar{A} \hookrightarrow \bar{B}$
in $A \hookrightarrow B$
finite étale G-cover
of affinoid spaces
(This is possible
via "Hensel" lemma)

The lifting is unique up to isomorphism. In fact the morphism $f_{(K)}$ extends (use Krasner-lemma) at the boundary of the formal fiber at x. And the germ of prolongation is unique so it is a Galois I_y-cover.



The main problem is so to compactify in a Galois way.
The local lifting problem says that we have Galois cover of open disks lifting



$$\mathrm{Spec}\, k[[z]] \longrightarrow \mathrm{Spec}\, k[[z]]^{I_y} = k[[t]]$$

Second Step: We prove a glueing lemma using Newton's theorem (the main point is that the extension $k((z))/k((t))$ is separable !). Conclude to the algebraicity using GAGA.

II. Local lifting for order p or p^2 automorphisms

It is possible to "lift" $k[[z]]/k[[z]]^G = k[[t]]$ in a Galois way as $A/R[[T]] = A^G$ for some A finite normal over $R[[T]]$(see Garuti [Ga]). The main problem is here to do this with a smooth A/R i.e. with good reduction over R.

For this purpose we need a numerical criteria for smoothness which is a particular case of a formula due to K.Kato (Duke M.J. 81 [Ka]).

Theorem(Kato). *Let $A/R[[T]]$ be a finite normal local ring such that A/π is reduced. We assume that $k((z)) = \mathrm{Fr}(A/\pi)/k((t))$ is separable and $\mathrm{rank}_{R[[T]]}A = \dim_{k((t))} k((z)) = n$. Let d_s be the degree of the different in the extension $k((z))/k((t))$. and d_η be the degree of the different in the extension $A \otimes_R K/R[[T]] \otimes_R K$. Then A is smooth over R i.e. $A \simeq R[[Z]]$ iff $d_\eta = d_s$.*

Application.

Order p^n case: We have Sekiguchi-Suwa [S-S] theory which shows the existence of a generic way to deform geometrically and in a Galois way a p^n-cyclic cover over $\operatorname{Spec} k((z)) \to \operatorname{Spec} k((z))^{<\sigma>}$. In order to deform in a smooth way we should be able to calculate differents; this is the main obstacle at the present to go further than $n = 2$.

Order p case([O-S-S]):

Let σ be an order p k-autmorphism of $k[[z]]$. By Artin-Schreier theory (abbreviation A-S) there exists $x \in k((z))$ such that $\sigma(x) = x + 1$ and $x^p - x = f(t)$. After a translation on x and a change of parameter we can take

$$\begin{cases} \sigma(x) = x + 1 \\ x^p - x = \frac{1}{t^m} \end{cases}$$

for some m, $(m, p) = 1$; we call $m + 1$ the Hasse-conductor or Weierstrass degree for σ.

Note that $z := x^{-\frac{1}{m}}$ is a parameter and that $\sigma(z) = z(1 + z^m)^{-\frac{1}{m}} = z(1 - \frac{1}{m}z^m + \cdots)$. By O-S-S theory we deform the A-S isogeny

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & \mathbb{G}_a & \longrightarrow & \mathbb{G}_a & \longrightarrow & 0 \\ & & & & x & \mapsto & x^p - x & & \end{array}$$

to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & \mathcal{G}^{(\lambda)} & \longrightarrow & \mathcal{G}^{(\lambda^p)} & \longrightarrow & 0 \\ & & & & x & \mapsto & \frac{(\lambda x+1)^p - 1}{\lambda^p} & & \end{array}$$

Let $R = W(k)[\zeta_{(1)}]$, $\lambda = \zeta_{(1)} - 1$. Let A be the integral closure of $R[[T]]$ in $F := \mathrm{Fr}R[[T]](X)$ where $\frac{(\lambda X+1)^p - 1}{\lambda^p} = \frac{1}{T^m}$ (this is a p-cyclic cover of $\mathbb{P}^1$ ramified in $T = 0$ and $T^m = -\lambda^p$ inside the open disc $|T| > 1$. So the generic different is

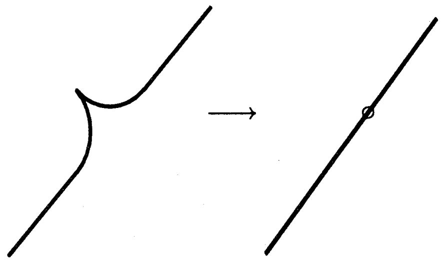
$$d_\eta = (m + 1)(p - 1)$$

which is known to be the different in the extension $k((z))/k((t))$. It is possible here to show that $Z := X^{-\frac{1}{m}}$ is a parameter for the open disc over R then $\sigma(Z) := \zeta Z(1 + (\zeta Z)^m)^{-\frac{1}{m}}$ lifts σ.

Remarks. 1. We could consider a lifting

$$\frac{(\lambda X+1)^p-1}{\lambda^p}=\frac{1}{(T-t_1)\cdots(T-t_m)}$$

with $t_i \in \pi R$, 2 by 2 distinct then $d_\eta = 2m(p-1) > d_s$. This cover has bad reduction and induces at the special fiber a cover with a cusp.



2. We will see in the next lecture that there are other non equivalent ways to lift an order p-automorphism.

Order p^2 case: Artin-Schreier-Witt theory (abbreviation A-S-W) gives

$$\begin{array}{ccccccc} 0 \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow & W_2 & \xrightarrow{F-1} & W_2 & \longrightarrow 0 \\ & (x_1,x_2) & \mapsto & (x_1^p-x_1, x_2^p-x_2-c(x_1^p,-x_1)) & \end{array}$$

where $c(x_1^p,-x_1) := \frac{(x_1^p-x_1)^p-(x_1^{p^2}+(-x_1)^p)}{p} \in \mathbb{Z}[x_1]$. After some translation we can find an Artin-Schreier representant

$$\begin{cases} x_1^p - x_1 = \frac{1}{t^{m_1}} & (m_1,p)=1 \\ x_2^p - x_2 - c(x_1^p,-x_1) = f_2\left(\frac{1}{t}\right) & \end{cases}$$

in such a way that $f_2\left(\frac{1}{t}\right) \in k\left[\frac{1}{t}, x_1^p - x_1\right]$ is written in a way which gives the different of the extension(see [G-M1] lemma 5.1).

Our main contribution was first to give an explicit formula for deforming A-S-W as the existence was proved in [S-S]; and secondly to provide in each case a lifting with the good different. We prove

Theorem([G-M1]). *Let $\lambda := \zeta_{(1)} - 1$, $\pi := \zeta_{(2)} - 1$, $\mu = \mathrm{Log}_p(1+\pi)$ the truncated logarithm in degree p (ζ_n is a compatible system of p^n th roots of 1).*

$$(y_1,y_2)=\left(\frac{(\lambda x_1+1)^p-1}{\lambda^p}, \frac{1}{\lambda^p}\{(\lambda x_2+\mathrm{Exp}_p\mu x_1)^p-(\lambda x_1+1)\mathrm{Exp}_p\mu^p y_1\}\right)$$

lifts A-S-W isogeny. (*)

Application:

$$\begin{cases} y_1 = \frac{1}{T^m} \\ y_2 = 0 \end{cases}$$

lifts the cover above with $f_2\left(\frac{1}{t}\right) = 0$.

In order to prove the smoothness it is sufficient to proceed by stage in the p^2-cyclic extension. We have only to look at the second stage above the open disc $|x_1| > 1$ ($x_1^{-\frac{1}{m_1}}$ is a parameter for the open disk in the first stage). It is easy to bound the generic different

(*) In particular $[(\mathrm{Exp}_p\mu x_1)^p - (\lambda x_1+1)\mathrm{Exp}_p\mu^p y_1]/\lambda^p$ lifts the cocycle $c(x_1^p,-x_1)$. In proving this we have to prove the congruence $p\mu - \lambda - p\frac{\mu^p}{\lambda^{p-1}} \equiv 0 \mod \pi^{p^2+1}$.

$d_\eta \le [1 \text{ (for } \infty \text{ pt)} \ + 1 \text{ (for } x_1 = -\frac{1}{\lambda}) \ + d^0_{x_1}\mathrm{Exp}_p(\mu^p y_1)](p-1) = (2+p(p-1))(p-1)$. The special differnt is $d_s = (m_2+1)(p-1)$ where $m_2 = d^0 c(x_1^p, -x_1) = p(p-1)+1$. So $d_\eta \le d_s$ and as always $d_\eta \ge d_s$ we conclude.

The general p^2-cyclic extension needs a good choice of f_2 in such a way we can read the different and for the lifting other involved formulas.

III. Other p-groups and Galois Inverse type conjecture

a) Obstructions to local lifting for $G = (\mathbb{Z}/p\mathbb{Z})^2$.

Theorem([G-M1]). *Let $G =< \sigma_1, \sigma_2 >\simeq (\mathbb{Z}/p\mathbb{Z})^2 \subset \mathrm{Aut}_k k[[z]]$ then we have the picture with $p+1$ intermediate p-cyclic extensions with Hasse conductors in the lower level $m_1+1 \le m_2+1 = m_3+1 = \ldots = m_{p+1}+1$ and in the upper level $m'_1 = m_2 p - m_1(p-1)$ and $m'_i = m_1$ for $i > 1$.*

$$\begin{tikzcd} & k[[z]] \arrow[dl, dash, "m'_1"'] \arrow[dr, dash, "m'_{p+1}"] & \\ k[[z_1]] \arrow[dr, dash, "m_1"'] & \ldots & k[[z_{p+1}]] \arrow[dl, dash, "m_{p+1}"] \\ & k[[t]] & \end{tikzcd}$$

If there is a local lifting then $p|m_1+1$ (this is stronger than Hasse-Arf congruences). Moreover, if

$$\begin{tikzcd} \mathrm{Spec}\, R[[Z_1]] \arrow[dr, "f_1"'] & & \mathrm{Spec}\, R[[Z_2]] \arrow[dl, "f_2"] \\ & \mathrm{Spec}\, R[[T]] & \end{tikzcd}$$

are liftings then the branch locus for $f_{1(K)}$ and $f_{2(K)}$ meet in $\frac{m_1+1}{p}(p-1)$ and this condition is an iff condition.

The proof is elementary, the congruence can be seen as follows. We know that m'_1+1 is the number of fix points for a lifting for σ_1, then the lifting of σ_2 acts freely on the m'_1+1 points (because the inertia groups are at most cyclic of order p) and so $p|m'_1+1$ and so $p|m_1+1$. The end is an application of Kato's criterion.

Remark. The congruences above have been generalised by J.Bertin [Be] for general groups. Moreover he proves that the Artin representation can be expressed in term of the regular representation and a representation of permutation (given by the action on the ramification locus) in particular it is rational.

b) We end this lecture by what we call :

"Inverse Galois type property".

Let G be a p-group; we say that G has the "Inverse Galois type property" if there is an R as above and a commutative diagram

$$\begin{tikzcd} \mathrm{Aut}_k k[[z]] & \mathrm{Aut}_R R[[Z]] \arrow[l] \\ G \arrow[u, hook] \arrow[ur, hook] & \end{tikzcd}$$

This is in some respect a strong form of Abhyankar's conjecture ([Ra]) and we prove
Theorem([M]). *Abelian p-groups of type $(p,\dots,p)$ have the Galois type property.*

Proof. In case $p=2$, $G=(\mathbb{Z}/2\mathbb{Z})^3$. The general case will be considered in next lecture.

Lemma. *The elliptic curve*

$$y^2=(1+\alpha_1x)(1+\alpha_2x)(1+(\sqrt{\alpha_1}+\sqrt{\alpha_2})^2x)$$

where $\alpha_i\in\mathbb{Z}_2^{\mathrm{ur}}$ are such that $\alpha_1\alpha_2(\alpha_1+\alpha_2)\neq 0 \mod 2$, has potentially good reduction at 2, an equation for the special fiber is

$$z^2-z=\bar{\alpha_1}\bar{\alpha_2}(\bar{\alpha_1}+\bar{\alpha_2})s^3$$

(a supersinguler elliptic curve as an étale 2-cyclic cover of $\mathbb{A}^1$).

Proof. Write

$$\begin{aligned}y^2=1&+(\alpha_1+\alpha_2+(\sqrt{\alpha_1}+\sqrt{\alpha_2})^2)x\\&+(\alpha_1\alpha_2+(\alpha_1+\alpha_2)(\sqrt{\alpha_1}+\sqrt{\alpha_2})^2)x^2+\alpha_1\alpha_2(\sqrt{\alpha_1}+\sqrt{\alpha_2})^2x^3.\end{aligned}$$

Let $\gamma=\alpha_1+\alpha_2+\sqrt{\alpha_1}\sqrt{\alpha_2}$ then

$$y^2=(1+\gamma x)^2+\alpha_1\alpha_2(\sqrt{\alpha_1}+\sqrt{\alpha_2})^2x^3.$$

Call

$$\begin{cases}x=2^{\frac{2}{3}}s\\ y=1+\gamma x-2z\end{cases}$$

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For the theorem consider the 3 elliptic curves

$$y_1^2=(1+\alpha_1x)(1+\alpha_2x)(1+(\sqrt{\alpha_1}+\sqrt{\alpha_2})^2x)$$

$$y_2^2=(1+\alpha_2x)(1+\alpha_3x)(1+(\sqrt{\alpha_2}+\sqrt{\alpha_3})^2x)$$

$$y_3^2=(1+\alpha_3x)(1+\alpha_1x)(1+(\sqrt{\alpha_3}+\sqrt{\alpha_1})^2x)$$

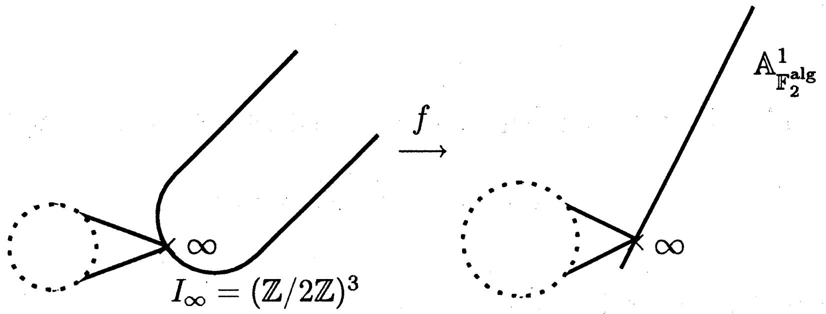
where $\alpha_i\in\mathbb{Z}_2^{\mathrm{ur}}$ and if $A_1=\overline{\alpha_1\alpha_2(\alpha_1+\alpha_2)}$, $A_2=\overline{\alpha_2\alpha_3(\alpha_2+\alpha_3)}$, $A_3=\overline{\alpha_3\alpha_1(\alpha_3+\alpha_1)}$ then if

$$\prod(\varepsilon_1A_1+\varepsilon_2A_2+\varepsilon_3A_3)\neq 0\qquad(\varepsilon_1,\varepsilon_2,\varepsilon_3)\in(\mathbb{Z}/2\mathbb{Z})^3-(0,0,0)$$

they have simultanously good reduction and the normalisation of the composition over $\mathbb{P}^1_{\mathbb{Q}_2^{\mathrm{ur}}}$ is a $(\mathbb{Z}/2\mathbb{Z})^3$-cover with good reduction and gives mod 2 the $(\mathbb{Z}/2\mathbb{Z})^3$ étale cover of the affine line generated by the three equations

$$\begin{cases}z_1^2+z_1=A_1s^3\\ z_2^2+z_2=A_2s^3\\ z_3^2+z_3=A_3s^3\end{cases}.$$

Calculate $d_s=4(2-1)(1+2+2^2)$, $d_\eta=\sharp\{\text{branch points}\}2^2(2-1)$ and the brach locus is $\{\infty,\alpha_1,\alpha_2,\alpha_3,(\sqrt{\alpha_i}+\sqrt{\alpha_j})_{i\neq j}\}$ so $d_s=d_\eta$.



Remark. (see [G-M2] II. 3.3.3) The "Inverse Galois type conjecture" is true for p^n-cyclic groups. The proof uses Lubin-Tate formal groups, we will indicate the method in the next lecture.

IV. References

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