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# Generators of topological groups *

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**Abstract**

We study various ways of generation of a topological group depending on the size and the topological properties of the set of generators.

## 1 Introduction

### 1.1 How to generate a topological group

Topological groups offer a rich choice of different ways of "generation" because of their two-fold nature. In the first place they are groups, so that one can consider the usual notion of generation – a subset $S$ of a group $G$ is said to *generate* $G$ if the smallest subgroup $\langle S\rangle$ of $G$ containing $S$ coincides with $G$. In the second place $G$ carries a topological structure, so that one can replace the equality $G=\langle S\rangle$ by the weaker condition $\langle S\rangle$ *is dense* in $G$. In such a case we refer to $S$ as to a *set of topological generators* and we say that $S$ *topologically generates* the group $G$. We start in §2 with the instances when the set $S$ that topologically generates $G$ has the smallest possible size (a singleton or a finite set). Then we consider in §3 the case when $S$ is a convergent sequence. Next come two different generalizations of a convergent sequence: a set with a single non-isolated point and a compact set. In §§4-7 we discuss the first aspect, in particular §5 deals with the case of closed discrete set of generators. Finally, in 8, we discuss the other generalization of the case of finite set of generators, namely compact sets of generators. In view of the large variety of results in this field most of the proofs are omitted.

### 1.2 Notation

We denote by $\mathbf{N}$ and $\mathbf{P}$ the sets of naturals and primes, respectively, by $\mathbf{Z}$ the integers, by $\mathbf{Q}$ the rationals, by $\mathbf{R}$ the reals, by $\mathbf{T}$ the unit circle group $\mathbf{R}/\mathbf{Z}$, by $\mathbf{Z}_p$ the $p$-adic integers ($p\in\mathbf{P}$). The cardinality of continuum $2^\omega$ will be denoted also by $\mathbf{c}$.

Let $G$ be a group. We denote by 1 the neutral element of $G$ and by $Z(G)$ the center of $G$. Topological groups are Hausdorff and completeness is intended with respect to the two-sided uniformity, so that every topological group $G$

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has a (Raĭkov) completion which we denote by $\hat{G}$. A group $G$ is *precompact* if $\hat{G}$ is compact, *pseudocompact* if every continuous real-valued function on $G$ is bounded, *countably compact* - if each open countable cover of $G$ admits a finite subcover. For topological groups $G$ and $H$ we denote by $c(G)$ the connected component of $G$, by $w(G)$ the weight of $G$ and by $GH$ the semidirect product of $G$ and $H$. For undefined symbols or notions see [19] or [20].

### 1.3 Some properties of the compact groups

The following facts about compact groups and their weight will be needed in the sequel. We give first a theorem that describes the structure of compact connected groups modulo compact connected abelian groups and compact connected Lie groups.

**Fact 1.1** (Varopoulos Theorem) *Every connected compact group $K$ is a quotient of a group of the form $A \times L$ with respect to some closed totally disconnected subgroup $N$ of $A \times Z(L)$, where $A$ is compact connected and abelian, while $L$ is a product of connected compact simple Lie groups.*

Note that every simple connected Lie group has a finite center, so that $Z(L)$ is totally disconnected as a product of finite groups. Moreover, $L/Z(L)$ is a product of a family of $w(L)$ metrizable groups, hence it has the same weight as $L$. Moreover, the projection $N_1$ of $N$ on $A$ is totally disconnected, so that $w(A/N_1) = w(A)$. As $N \subseteq N_1 \times Z(L)$ the group $K \cong (A \times L)/N$ projects onto the group $A/N_1 \times L/Z(L)$ that has weight $w(A) \cdot w(L) = w(K)$. Consequently $w(K/D) = w(K)$ for every closed totally disconnected subgroup $D \subseteq Z(L)$.

On the other hand, the center of $K$ coincides with the image of the center of $A \times L$, hence it is isomorphic to $[A \times Z(L)]/N$. Hence its connected component is isomorphic to $A/N \cap A$. One can choose a representation with $A \cap N = 1$ in order to have $c(Z(K)) \cong A$.

The commutator subgroup $K'$ is coincides with the image of $L$, so it is isomorphic to $L/N \cap L$. A center-free connected compact group $K$ has a very simple form: $K \cong \prod_{i \in I} L_i$, where each $L_i$ is a connected compact simple Lie group with trivial center (i.e., algebraically simple).

**Fact 1.2** (Lee's theorem [33]) *Every compact group $G$ admits a totally disconnected compact subgroup $H$ with $G = c(G)H$ and such that $D = H \cap c(G) \subseteq Z(c(G))$. Consequently, $w(G/D) = w(G)$ and $G/D \cong c(G)/DF$ with a compact totally disconnected group $F$.*

The equality $w(G/D) = w(G)$ follows from the fact that for the connected compact subgroup $K = c(G)$ of $G$ we have:

- $w(G) = w(K)w(G/K)$, and
- $w(K) = w(K/D)$ (the proof of this fact was given above).

Hence, $w(G/D) = w(G/K) \cdot w(K/D) = w(G/K) \cdot w(K) = w(G)$.

## 2 When the set of generators is very small

### 2.1 Monothetic groups

The first natural question is what can we say when $S$ is as simple as possible, e.g. a singleton. A topological group having a dense cyclic subgroup is called *monothetic*. Clearly, such a group must have the following two properties:

(a) $G$ is abelian;

(b) $w(G) \leq \mathbf{c} = 2^\omega$.

Let us note that while (b) has a purely topological nature, the first restraint has a purely algebraic nature, due to the fact that cyclic groups are abelian and the fact that a Hausdorff group having a dense abelian subgroup must be abelian. We shall see below (Theorem 2.3) that one can invert this under some conditions, i.e., (a)&(b) imply the group is monothetic.

**Example 2.1** The circle group $\mathbf{T}$ as well as all its powers $\mathbf{T}^\alpha$, $\alpha \leq \mathbf{c}$, are monothetic. This follows from the following theorem of Kroneker:

**Theorem 2.2** *Let $\beta_1, \ldots, \beta_n$ be real numbers such that $1, \beta_1, \ldots, \beta_n$ are rationally independent. Then*

$$\langle(\beta_1, \ldots, \beta_n)\rangle + \mathbf{Z}^n \textit{ is dense in } \mathbf{R}^n \tag{1}$$

Now to see that $\mathbf{T}^\mathbf{c}$ is monothetic take a Hamel base $B = \{\beta_\alpha\}_{\alpha<\mathbf{c}}$ of $\mathbf{R}$ over $\mathbf{Q}$ with $\beta_0 = 1$. Let $q : \mathbf{R} \longrightarrow \mathbf{T} = \mathbf{R}/\mathbf{Z}$ be the canoinical quotient map. Then the function $\beta : \mathbf{c} \longrightarrow \mathbf{T}$ defined with $\beta(\alpha) = q(\beta_{\alpha+1})$ for $\beta < \mathbf{c}$ produces an element $\beta \in \mathbf{T}^\mathbf{c}$ such that the cyclic subgroup $\langle\beta\rangle$ is dense in $\mathbf{T}^\mathbf{c}$.

Another consequence of (1) is that $\mathbf{R}^n$ has a dense $n+1$-generated subgroup (but no $n$-generated subgroup can be dense, see [2]).

Another important monothetic group is the compact Pontryagin dual $\mathbf{K} = Hom(\mathbf{Q}, \mathbf{T})$ of the rationals.

Since every compact connected abelian group of weight $\leq \mathbf{c}$ is monothetic (??), we obtain:

**Theorem 2.3** *A compact connected group is monothetic iff (a) and (b) are satisfied.*

Let us recall now the following well known fact: there exists a (projectively) universal monothetic compact group, namely $M = \mathbf{K}^\mathbf{c} \times \prod_p \mathbf{J}_p$ (i.e., every compact monothetic group is isomorphic to a quotient of $M$).

## 2.2 Topologically finitely generated groups

The next most simple case comes when the set $S$ of topological generators is finite. Clearly, every group with a dense finitely generated subgroup must have weight $\leq \mathbf{c}$.

**Theorem 2.4** (Kuranishi's theorem) *Every semisimple compact connected Lie group has a dense 2-generated subgroup.*

Hofmann and Morris [28] extended this theorem to the case of *arbitrary* compact connected groups that satisfy the obvious necessary considiton of having weight $\leq \mathbf{c}$.

Here a question may arise about the importance of connectedness in these results. In our next comments we show that connectedness is indeed relevant. It turns out that for topologically $n$-generated countably compact groups the connected part "prevail" in appropriate sense.

In the sequel we denote by $F_n$ the free group of $n$ generators.

**Theorem 2.5** (Hall's Theorem) *The intersection of the normal subgroups of finite index of $F_n$ is trivial.*

Consequently, the family of all normal subgroups of finite index of $F_n$ is a local base at 1 of a group topology $\tau_H$, the *profinite topology* of $F_n$. This topology is precompact, i.e., the completion $\hat{F}_n$ of $(F_n, \tau_H)$ is compact. This group is metrizable since there are only countably many finite-index subgroups of $F_n$ (indeed, every such subgroup is finitely generated by Nielsen-Schreier's theorem [44, Theorem 6.1.1]). Moreover, every precompact $n$-generated group with local base at 1 consisting of open normal subgroups is a continuous homomorphic image of $(F_n, \tau_H)$. Therefore, every compact totally disconnected topologically $n$-generated group is a quotient group of $\hat{F}_n$. This proves the following:

**Theorem 2.6** *For a totally disconnected countably compact group $G$ and $n \in \mathbb{N}$ TFAE:*

*(a) $G$ is topologically $n$-generated;*

*(b) is a quotient group of $\hat{F}_n$.*

*In particular, topologically finitely generated countably compact groups are metrizable, hence compact.*

Now we see that the connected component $c(G)$ of a topologically finitely generated countably compact group is a $G_\delta$-subgroup:

**Corollary 2.7** *If a countably compact group is topologically finitely generated then $G/c(G)$ is metrizable (hence, compact).*

# 3 Generating a topological group by a convergent sequence

## 3.1 Generating a compact group by a convergent sequence

A compact metrizable group may fail to have a finite set of topological generators even in very simple cases as that of the group $G = \{0,1\}^\omega$. Indeed, here the finitely generated subgroups are finite, so cannot be dense. On the other hand, if we take $s_n$ to be the sequence $(\underbrace{0,\ldots,0,1}_{n},0\ldots,0,\ldots) \in G$ then one can easily see that $s_n \longrightarrow 0$ in $G$ and the set $S = \{s_n\}$ is a set of topological generators. In other words, the group $G$ is generated by a convergent sequence. This is not surprizing since:

**Theorem 3.1** *Every compact metrizable group is topologically generated by a convergent sequence.*

Before discussing the proof of this theorem let us note that it suffices to consider only sequences that converge to 0. Indeed, if the convergent sequence $S = \{s_n\} \longrightarrow g$ generates a dense subgroup of $G$, then the sequence $\hat{S} = \{s_n g^{-1}\} \cup \{g\}$ converges to 1 and generates a dense subgroup as well.

The proof of Theorem 3.1 in the abelian case is an easy consequence of the fact that such groups are quotients of the group $(\mathbf{K} \times \prod_p \mathbf{J}_p)^\omega$. In the general case it is a consequence of a more general theorem we give below (see Theorem 3.4). Prior to passing to that more general result we need the following remark. Clearly, a non-separable group cannot be topologically generated by a (convergent) sequence. In order to eliminate this irrelevant cardinality restraint we consider also *supersequences* $S = \{s_\alpha\}$ converging to 1, i.e., sets $S$ such that $S \setminus \{1\}$ is discrete and $S \cup \{1\}$ is compact. We admit here finite sets $S$, i.e., eventually constant convergent sequences. In case $S$ is infinite, this means that $S \cup \{1\}$ is the one-point Alexandrov compactification of the discrete set $S \setminus \{1\}$.

It was proved by Douady [17, Theorem 1.3] (see also [21, Proposition 15.11]) that every infinite Galois group (i.e., compact totally disconnected topological group) has a system $S$ of generators that *converges to* 1. Denote by $\mathcal{S}eq$ the class of topological groups that have a dense subgroup generated by a convergent supersequence. Clearly, the class $\mathcal{S}eq$ contains the class of topologically finitely generated groups.

**Proposition 3.2** *Let $f: G \longrightarrow H$ be a continuous homomorphism. If $G \in \mathcal{S}eq$ then also $H \in \mathcal{S}eq$ whenever $f(G)$ is a dense in $H$.*

This implies in particular that if $G \in \mathcal{S}eq$ is dense in $H$ then also $H \in \mathcal{S}eq$. The class $\mathcal{S}eq$ has the following nice closure properties:

**Proposition 3.3** *The class $\mathcal{S}eq$ is closed under taking:*

1. *direct products*
2. *continuous homomorphic images (in particular, quotients)*
3. *inner products: if $G = NH$ and both subgroups $N$ and $H$ of $G$ are in $\mathcal{S}eq$ then also $G \in \mathcal{S}eq$.*

Hofmann and Morris [28] established the fact (even if in different terms) that every compact group is topologically generated by a convergent sequence, i.e.

**Theorem 3.4** *The class $\mathcal{S}eq$ contains all compact groups.*

We shall briefly sketch their proof of the theorem. As mentioned above, this was already known in the totally disconnected case ([17, Theorem 1.3]).

**Step 1.** $\mathcal{S}eq$ contains all compact abelian groups.
Indeed, every compact abelian group is a quotient of a product $\prod_i M_i$ where each $M_i$ is a compact monothetic group (in fact, either $\mathbf{J}_p$ or $\mathbf{K}$). Now the properties from Proposition 3.3 apply.

**Step 2.** $\mathcal{S}eq$ contains all compact connected groups.
Every compact connected group $G$ coincides with the product $Z(G)G'$, where $Z(G)$ is the center of $G$ and $G' = \langle [a,b] : a, b \in G \rangle$ is the commutator subgroup of $G$. By Step 1 $Z(G) \in \mathcal{S}eq$. On the other hand, by Fact 1.1 $G'$ is a quotient of a product of compact connected simple Lie groups $L_i$. So by Kuranishi's theorem $L_i \in \mathcal{S}eq$ and again Proposition 3.3 applies.

In the general case $G = c(G)H$ for some totally disconnected subgroup $H$. So by Step 2 $c(G) \in \mathcal{S}eq$ and $H \in \mathcal{S}eq$ by Douady's theorem. By Proposition 3.3 also $G \in \mathcal{S}eq$.

## 3.2 Countably compact groups generated by a convergent supersequence

Here we shall discuss countably compact groups in $\mathcal{S}eq$. Obviously, a topological group $G$ which is not topologically finitely generated belongs to $\mathcal{S}eq$ only if $G$ contains non-trivial convergent sequences. In particular, an infinite torsion topological abelian group $G \in \mathcal{S}eq$ must contain non-trivial convergent sequences. In view of the example (requiring MA) of van Douwen of a countably compact subgroup of $\{0,1\}^{\mathbf{c}}$ without non-trivial convergent sequences. one immediately concludes that countable compactness alone cannot guarantee the existence of a topologically generating convergent sequence. This example was given for the first time in [4]. Various ZFC examples of $\omega$-bounded groups $H \notin \mathcal{S}eq$ were given in [14] (a group $G$ is *$\alpha$-bounded* if every subset of size $\alpha$ of $G$ is contained in a compact subset of $G$):

**Example 3.5** (1) ([14, Theorem 2.8]) For every infinite cardinal $\alpha$ there exists a connected, locally connected $\alpha$-bounded abelian topological group $G \notin \mathcal{S}eq$.

(2) ([14, Corollary 2.9]) There exists a non-separable, connected and locally connected abelian group $H \notin \mathcal{S}eq$ with $|H| = \mathbf{c}$ such that $H^\omega$ is countably compact.

(3) ([14, Theorem 2.11]) There exists an $\omega$-bounded dense subgroup $G \notin \mathcal{S}eq$ of $\{0,1\}^{\mathbf{c}}$.

The above examples show that countable compactness of $G$ may fail to guarantee $G \in \mathcal{S}eq$. Our aim will be to see how can the situation change if we impose on the group also *minimality*. A topological group $(G,\tau)$ is called *minimal* if $\tau$ is a minimal element of the partially ordered (with respect to inclusion) set of Hausdorff group topologies on the group $G$.

**Theorem 3.6** ([15, Theorem 4.2.1]) *$\mathcal{S}eq$ contains all connected abelian groups that contain a dense countably compact minimal group. In particular, every connected countably compact minimal abelian group has a generating convergent supersequence.*

The restriction on the group $G$ to be abelian can probably be removeded from Theorem 3.6, but we have no proof at hand.

**Theorem 3.7** *Let $G$ be a countably compact minimal abelian group. Then $c(G) \in \mathcal{S}eq$. If $G \in \mathcal{S}eq$, then also $G/c(G) \in \mathcal{S}eq$.*

Since both $c(G)$ and $G/c(G)$ are minimal (by [8]), the above theorem reduces of the study of the general countably compact minimal abelian groups in $\mathcal{S}eq$ to case of totally disconnected ones.

Now we show that a minimal countably compact abelian group need not have a generating convergent sequence (compare with Theorem 3.6).

**Example 3.8** There exists a totally disconnected $\omega$-bounded (and hence countably compact) minimal abelian group $H \notin \mathcal{S}eq$. To get an example take the inverse image $H$ under the canonical homomorphism $\mathbf{Z}(4)^{\mathbf{c}} \longrightarrow \mathbf{Z}(2)^{\mathbf{c}}$ of the subgroup $G$ of $\mathbf{Z}(2)^{\mathbf{c}}$ constructed as in Example 3.5 (3). By Proposition 3.2 $H \notin \mathcal{S}eq$. Minimality of $H$ follows from the minimality criterion for dense subgroups (see Theorem 4.12 or [11, Chap. 4]).

### 3.3 The sequential generating rank

For $G \in \mathcal{S}$ set

$$seq(G) := \min\{|S| : S \subseteq G \text{ generating supersequence of } G\}.$$

Since $|S| \leq \psi(G)$ for every convergent supersequence $S$ in $G$, we have

$$d(G) \leq \max\{\omega, seq(G)\} \leq \psi(G). \tag{2}$$

The following fact was proved first by Hofmann and Morris [28, Theorem 4.14] in the case of compact non-monothetic groups $G$. Recently Shakhmatov and the author [12] succeeded to find a new proof that works for *all* topological groups:

**Theorem 3.9** [12]) *$seq(G)^\omega \geq w(G)$ for every $G \in \mathcal{S}eq$.*

Actually, when $G$ is compact and connected then $seq(G)$ turns out to be the least cardinal $\kappa$ such that $\kappa^\omega \geq w(G)$, in particular $seq(G)$ depends only on the weight of the group $G$ ([12]).

For every $\alpha > \mathbf{c}$ there exists a compact group $G$ such that $seq(G) \geq \alpha$ and $w(G) = \alpha^\omega$ [29, Corollary 2.16]. Note that for compact $G$ with $w(G) \leq \mathbf{c}$ one has $seq(G) = 2$ when $G$ is connected and non-abelian; otherwise $\max\{\omega, seq(G)\} =$

$w(G/c(G))$ when $G/c(G)$ is infinite (clearly, for finite $G/c(G)$ one has $seq(G/c(G)) \leq seq(G) \leq seq(G/c(G)) + 2$).

The argument to prove Proposition 3.3 proves also the equality

$$seq(G_1 \times G_1) = seq(G_1)seq(G_2) \tag{1}$$

that obviously extend to all finite direct products. In the case of infinite products one has the inequalities that follow from the above properties:

$$\log |I| \cdot sup\{seq(G_i) : i \in I\} \leq seq(\prod_{i \in I} G_i) \leq |I| \cdot sup\{seq(G_i) : i \in I\}.$$

This becomes an equality in the case of countably infinite products. This formula is available also in the case of $\sum$-products and $\sigma$-products.

Analogously, the argument to prove Proposition 3.2 leads to $seq(H) \leq seq(G)$ when there exists a continuous homomorphism $f: G \longrightarrow H$ such that $f(G)$ is a dense in $H$.

# 4 The suitable sets

The above instances motivated Hofmann and Morris [28] to introduce the notion of a *suitable set* of a topological group $G$, this is a discrete subset $S$ of $G$ that generates a dense subgroup of $G$ and $S \cup \{1\}$ is closed in $G$. In other words, the set $S' = S \cup \{1\}$ has again at most one adherence point (as in the case of a convergent supersequence $S \longrightarrow 1$), namely 1. These authors extended Douady's theorem by proving that every *locally compact* group has a suitable set (but it need not be a converget supersequence any more!).

**Theorem 4.1** (Hofmann–Morris [28]) *Every locally compact topological group has a suitable set.*

Later they proved a much stronger result for connected groups of weight $> \mathbf{c}$ (Theorem 4.6).

Here we see that suitable sets in countably compact groups give nothing new. It turns out that they are either finite or a non-trivial convergent supersequence:

**Corollary 4.2** ([15, Proposition 2.2]) *Let $G$ be a countably compact group. Then:*

*(a) $G$ has a suitable set iff $G \in \mathcal{S}eq$.*

*(b) $G$ has a closed suitable set iff $G$ has a finitely generated dense subgroup. In such a case $G/c(G)$ is compact metrizable.*

We denote by $\mathcal{S}$ the class of groups having a suitable set set. Sometimes suitable sets $S$ *algebraically generate* the group $G$. In such a case we refer to $S$ as a *generating suitable set* and denote by $\mathcal{S}_g$ the class of groups having such a set [15]. There are few examples of groups in $\mathcal{S}_g$: all countable groups [4] (see also §6.1).

More detail on suitable sets can be found in the paper [14, 2, 15, 48]).

## 4.1 The generating rank

Following [28], for $G \in \mathcal{S}$ set

$$s(G) := \min\{|S| : S \subseteq G \text{ suitable }\}.$$

Then

$$d(G) \leq \max\{\omega, s(G)\} \leq w(G). \tag{3}$$

The first inequality is obvious, for the second one should exploit the fact that $S \setminus \{1\}$ is discrete.

Mel'nikov [34] proved the equality $\max\{\omega, s(G)\} = w(G)$ in the case of totally disconnected compact groups. For reader's convenience we give a proof here.

**Theorem 4.3** (Mel'nikov) *Every totally disconnected compact group $G$ satisfies $\max\{\omega, s(G)\} \leq w(G)$.*

*Proof.* To show that $\max\{\omega, s(G)\} \geq w(G)$ it suffices to note that if $S$ is a supersequence convergent to 1 in such a group assigning to each open normal subgroup $N$ of $G$ the finite set $F_N := S \setminus N$ one obtains a countably many-to-one map from the filter $\mathcal{N}$ of all open normal subgroups of $G$ to $[S]^{<\omega}$. Indeed, let $N_0$ be the closed normal subgroup generated by $N \cap S$. Since $\langle S \rangle$ is dense in $G$, it follows that $NF_N = G$ and $G/N_0$ is topologically finitely generated, hence metrizable by Theorem 2.6. Then if $N'$ is an open normal subgroup of $G$ such that $S \setminus N' \subseteq F_N$ then $N'$ contains the set $N \cap S$ and consequently also $N_0$. Clearly, such subgroups $N'$ are in bijective correspondence with the open normal subgroup of the quotient $G/N_0$ that are countably many. QED

It was shown in [14] that $\max\{\omega, s(G)\} \leq L(G)\psi(G)$ for every group $G \in \mathcal{S}$. In particular this gives

$$G \in \mathcal{S} \Rightarrow d(G) \leq L(G)\psi(G). \tag{4}$$

This helped in finding a ZFC example of a group without a suitable set in [14]. It was based on the following example found by Okunev and Tamano [36]:

**Example 4.4** There exists a $\sigma$-compact separable space $X$ with $nw(X) > \omega$ and $C_p(X)$ Lindelöff. Then $L = C_p(X)$ is not separable, while $\psi(L) = L(L) = \omega$, so that by (4) $L$ has no suitable set.

Another example with stronger property can be obtained under the assumption of $\diamondsuit$.

**Example 4.5** ([14]) Ivanov proved in [32] under the assumption of $\diamondsuit$ there exists a compact non-metrizable space $X$ such that all $X^n$ are hereditarily separable. Now $L = C_p(X)$ is hereditarily Lindelöf and non-separable. So for every dense subgroup $H$ of $L$ we have $d(H) > \omega = L(H)\psi(L)$. Again by (4) $H$ has no suitable set.

In the case of locally compact groups one has the general Theorem 4.1 and the following more precise form:

**Theorem 4.6** (Hofmann–Morris [30]) *If $G$ is a locally compact connected group with $w(G) > \mathbf{c}$ then the arc component of $G$ contains a suitable set of $G$ of cardinality $s(G)$.*

In case when the vector subgroup splits one has:

**Theorem 4.7** (Cleary–Morris [2]) *For a connected compact group $G$ with $w(G) \leq \mathbf{c}$ $s(\mathbf{R}^n \times G) = n + 1$.*

The class $\mathcal{S}$, similarly to the class $\mathcal{S}eq$, is closed under taking (semi)direct products, $\sum$-products and $\sigma$-products. More precisely:

$$s(\prod_{i \in I} G_i) \leq |I| sup\{s(G_i) : i \in I\}.$$

Analogous formula is available in the case of $\sum$-products and $\sigma$-products.

Unlike the class $\mathcal{S}eq$, the class $\mathcal{S}$ fails to be closed under taking arbitrary quotients (this property fails even for local homeomorphisms, cf. Example 4.11). On the other hand, $\mathcal{S}$ is closed under closed homomorphic images; more precisely, for a continuous closed surjective homomorphism $h : G \longrightarrow H$ one has $s(H) \leq s(G)$ ([15]). The closedness of $h$ is essential here, this property fails even when $\ker h$ is discrete (cf. Example 4.11).

## 4.2 Separable groups

Separable groups with countable pseudocharacter (more generally, $nw(G) = \omega$) admit a suitable set.

**Theorem 4.8** *Let $G$ be a separable topological group. Then $G$ has a suitable set in the following case:*
*(a) if $G$ is of countable pseudocharacter;*
*(b) if $G$ is not precompact; in this case $G$ has a closed suitable set.*

Item (b) is a particular case of the following more general fact observed in [4]. These authors ([4, Definition 5.3]) defined the *boundedness number* $b(G)$ of a topological group $G$ as

$$b(G) := \min\{\kappa : (\exists \text{ open } U \subseteq G)(\forall F \in [G]^{<\kappa})\ G \neq FU\}.$$

Then $b(G) \leq d(G)^+$ for every topological group $G$ [4, Theorem 5.5].

**Theorem 4.9** ([4, Theorem 5.7]) *If $d(G) < b(G)$ then $G \in \mathcal{S}$ (actually, $G$ has a closed suitable set) and $s(G) = d(G)$.*

This theorem shows that a separable group without a suitable set must be precompact. Actually, it can be shown that such a group must be pseudocompact ([14, Corollary 3.8]).

Since a topological group with a countable network is separable and has countable pseudocharacter we get from (a):

**Corollary 4.10** ([14, Corollary 3.10]) *Every topological group with a countable network has a suitable set.*

Note that for an open subgroup $H$ of a topological group $G$ one obviously has $[G : H] < b(G)$. Since $d(G) \leq |G|$ for every $G$ we conclude that if $H$ is an open subgroup of $G$ with $[G : H] = |G|$ then $G$ has a closed suitable set by Theorem 4.9. In this way we get the following:

**Example 4.11** Let $H$ be any topological group and let $H_d$ denote the group $H$ equipped with the discrete topology. Then the group $G = H \times H_d$ has a closed suitable set as $H$ is an open subgroup of $G$ with $[G : H] = |G|$. If we choose $H$ without a suitable set, then $H \cong G/H_d$ is locally homeomorphic to $G$ that has a (closed) suitable set.

## 4.3 Groups close to being metrizable

Comfort, Morris, Robbie, Svetlichny and Tkačenko [4, Theorem 6.6] proved that every metrizable topological group $G$ has a suitable set. Recently Okunev and Tkachenko [37] found a nice unifying generalization of this fact and Theorem 4.1. It is based on the notion of an *almost metrizable* topological group introduced by Pasynkov [38] – a topological group $G$ is said to be *almost metrizable* if it contains a non-empty compact set $K$ of countable character in $G$. Clearly, all locally compact and all metrizable topological groups are almost metrizable

[38]. It was proved in [37] that every almost metrizable group has a suitable set. Note that the line of this generalization of compactness is transversal to the line of countable compactness considered in §3.2. In fact, as shown in [38], an almost metrizable topological group $G$ has a compact subgroup $N$ such that the quotient space $G/N$ is metrizable. Since every pseudocompact metrizable space is compact, we conclude that every pseudocompact (in particular, countably compact) almost metrizable group is compact.

In connection with metrizable and close to being metrizable groups we mention also the following result: if a group $G$ is a countable unions of closed metrizable subspaces then $G$ has a suitable set ([14, Corollary 3.13], in case $G$ is not compact that set can be chosen closed).

## 4.4 Free topological groups with or without suitable sets

The first ZFC example of a topological group without a suitable set was given in the framework of free topological groups. It was proved in [4] that the free abelian topological group $A(\beta\omega \setminus \omega)$ has no suitable set. This can be put in a more general form (see [4]). However, the free topological group over a compact space often has a suitable set (see (e) below).

The free topological group $F(X)$ of a Tychonov space $X$ has a suitable set in many cases. Namely, when $X$ is:

(a) separable (holds for $A(X)$ as well, cf. [4]);

(b) metrizable ([14]);

(c) paracompact $\sigma$-space (i.e., has a $\sigma$-discrete network) [47]);

(d) has at most one non-isolated point ([48]);

(e) compact with one of the following properties:

- ordinal space ([48, 37]);
- dyadic space [37];
- polyadic space (=continuous image of a product of convergent supersequences) [37].

## 4.5 Minimal groups

We say that a topological group $G$ is *totally minimal* if every Hausdorff quotient group of $G$ is minimal. To recall a criterion for (total) minimality of dense subgroups we need the following definition. A subgroup $H$ of a topological group $G$ is *totally dense* if for every closed normal subgroup $N$ of $G$ the intersection $H \cap N$ is dense in $N$. For the minimality criterion of dense subgroups we need another notion: a subgroup $H$ of a topological group $G$ is *essential* if every non-trivial closed normal subgroup $N$ of $G$ non-trivially meets $H$.

**Theorem 4.12** *Let $G$ be a Hausdorff topological group and $H$ be a dense subgroup of $G$. Then:*

*(1)* ([9]) *$H$ is totally minimal iff $G$ is totally minimal and $H$ is totally dense in $G$.*

*(2)* ([?]) *$H$ is minimal iff $G$ is minimal and $H$ is essential in $G$.*

**Theorem 4.13** *A totally minimal group $G$ has a suitable set in the following cases:*

- ([15, Theorem 4.1.4]) *$G$ is abelian;*

- ([15, Theorem 4.1.5]) *$G$ is connected and precompact.*

The proofs of both items is based on the fact that metrizable groups admit suitable a set and the group $G$ is totally dense in a product of metrizable groups.

It is not clear whether "totally minimal" can be replaced by "minimal" in Theorem 4.13, asking the group to satisfy the conjunction of both conditions in the theorem. Theorem 3.6 shows that this is possible if one adds also countable compactness. The role of connectedness was shown in Example 3.8.

# 5 Closed discrete generators (closed suitable sets)

Here we discuss when $G \in \mathcal{S}$ has a *closed* suitable set. We denote by $\mathcal{S}_c$ the subclass of these groups in $\mathcal{S}$. We offer here some recent unpublished results from [16].

**Proposition 5.1** *Let $G$ be a topological group. Suppose that $H$ is a closed normal subgroup of $G$ such that $G/H$ contains a* closed *suitable set $\Sigma$. If $d(H) \leq |\Sigma|$, then $G$ has a* closed *suitable set.*

Note that the requirement $H \in \mathcal{S}$ is no longer needed. To pay for this however, we need that the *density* of $H$ not be too large.

**Definition 5.2** For $G \in \mathcal{S}$ (resp., $G \in \mathcal{S}_c$) denote by $\sigma_c(G)$ the minimum cardinality of a closed suitable set for $G$. Similarly, let $\Sigma(G)$ (resp. $\Sigma_c(G)$) the minimum cardinal $\alpha$ such that, if $S$ is a suitable set for $G$, then $|S| < \alpha$ (resp., the minimum cardinal $\alpha$ such that, is $S$ is a closed suitable set for $G$, then $|S| < \alpha$). For $G \notin \mathcal{S}$ (resp., $G \notin \mathcal{S}_c$) set $s(G) = \infty$ and $\Sigma(G) = 0$ (resp., $\sigma_c(G) = \infty$ and $\Sigma_c(G) = 0$).

Note that:

1. $\Sigma(G) > 0 \Longrightarrow \Sigma(G) \geq \aleph_0$ (resp., $\Sigma_c(G) > 0 \Longrightarrow \Sigma_c(G) \geq \aleph_0$).
2. ([15, 2.7]) A $\sigma$-compact group $G \in \mathcal{S}_c$ satisfies $\sigma_c(G) \leq \aleph_0$; and $\Sigma_c(G) \leq \aleph_1$.
3. ([15, 3.4.5] and [48, 21]) $s(G) \leq s(H)\sigma_c(G/H)$ (resp. $\sigma_c(G) \leq \sigma_c(H)\sigma_c(G/H)$) when all values involved are $\neq \infty$. If so, then $\Sigma(G) \geq \Sigma(H)\Sigma_c(G/H)$ (resp. $\Sigma_c(G) \geq \Sigma_c(H)\Sigma_c(G/H)$).

**Theorem 5.3** ([16]) *Let the group $G$ have a closed normal subgroup $H$ with $G/H \in \mathcal{S}_c$;*

*(a) if $d(H) \leq \sigma_c(G/H)$, then $\sigma_c(G) \leq \sigma_c(G/H)$;*

*(b) if $d(H) < \Sigma_c(G/H)$, then $G \in \mathcal{S}_c$ and $\Sigma_c(G/H) \leq \Sigma_c(G)$;*

*(c) if $H$ is discrete, then always $G \in \mathcal{S}_c$ with $\sigma_c(G) \leq |H \setminus \{1\}| \cdot \sigma_c(G/H)$.*

Let us note that the first item of this theorem is a substantial improvement of [15, Theorem 3.4.5 (b)] (in the case $d(H) \leq \sigma_c(G/H)$ !) where the conclusion $G \in \mathcal{S}_c$ is obtained only under the condition $N \in \mathcal{S}_c$ and the proof gives the weaker inequality $\sigma_c(G) \leq \sigma_c(H) \cdot \sigma_c(G/H)$. See also 5.6 *infra.*

Notice that $s(\mathbf{T}) = \sigma_c(\mathbf{T}) = 1$, $\Sigma_c(\mathbf{T}) = \aleph_0$, whereas $\Sigma(\mathbf{T}) = \aleph_1$. Also, $s(\mathbf{R}) = 2$, and $\Sigma(\mathbf{R}) = \aleph_1$. Motivated by the fact that in these two examples $s(G) < \aleph_0$ and $\Sigma(G) = \aleph_1$ one can ask:

**Question 5.4** ([16]) Let $G$ be a topological group such that $s(G) < \aleph_0$. Does it follow that $\Sigma(G) = \aleph_1$? For example, is there a topologically finitely generated topological group, without *infinite* suitable sets?

M. Tkachenko observed that the answer is "Yes" under the assumption of CH (there exists a countably compact monothetic group without converging sequences [46]).

As seen above, $\mathbf{T}$ (or any other compact monothetic group) is a counterexample for the above question if we replace $s$ and $\Sigma$ by $\sigma_c$ and $\Sigma_c$, resp.; *i.e.* $\mathbf{T}$ is a topologically finitely generated topological group, without infinite *closed* suitable sets. See also 6.5 *infra.*

The next theorem from [16] answers positively [15, Question 3.4.3 (b)] in the case of a *discrete* divisor subgroup $H$.

**Theorem 5.5** *Let $G$ be a topological group. Suppose that $H$ is a discrete normal subgroup of $G$ with $G/H \in \mathcal{S}$. Then $G \in \mathcal{S}$.*

A similar result to Theorem 5.3 (a) *supra* follows:

**Corollary 5.6** *Let $G$ be a topological group. Suppose that $H$ is a discrete normal subgroup of $G$ with $G/H \in \mathcal{S}$. Then $s(G) \leq |H \setminus \{1\}| \cdot s(G/H)$.* QED

# 6 Generators in Bohr topologies

Here we discuss topological groups equipped with the *Bohr topology.* Given a topological group $(G, \tau)$, consider the weakest group topology $\tau^+$ on $G$ which makes all $\tau$-continuous homomorphisms of $G$ to compact groups $\tau^+$-continuous. The new topology $\tau^+$ is called the *Bohr topology* on $G$. Clearly, $\tau^+$ is weaker that $\tau$ and the group $G^+ = (G, \tau^+)$ is precompact. However, the topology $\tau^+$ need not be Hausdorff. The group $G$ is said to be *maximally almost periodic* (MAP) when $G^+$ Hausdorff.

## 6.1 Abelian groups

All locally compact abelian groups are MAP. Furthermore, the continuous homomorphisms $G \longrightarrow \mathbf{T}$ of a locally compact abelian group $G$ separate the elements of $G$. It is also known that for an abelian group $(G, \tau)$, the Bohr topology $\tau^+$ on $G$ is the weakest one which makes the $\tau$-continuous homomorphisms to the circle group $\tau^+$-continuous.

There are many precompact topological groups which have no suitable set; one can even find an $\omega$-bounded minimal abelian group which is not in $\mathcal{S}$ (see Example 3.5). On the other hand, every locally compact group has a suitable set by Theorem 4.1. The following result (proved independently in [15, 48] shows that the functor $^+$ assigning to a group $G$ its modification $G^+$ preserves the latter property of locally compact abelian groups.

**Theorem 6.1** *$G^+$ has a suitable set for every locally compact abelian group $G$.*

If a group $G$ is discrete, we follow van Douwen and write $G^\#$ instead of $G^+$. One can prove that $G^\#$ has a closed generating suitable set for every discrete abelian group, i. e., $G^\# \in \mathcal{S}_g$ ([15, Theorem 5.7]).

Theorem 6.1 can be generalized as follows. The functor $^+$ preserves arbitrary products of locally compact Abelian group, that is, $(\prod_{i \in I} G_i)^+ \cong \prod_{i \in I} G_i^+$ for every family $\{G_i : i \in I\}$ of locally compact abelian groups (see [15, Theorem 5.1 (d)]). This fact along with Theorem 6.1 and the stability of $\mathcal{S}$ under products gives:

**Theorem 6.2** ([15, Corollary 5.9]) *Let $G$ be a cartesian product of locally compact abelian groups. Then $G^+ \in \mathcal{S}$.*

Let us mention here that the question whether every abelian topological group satisfying Pontryagin duality admits a suitable set ([15, Question 5.10 (b)], see also [47, Problem 5.4 (b)]) has a negative answer. This question was motivated by the above corollary and the fact that LCA groups as well as all their products satisfy Pontryagin duality.

**Example 6.3** ([16]) Let $X$ be a compact zero-dimensional space. Then the free abelian topological group $A(X)$ satisfies Pontryagin duality according to a theorem of Pestov [39] (see also [24] for a generalization). On the other hand, for the compact zero-dimensional space $X = \beta D$, where $D$ is a discrete space of cardinality $\aleph_1$, its group $A(X)$ has no suitable sets [4, Corollary 3.10]. QED

The results from §5 about lifting of closed suitable sets can be applied to answer a question from [15, Question 5.10(a)] (see also [47, Problem 5.4 (a)]).

**Theorem 6.4** ([16]) *Let $G$ be a locally compact Abelian group. Then $G \in \mathcal{S}_c$ iff $G^+ \in \mathcal{S}_c$.*

**Remark 6.5** [15, Proposition 2.8] implies that for a locally compact non-compact group $G$, $G \in \mathcal{S}_c \iff d(G) < b(G)$. Note that according to the inequality $b(G) \leq d(G)^+$ in the general case Theorem 4.9), this yields $G \in \mathcal{S}_c$ iff $b(G) = d(G)^+$.

## 6.2 Non-abelian groups

Here we mention another application of lifting closed suitable sets to the Bohr topology. A topological group $G$ is said to be *Moore* if any continuous unitary irreducible representation is finite dimensional. We refer the reader to §22 of [26] for an explanation of the terminology. The class [Moore] of *locally compact* Moore groups contains all locally compact Abelian and all compact groups, and it is closed under the operations of forming closed subgroups, Hausdorff quotients, and finite products and extensions (Roberston [42]). It is true that [Moore] $\subset$ [MAP], where [MAP] denotes the class of *locally compact* MAP groups. If $G$ is a [MAP] group such that the closure $G'$ of the *commutant* subgroup of $G$ is compact, then $G$ is called a *Takahashi* group. The class of locally compact Takahashi groups is denoted by [Tak]. As in 6.6, if $G$ is a [Tak] group, then we denote by $G'$ the closure of the commutator subgroup of $G$. Then $G'$ is compact, and $G/G'$ is a locally compact Abelian group, so that [Tak] $\subseteq$ [Moore].

**Theorem 6.6** ([16]) *Let $G$ be a* [Moore] *group. Then $G^+ \in \mathcal{S}$.*

The above answers positively a conjecture in [48, Remark 26.5].

**Example 6.7** (a) Another source of [MAP] groups contained in $\mathcal{S}$ is given by the class of the so called *van der Waerden* groups. A compact group is said to be a vdW group, if every algebraic homomorphism into a compact group is continuous. It follows that a compact group $G \in$ vdW $\iff$ $G_d^+ = G$, where $G_d$ denotes the underlying group of $G$ equipped with its discrete topology. By Remus and Trigos-Arrieta [43, Corollary 1], and the paragraph following Question 1 in (loc. cit.), we have that an infinite vdW group equipped with its discrete topology cannot be Moore. However since $G_d^+ = G$ is compact, (discretized vdW)$^+ \subset \mathcal{S}$.

(b) Consider the discrete group $G = \oplus_{n<\omega} \mathcal{S}_3$, where $\mathcal{S}_3$ is the symmetric group. Being countable, $G$ is not a vdW group. Moreover, in page 207 of Heyer [25] it is shown that $G$ is not Moore. Since $G$ is countable, $G^+ \in \mathcal{S}_c$ follows from [4, Theorem 2.2].

**Remark 6.8** • The class [Moore] is disjoint from the class of *infinite* discrete vdW groups. As shown above, when equipped with their Bohr topologies, both classes are contained in $\mathcal{S}$. The example in (b) is contained in neither one of the above, yet belongs to $\mathcal{S}$. Hence, a natural line of research is to investigate the relation of [MAP] groups equipped with their Bohr topology, and the class $\mathcal{S}$.

- Notice that the (Abelian) free topological group on any space $X$ is MAP, and it is locally compact if and only if $X$ is discrete (Dudley [18]). Thus, if we drop the requirement on the groups to be locally compact, then 6.3 is an example of a MAP group such that $G^+ \notin \mathcal{S}$. For another (trivial) example, consider any *totally bounded* group that do not belong to $\mathcal{S}$ ([4], [14], [15], and [48]).

Here is a version of 6.4 for [Moore].

**Theorem 6.9** ([16]) *Let $G$ be a* [Tak] *group. Then $G \in \mathcal{S}_c \iff G^+ \in \mathcal{S}_c$.*

**Theorem 6.10** ([16]) *Let $G$ be a* [Moore] *group. Then $G \in \mathcal{S}_c \iff G^+ \in \mathcal{S}_c$.*

**Remark 6.11** Let $G \in$ [Tak]. Then $G \in \mathcal{S}_c \Longrightarrow G/G' \in \mathcal{S}_c$ and $G^+ \in \mathcal{S}_c \Longrightarrow G^+/G' \in \mathcal{S}_c$ by [15, 3.4.2], $G/G' \in \mathcal{S}_c \iff G^+/G' \in \mathcal{S}_c$ by 6.4, and $G^+ \in \mathcal{S}_c \iff G \in \mathcal{S}_c$ by Theorem 6.9. If $d(G') < \Sigma_c(G/G')$, then Proposition 5.1 would imply $G/G' \in \mathcal{S}_c \Longrightarrow G \in \mathcal{S}_c$ and $G^+/G' \in \mathcal{S}_c \Longrightarrow G^+ \in \mathcal{S}_c$, hence in this case, all four properties of $G$ are equivalent. The condition $d(G') < \Sigma_c(G/G')$ is necessary: Let $L$ be any *simple* compact *Lie* group, and take $G := \mathbf{Z} \times L^{\mathbf{c}^+}$. Then $G' = \{0\} \times L^{\mathbf{c}^+}$, hence $G \in$ [Tak]. By 6.4 $\{G/G',\, G^+/G'\} \subset \mathcal{S}_c$, yet $\{G,\, G^+\} \cap \mathcal{S}_c = \emptyset$ by [15] (2.7 (a)).

# 7 Totally suitable sets

Here we report result from [16] on a class contained in $\mathcal{S}$: call a suitable set $S$ in a topological group $G$ *totally suitable* if it has the additional property that $\langle S \rangle$ is totally dense in $G$. Let $\mathcal{S}_t$ denote the class of all groups having a totally suitable set. Obviously $\mathcal{S}_g \subseteq \mathcal{S}_t \subseteq \mathcal{S}$. We start the next subsection with two examples showing that both inclusions and are proper even in the case of compact abelian groups.

## 7.1 Totally suitable sets in compact abelian groups

**Example 7.1** (a) The circle group $\mathbf{T}$ has a totally suitable set. In fact, let $S$ be the set of all points of the form $x_n = 1/n!$ in $\mathbf{T}$. Since $x_n \longrightarrow 0$, clearly $S$ is a suitable set. Since $S$ generates $\mathbf{Q}/\mathbf{Z}$, which is totally dense in $\mathbf{T}$, this proves $\mathbf{T} \in \mathcal{S}_t$. We leave to the reader the extension of this argument to $G = \mathbf{T}^n$. Another (easy) example is that $\mathbf{Z}_p \in \mathcal{S}_t$ [11, Theorem 3.5.3]. More generally, *every* suitable set in $\mathbf{Z}_p$ is totally suitable (loc. cit). See §7.3 *infra.*

(b) $G = \mathbf{Z}_p^2 \notin \mathcal{S}_t$, consequently no compact abelian group containing a copy of $\mathbf{Z}_p^2$ can be in $\mathcal{S}_t$. Indeed, every totally dense subgroup of $G$ has cardinality $\mathbf{c}$ [41], while every suitable set of $G$ must be countable. Another easy example is $G = \mathbf{Z}(p)^\omega \notin \mathcal{S}_t$, where $\mathbf{Z}(p)$ is the cyclic group of order $p$ (again every suitable set of $G$ must be countable while no proper subgroup of $G$ can be totally dense). From the "connected end" one can show that no infinite power $G = \mathbf{T}^\sigma \in \mathcal{S}_t$. In fact, every totally dense subgroup of $G$ has cardinality $2^\sigma = |G|$ ([11]), while suitable sets have cardinality $\leq \sigma$.

Putting together a) and b) we see that $\mathcal{S}_t$ is not closed even under taking Cartesian squares. On the other hand, taking into account that surjective continuous homomorphisms preserve total density of subgroups, we extend [15, Theorem 3.4.1] to $\mathcal{S}_t$:

**Proposition 7.2** *$\mathcal{S}_t$ is closed under taking closed continuous homomorphic images.*

By means of this proposition and the fact that a group in $\mathcal{S}_t$ has no copies of $\mathbf{Z}_p^2$ for any prime $p$ we prove:

**Theorem 7.3** *Let $G \in \mathcal{S}_t$ be a compact abelian group. Then $G$ is finite-dimensional and $G/c(G) \cong \prod_p G_p$ where each group $G_p$ is either a finite $p$-group or a product $\mathbf{Z}_p \times F_p$ where $F_p$ is a finite $p$-group. In particular, $G$ is metrizable.*

The above theorem yields that for any prime $p$ the power $\mathbf{Z}(p)^\alpha \in \mathcal{S}_t$ iff $\alpha$ is finite. The conclusion of Theorem 7.3 enables us to claim that compact abelian groups $G \in \mathcal{S}_t$ are generated by a converging sequence $s_n \longrightarrow 0$. In particular, this means that $G$ admits a countable totally dense subgroup. The class $\mathcal{K}$ of compact abelian groups $G$ with this property is described in [11, p. 141] (where it is denoted by $\mathcal{K}'$): a compact abelian group $G$ belongs $\mathcal{K}$ iff for every prime $p$ the group $G$ has no copies of $\mathbf{Z}_p^2$ and $\mathbf{Z}(p)^\omega$. The inclusion

$$\mathcal{S}_t \cap \{\text{compact abelian groups}\} \subseteq \mathcal{K}$$

helps to describe the compact abelian groups in $\mathcal{S}_t$. For $G \in \mathcal{K}$ define the *support* of $G$ as $\pi(G) := \{p \in \mathbf{P} : \mathbf{Z}_p \text{ embeds in } G\}$. In these terms one has:

**Theorem 7.4** ([16]) *Let $G \in \mathcal{K}$ be a (compact) connected group with $|\pi(G)| < \infty$. Then $G \in \mathcal{S}_t$.*

It is not clear whether Theorem 7.4 generalizes to *all* connected groups in $\mathcal{K}$. It seems that *all connected groups of $\mathcal{C}$ are in $\mathcal{S}_t$* without any restriction on $|\pi(G)|$, but no proof is available even in the particular case of $G = \mathbf{K}$. Note that $\mathbf{K} \in \mathcal{S}_t$ would imply that $\mathcal{S}_t$ contains all one-dimensional compact connected abelian groups by virtue of Proposition 7.2.

We prove below that for $G \in \mathcal{K}$ with $c(G) \in \mathcal{S}_t$ one has $G \in \mathcal{S}_t$.

The class of groups $G \in \mathcal{K}$ with $\pi(G) = \emptyset$, known as *exotic tori* (cf. [10, 11]), presents a good approximation of the usual tori $\mathbf{T}^n$. These are the compact abelian groups $G$ such that the torsion subgroup $t(G)$ is totally dense in $G$ (or, equivalently, $t(G)$ is a dense and minimal subgroup of $G$, cf. [10]). A compact abelian group is an exotic torus iff it admits subgroups isomorphic to the $p$-adic integers $\mathbf{Z}_p$ for no prime $p$ ([10, 11]).

The class of connected exotic tori is quite large – there are $\mathbf{c}$ many pairwise non-homotopically-equivalent connected one-dimensional exotic tori ([10, 11]). In general, an exotic torus $G$ need not be in $\mathcal{K}$. One has $G \in \mathcal{S}_t$ iff $t(G)$ is countable. This surely occurs when the exotic torus $G$ is connected, then $t(G) \cong (\mathbf{Q}/\mathbf{Z})^n$ where $n = \dim G$ is finite ([10]).

The next corollary obviously follows from Theorem 7.4 and generalizes our observation $\mathbf{T}^n \in \mathcal{S}_t$ in Example 7.1(b). It shows that our main conjecture is true for connected exotic tori.

**Corollary 7.5** *Every connected exotic torus is in $\mathcal{S}_t$.*

## 7.2 Factorization in $\mathcal{S}_t$

It is easy to see that no infinite powers of compact groups belong to $\mathcal{S}_t$. On the other hand, Theorem 7.3 yields that if a product $\prod_i G_i \in \mathcal{S}_t$, then only countably many groups $G_i$ can be non-trivial. Moreover, $\mathcal{S}_t$ is closed under those finite products that do not lead out of the class $\mathcal{K}$:

**Theorem 7.6** *For $G_1, G_2 \in \mathcal{S}_t$ one has $G_1 \times G_2 \in \mathcal{S}_t$ iff $\pi(G_1) \cap \pi(G_2) = \emptyset$.*

The above theorem gives:

**Corollary 7.7** *Let $G$ be a compact abelian group. Then the following conditions are equivalent for $G$:*

1. *$G \times H \in \mathcal{S}_t$ for every $H \in \mathcal{S}_t$;*
2. *$G^2 \in \mathcal{S}_t$;*
3. *$G$ is an exotic torus and $G \in \mathcal{S}_t$.*

Indeed, since $\mathcal{S}_t$ is closed under quotients of compact groups by Proposition 7.2, for a compact abelian group $G \in \mathcal{S}_t$ one has $G^2 \in \mathcal{S}_t$ iff $G$ is an exotic torus.

The following useful formula is available for every compact abelian group $G$ and closed subgroup $G_1$ of $G$:

$$\pi(G) = \pi(G_1) \cup \pi(G/G_1), \tag{1}$$

therefore, $(G_1 \in \mathcal{K}) \wedge (G/G_1 \in \mathcal{K})$ imply $G \in \mathcal{K}$ if and only if $\pi(G_1) \cap \pi(G/G_1) = \emptyset$. Every compact abelian group $G$ can be written as $G = G_z \times G_0$, where $G_z = \prod_{p \in \pi(G) \setminus \pi(c(G))} \mathbf{Z}_p$, $G_0 \supseteq c(G)$ and $\pi(G_0) = \pi(c(G_0)) = \pi(c(G))$. Since every group of the type $G_z$ is in $\mathcal{S}_t$ and $\pi(G_z) \cap \pi(G_0) = \emptyset$, Theorem 7.6 yields $G \in \mathcal{S}_t$ iff $G_0 \in \mathcal{S}_t$. This is why, from now on we shall assume that $\pi(G) = \pi(c(G))$.

**Question 7.8** Does Theorem 7.6 hold for *extensions* instead of direct products? In other words, if $G$ has a closed normal subgroup $G_1 \in \mathcal{S}_t$ such that $G_2 = G/G_1 \in \mathcal{S}_t$ is it true that also $G \in \mathcal{S}_t$ (note that according to the above remark $\pi(G_1) \cap \pi(G/G_1) = \emptyset$ is a necessary condition for this)?

In view of Theorem 7.4 the answer is "Yes" for connected groups with finite support. The answer is "Yes" also in the case when $G_1 = c(G)$:

**Theorem 7.9** *Let $G \in \mathcal{C}$ be a compact group with $c(G) \in \mathcal{S}_t$. Then $G \in \mathcal{S}_t$.*

The next corollary strengthens Theorem 7.4.

**Corollary 7.10** *If $\pi(c(G))$ is finite for some group $G \in \mathcal{K}$ then $G \in \mathcal{S}_t$.*

These examples provide a large class of groups in $\mathcal{S}_t$.

## 7.3 Compact abelian groups where all suitable sets are totally suitable

Note that every sequence converging to 0 in $\mathbf{T}$ is a suitable set, hence not every suitable set is totally suitable. On the contrary, in a group like $G^{\#}$ every suitable set is also totally suitable. This suggests the following

**Problem 7.11** *Characterize the groups in which* every *suitable set is also totally suitable.*

Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ of groups in which every suitable set is also totally suitable. Obviously, $\mathbf{T} \notin \mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{S}_t$. Moreover, $\mathbf{Z}_p \in \mathcal{T}$ for every prime $p$ (7.1 (b)), while $\mathbf{Z}_p \times \mathbf{Z}_q \notin \mathcal{T}$ for all pairs of primes $p, q$ (because the singleton $(1,1) \in \mathbf{Z}_p \times \mathbf{Z}_q$ forms a suitable set in case $p \neq q$ that is not totally suitable).

**Example 7.12** • For every infinite subset $A \subseteq \mathbf{P}$, $G_A := \prod_{p \in A} \mathbf{Z}(p) \in \mathcal{S}_t \setminus \mathcal{T}$. More generally, no finite power of $G_A$ belongs to $\mathcal{T}$.

- For every $p \in \mathbf{P}$ and every non-trivial finite abelian group $F$, $G = \mathbf{Z}_p \times F \notin \mathcal{T}$. In fact, let $F = \{f_1, \ldots, f_n\}$ and let $\xi_1, \ldots, \xi_n \in \mathbf{Z}_p \setminus p\mathbf{Z}_p$ be independent. Then $S := \{(\xi_1, f_1), \ldots, (\xi_n, f_n)\}$ is a suitable set of $G$ that is not totally suitable, since $\langle S \rangle \not\supseteq t(G) = \{0\} \times F$. (The density of $H = \langle S \rangle$ is ensured by the fact that $(\xi_i, 0) \in H$ for some $i$. Hence $cl(H) \supseteq \mathbf{Z}_p \times \{0\}$. To conclude observe that the second projection $G \longrightarrow F$ sends $H$ *onto* $F$.)
- For every infinite subset $A \subseteq \mathbf{P}$, $G_A := \prod_{p \in A} \mathbf{Z}_p \notin \mathcal{T}$ since $G_A$ is monothetic, while the only monothetic compact group with a totally dense infinite *cyclic* subgroup is $\mathbf{Z}_p$.
- Now assume that $G \in \mathcal{T}$ is connected. Then by the inclusion $\mathcal{T} \subseteq \mathcal{S}_t$, and by Theorem 7.3, $G$ is metrizable. Hence $G$ is *monothetic*. By the final remark in the previous item, $G$ cannot be in $\mathcal{T}$ – a contradiction.

It turns out that these examples pretty much characterize all compact abelian groups in $\mathcal{T}$ and yield the following surprising characterization of the $p$-adic integers:

**Theorem 7.13** *Let $G$ be an infinite compact abelian group. Then all suitable sets of $G$ are totally suitable iff $G \cong \mathbf{Z}_p$ for some prime $p$.*

*Proof.* We prove first that $\mathcal{T}$ is closed under taking quotients. In fact, let $G \in \mathcal{T}$ and let $f : G \longrightarrow N$ be a continuous surjective homomorphism. Take a suitable set $S$ of $N$. Then $S$ is either a finite set or a converging sequence since our groups are compact metrizable (4.2). Therefore, we can find a set $S_1$ in $G$ with similar properties with $f(S_1) = S$. Now $\ker f$ is a compact metrizable group. By Theorem 3.1 there exists a convergent sequence $S_2$ generating $\ker f$. Then $S_0 = S_1 \cup S_2$ is a convergent sequence generating $G$. Now $G \in \mathcal{T}$ yields that the subgroup $\langle S_0 \rangle$ of $G$ is totally dense. Hence the subgroup $\langle S \rangle$ of $N$ (as a homomorphic image of $\langle S_0 \rangle$) is totally dense as well. Thus $N \in \mathcal{T}$. Since $\mathbf{T} \notin \mathcal{T}$, this proves that every group in $\mathcal{T}$ is totally disconnected. The above example shows that the unique totally disconnected compact groups in $\mathcal{T}$ are the groups $\mathbf{Z}_p$ for some prime $p$. QED

## 7.4 When the group is not compact abelian

We discuss first totally suitable sets in some non-compact abelian groups that are still close to being compact.

As far as LCA groups are concerned, it is easy to see that $\mathbf{R}^n \in \mathcal{S}_t$ iff $n = 0$. More generally, a metrizable separable LCA group in $\mathcal{S}_t$ cannot contain non-trivial vector subgroups. One can prove that *if $G \in LCA$ has no vector subgroups and for some compact open subgroup $K$ of $G$ the quotient $G/K$ is not torsion, then $G \in \mathcal{S}_t$ iff $G \in \mathcal{S}_g$.* Thus one is left with LCA groups $G$ that are covered by compact open subgroups $K_\alpha$ such that $G/K_\alpha$ is torsion. For every $\alpha$ the set $S_\alpha := S \cap K_\alpha$ is a supersequence converging to 0 in $K_\alpha$, so that $|S_\alpha| \leq w(K_\alpha) = w(K)$, consequently, $|S| \leq w(K)$ as the open set $K_\alpha$ contains all but finitely many elements of $S$. Then $|H| \leq w(K)$ and $H \cap K$ is totally dense in $K$. Then one can prove that either $w(K) = \omega$ (i.e., $G$ is metrizable), or $w(K)$ is a strong limit cardinal[1] of countable cofinality.

One can generalize the statement "$G$ is metrizable" in Theorem 7.3 in the more general case of a countably compact minimal abelian group (every compact group has these two properties) as follows.

**Theorem 7.14** *Let $G \in \mathcal{S}_t$ be a countably compact minimal abelian group. Then $G$ is metrizable, hence compact.*

As far as totally suitable sets in non-abelian groups are concerned we note that $G \in \mathcal{S}$ iff $G \in \mathcal{S}_t$ when $G$ is topologically simple (has no proper closed normal subgroups). Hence the infinite symmetric group $S(X)$, as well as all simple compact connected Lie groups are in $\mathcal{S}_t$. Actually, all products of such groups are in $\mathcal{S}_t$. In contrast with the abelian case, now many non-compact separable metrizable LC groups (as $SL_n(\mathbf{R})$ etc.) belong to $\mathcal{S}_t$ (note that such groups contain copies of $\mathbf{R}$).

# 8 Compact generation of topological groups

When the set of generators $S$ is compact we speak of *compactly generated* group. Note that $S^{-1}$, as well as all powers $S^n$, are compact along with the set $S$. Hence, with $S_0 := S \cup S^{-1}$, one can see that $G = S_0 \cup S_0^2 \cup \ldots \cup S_0^n \ldots$. Hence $G$ is $\sigma$-compact. This proves the implication

$$\text{compactly generated} \Longrightarrow \sigma\text{-compact}. \tag{6}$$

In order to better realize the implication (6) let us recall that a LCA group $G$ is:

(a) compactly generated iff $G \cong \mathbf{R}^n \times \mathbf{Z}^m \times K$, where $n, m \in \mathbf{N}$ and $K$ is a compact abelian group.

(b) $\sigma$-compact iff $G \cong \mathbf{R}^n \times H$, where $H$ contains an open compact subgroup $K$ with $|H/K| \leq \omega$.

Let us note that (a) and (b) are quite different even in the case when $G$ is discrete – then $n = 0$ in both cases and (a) means that $G$ is finitely generated, while (b) entails only that $G = H$ is countable.

The precise relation in (6) was determined recently by Fujita and Shakhmatov [22] in the case of metric groups.

They observed that a compactly generated group $G$ must necessarily satisfy the following condition

$$\text{for every open subgroup } H \leq G \text{ there exists a finite } F \subseteq G \text{ with } G = \langle F \cup H \rangle. \tag{FS}$$

In other words, (FS) says that $G$ is finitely generated modulo every open subgroup. In the case of an abelian group $G$ this means precisely that every discrete quotient $G/H$ of $G$ is finitely generated. In the following example we collect several sufficient conditions that imply (FS).

**Example 8.1** (FS) follows from each of the following conditions:

- $G$ is a (dense subgroup of a) connected group;
- $G$ has no proper open subgroups;

[1] i.e., $\lambda < w(G)$ always yields $2^\lambda < w(G)$.

- $G$ is precompact.

Clearly, (a) implies (b), but in general (b) does not imply (a) even for totally disconnected complete metric groups (Stevens [45]).

**Theorem 8.2** (Fujita–Shakhmatov [22]) *A metric group $G$ is compactly generated iff $G$ is $\sigma$-compact and satisfies (FS).*

Here *metrizable* can be replaced by almost metrizable, but it cannot be removed completely as the following example shows:

**Example 8.3** The group $G = \mathbf{Q}^{\#}$ is countable and precompact, hence satisfies (FS). Nevertheless, $G$ is not compactly generated. In fact, by Glicksberg's theorem [23] the only compact subsets of $G^{\#}$ are the finite ones. Since $G$ is not finitely generated, it cannot be compactly generated either.

The main tool in proving Theorem 8.2 is the following technical results that illustrates the power of the condition (FS) in the case of metrizable groups:

**Lemma 8.4** ([22, Theorem 7]) *Let $G$ be a metric group that satisfies (FS). Then for every countable subset $D \subseteq G$ there exists a convergent sequence $S$ with $D \subseteq \langle S \rangle$.*

## 8.1 Topological compact generation

Now we require that the group $G$ has a compact set $S$ of *topological* generators and consider the class $\mathcal{C}$ of all groups $G$ with this property. This gives a natural generalization of the class $\mathcal{S}eq$. Now (6) may fail. Let is recall, that according to [22] for a $\sigma$-compact group "compactly generated" is equivalent to "topologically compactly generated", i.e., "compactly generated" is always equivalent to "$\sigma$-compact and topologically compactly generated". Hence the class $\mathcal{C}$ contains $\mathcal{S}eq$ but need not contain all $\sigma$-compact groups.

What is important here is that the necessary condition (FS) remains valid for topologically compactly generated groups. Indeed, it is easy to see that $G \in \mathcal{S}eq$ implies $(FS)$ as well: if $S$ is a supersequence that topologically generates $G$ then for every open subgroup $H$ of $G$ the complement $F = S \setminus S$ is finite as $S \longrightarrow 1$ and $H$ is a nbd of 1. Then the subgroup $\langle F \cup H \rangle$ is both dense (contains the dense subgroup $\langle S \rangle$) and open (contains the open subgroup $H$). Thus $\langle F \cup H \rangle$ must be also closed and coincide with $G$. The necessary condition (FS) remains valid for topologically compactly generated groups too. Indeed, this follows from the next lemma or the following direct argument. If $K$ is a compact set that topologically generates $G$ then for every open subgroup $H$ of $G$ there exist finitely many translate $\{aH : a \in F\}$ that cover $K$. Then the subgroup $\langle F \cup H \rangle$ is both dense (contains the dense subgroup $\langle K \rangle$) and open (contains the open subgroup $H$). Thus $\langle F \cup H \rangle$ must be also closed and coincide with $G$.

**Lemma 8.5** *Let $G_1$ be a dense subgroup of $G$. Then $G$ satisfies (FS) iff $G_1$ satisfies (FS).*

*Proof.* Assume $G$ satisfies (FS) and let $H$ be an open subgroup of $G_1$. Then its closure $\overline{H}$ in $G$ is open, so there exists a finite subset $F \subseteq G$ such that $F \cup \overline{H}$ generates $G$. For every $f \in F$ pick an element $g_f \in G_1 \cap f\overline{H}$ (it exists by the density of $G_1$) and let $F_1 := \{g_f : f \in F\}$. Then the finite set $F_1 \subseteq G_1$ has the property $G = \langle F_1 \cup \overline{H} \rangle$. We shall see that $A = \langle F_1 \cup H \rangle$ coincides with $G_1$. Indeed, $A$ is dense in $G = \langle F_1 \cup \overline{H} \rangle$ as every element of $\overline{H}$ is a limit if a net of elements of $H$. Thus $A$ is dense in $G_1$ too. But $A$ contains an open subgroup of $G_1$. Hence $A$ is also closed in $G_1$. Thus $A = G_1$.

Now assume that $G_1 \in (FS)$. Let $H$ be now an open subgroup of $G$. Then there exists a finite $F \subseteq G_1$ such that $F$ generates $G_1$ along with $G_1 \cap H$. Now the subgroup $B$ of $G$ generated by $H$ and $F$ contains $G_1$, hence it is dense. On the other hand, it contains $H$ hence it is also closed. Thus $B = G$. Therefore $G \in (FS)$. QED

Since every compactly generated group satsifies (FS) and since every $G \in \mathcal{C}$ contains a dense compactly generated subgroup, the Lemma 8.5 gives:

**Corollary 8.6** *$G \in (FS)$ for every $G \in \mathcal{C}$.*

Lemma 8.4 gives :

**Lemma 8.7** *A separable metric group that satisfies (FS) is topologically generated by a convergent sequence.*

Then one can easily obtain the following:

**Theorem 8.8** *For a metrizable group $G$ the followig are equivalent:*

*(a) $G$ is separable and satisfies (FS);*

*(b) $G$ is topologically compactly generated;*

*(c) $G$ is topologically generated by a convergent sequence.*

Without "metrizable" (a) does not imply (b) (see Example 8.3). In general (b) does not imply $\sigma$-compact. However, it seems plausible that (c) and (b) remain equivalent if (b) is replaced by a stronger form:

**Conjecture 8.9** (Dikranjan-Shakhmatov) *$G$ is topologically generated by a compact metrizable set iff $G$ is topologically generated by a converging sequence.*

This conjecture is true when $G$ is generated by a compact connected metrizable set $S$ (consider $F(S)$ - the free abelian topological group and the dense homomorphism $F(S) \longrightarrow G$; it suffices to prove it for $F(S)$).

## 8.2 The $k$-generating rank

For $G \in \mathcal{C}$ set

$$k(G) := \min\{w(K) : K \subseteq G \text{ generating compact set of } G\}.$$

Note that convergent supersequences are compact, so that $\mathcal{S}eq \subseteq \mathcal{C}$ and $seq(G) \geq k(G)$ if the group $G$ is in $\mathcal{S}eq$. Since $d(K) \leq w(K)$ for every compact set $K$, we get

$$d(G) \leq k(G) \leq seq(G) \leq \psi(G)$$

in case $k(G)$ is infinite.

The invariant $k(G)$ appears for the first time implicitely in [40, Theorem] where the following theorem is proved:

**Theorem 8.10** ([40]) *Let $G$ be a locally compact abelian group. Then $k(H) = w(H)$ for every closed subgroup of $G$ iff $c(G)$ is metrizable.*

The next proposition gives properties similar to those of $\mathcal{S}eq$.

**Proposition 8.11** *The classe $\mathcal{C}$ is closed under taking:*

1. *direct products, $\sum$-products and $\sigma$-products.*

2. *continuous homomorphic images (in particular, quotients)*
3. *inner products: if $G = NH$ and both subgroups $N$ and $H$ of $G$ are in $\mathcal{C}$ then also $G \in \mathcal{C}$.*

In analogy with (1) one can prove that $k(G_1 \times G_1) = k(G_1)k(G_2)$.

Let us note that according to Theorem 8.8 the metrizable groups in $\mathcal{C}$ are separable. Hence if a product of metrizable groups belongs to $\mathcal{C}$, then these groups are necessarily separable.

**Lemma 8.12** *Let $G = \prod_{i\in I} K_i$ be a product of non-trivial precompact metrizable groups $K_i$. Then $G \in \mathcal{C}$ and $k(G)^\omega \geq w(G)$ unless $G$ is monothetic.*

*Proof.* Assume $G$ is not monothetic. $G \in \mathcal{C}$ as $K_i$ (being precompact) satisfies $(FS)$ for every $i \in I$. In case the group $G$ is metrizable, or more generally, $|I| \leq \mathbf{c}$, $w(G) \leq \mathbf{c} = k(G)^\omega$ as $G$ is not monothetic.

Now we assume that $|I| \geq \mathbf{c}$ Let us note first that without loss of generality each group $K_i$ can be assumed complete, hence compact. Moreover, as there are at most $\mathbf{c}$ many paiwise non-isomorphic compact metrizable groups $\{K_j\}_{j<\mathbf{c}}$, it is not restrictive to assume that *all* groups $K_i$ are isomorphic to a fixed one $H$. Indeed, if $G = \prod_{j<\mathbf{c}} H_j^{\alpha_j}$, then for every $j < \mathbf{c}$ we can get $k(G)^\omega \geq \alpha_j$ noting that the projection $G \longrightarrow H_j^{\alpha_j}$ gives $k(G) \geq k(H_j^{\alpha_j})$. Since also $k(G)^\omega \geq \mathbf{c}$, we get immediately $k(G)^\omega \geq w(G)$. Now, when $G = H^\alpha$ one can argue as in the proof of [28, Lemma 4.10] where $H$ is a compact Lie group and $S$ is a compact generating set of $G$ containing 1. For the sake of completeness we shall sketch briefly that proof in the case of a general compact metric group $H$. Let $S$ be an arbitrary compact generating set of $G$. Set $B = H^{C(S,H)}$ and note that $S$ can be considered in a natural way as a subgroup of $B$ via the evaluation maps $ev_s$, where $s \in S$ and $ev_s(f) = f(s)$ for every $f \in C(S,H)$. Note that there is a natural projection $\pi : B \longrightarrow G$ defined by $\pi(\varphi)(i) := \varphi(p_i|_Y)$ for $\varphi \in B$, i.e., $\pi(\varphi)(i)$ is simply the evaluation of the map $\varphi : C(S,H) \longrightarrow H$ at the function $p_i|_Y \in C(S,H)$. Note that the so defined $\pi$ is identical on $S$. Moreover, $\pi$ is a continuous homomorphism of compact groups. Since the image $S = \pi(S)$ generates a dense subrgoup of $G$, the homomorphism $\pi$ is also surjective as $\pi(B)$ is a closed subgroup of $G$ that contains a dense subgroup. Hence we have $|C(S,H)| \geq w(B) \geq w(G) = \alpha$. On the other hand, one can prove that $|C(S,H)| \leq w(S)^\omega$ using the fact that $H$ is a subspace of $I^\omega$ and $|C(S,I)| \leq w(S)^\omega$. A proof of this probably well known fact can be found for example in [27, Proposition 1.4.1]). QED

In this way we get a second proof of (4) in the case of precompact groups.

**Remark 8.13** It seems plausible that a more general version of the lemma can be proved. Let us note that the groups $G$ involved in this section have $b(G) \leq \omega_1$ (*$\omega$-totally bounded* in the sense of Guran, it can be proved that they are subgroups of products of separable metrizable groups). This suggests the question: *do $\omega$-totally bounded groups with (FS) belong to $\mathcal{C}$ ?* Note that all works for such groups as well in case they are *dense* subgroups of products of separable metrizable groups. But also a second point is important: we used in the above proof that fact that $\pi$ does not increase the weight – this works for precompact groups.

Applying this lemma one can easily prove that the inequality $k(G)^\omega \geq w(G)$ holds for every group $G$ that adimts a dense continuous homomorphism into a product of $w(G)$ many non-trivial precompact metrizable groups.

**Theorem 8.14** *The inequality $k(G)^\omega \geq w(G) \geq seq(G) \geq k(G)$ is satisfied by a topological group $G \in \mathcal{C}$ that has one of the following properties:*

- *G is compact connected;*
- *G is precompact abelian.*

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