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# On the theory of commuting symbolic dynamical systems and its generalization

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We consider dynamical systems $(X, f)$, where $X$ is a compact metric space and $f : X \to X$ is an onto continuous map. Two dynamical systems $(X, f)$ and $(X, g)$ are said to *commute* if $fg = gf$.

Let $(X, f)$ and $(Y, g)$ be two dynamical systems. An onto continuous map $\phi : X \to Y$ with $\phi f = g\phi$, is called a *factor map* of $(X, f)$ onto $(Y, g)$, and a homeomorphism $\phi : X \to Y$ with $\phi f = g\phi$ is called a *conjugacy* of $(X, f)$ onto $(Y, g)$. If two dynamical systems $(X, f)$ and $(Y, g)$ have a conjugacy between them, then we say that they are *conjugate* and write $(X, f) \simeq (Y, g)$.

A continuous map $\varphi : X \to X$ with $\phi f = f\phi$ is called an *endomorphism* of $(X, f)$ and a self-conjugacy $\varphi : (X, f) \to (X, f)$ is called an *automorphism* of $(X, f)$. Hence if two dynamical systems $(X, f)$ and $(X, g)$ commute, then $g$ is an onto endomorphism of $(X, f)$, and if $g$ is 1-1 in addition, then $g$ is an automorphism of $(X, f)$.

Let $X$ be a compact metric space endowed with metric $d_X$. Let $f : X \to X$ be a continuous map. A bisequence $(x_j)_{j\in\mathbf{Z}}$ of points $x_j$ of $X$ is called an *orbit* of $f$ if $f(x_j) = x_{j+1}$ for all $j \in \mathbf{Z}$. For $\epsilon > 0$, we say that $f$ is $\epsilon$-*expansive* or call $\epsilon$ an *expansive constant* for $f$ if for any orbits $(x_j)_{j\in\mathbf{Z}}$ and $(x'_j)_{j\in\mathbf{Z}}$ of $f$, if $d_X(x_j, x'_j) \leq \epsilon$ for all $j \in \mathbf{Z}$, then $(x_j)_{j\in\mathbf{Z}} = (x'_j)_{j\in\mathbf{Z}}$. We say that $f$ is *positively* $\epsilon$-*expansive* if for any $x, x' \in X$ if $d_X(f^j(x), f^j(x')) \leq \epsilon$ for all $j \geq 0$, then $x = x'$. We say that $f$ is *expansive* if $f$ is $\epsilon$-expansive for some $\epsilon > 0$, and we also say that $f$ is *positively expansive* if $f$ is positively $\epsilon$-expansive for some $\epsilon > 0$.

For $\delta > 0$, a bisequence $(y_j)_{j\in\mathbf{Z}}$ of points $y_j$ of $X$ is called a $\delta$-*pseudo-orbit* of $f$ if $d_X(f(y_j), y_{j+1}) \leq \delta$ for all $j \in \mathbf{Z}$. For $\epsilon > 0$, an orbit $(x_j)_{j\in\mathbf{Z}}$ of $f$ is said to $\epsilon$-*trace* $(y_j)_{j\in\mathbf{Z}}$ if $d(x_j, y_j) \leq \epsilon$ for all $j \in \mathbf{Z}$. We say that $f$ has the *pseudo-orbit tracing property* (POTP) if for any $\epsilon > 0$, there is $\delta > 0$ such that any $\delta$-pseudo-orbit of $f$ is $\epsilon$-traced by some orbit of $f$.

An onto continuous map $f : X \to X$ of a metric space is said to be *topologically transitive* [topologically mixing] if for any nonempty open sets $U, V \subset X$, there is $n \in \mathbf{N}$ such that $f^n(U) \cap V \neq \emptyset$ [there is $N \in \mathbf{N}$ such that for all $n \geq N$ $f^n(U) \cap V \neq \emptyset$].

Let $A$ be an alphabet (i.e., a finite set of symbols). Let $X_A = A^{\mathbf{Z}} = \{(a_j)_{j\in\mathbf{Z}} \mid a_j \in A\}$ (with the product topology of the discrete topology on $A$). We define a metric $d$ on $X_A$ as follows: for $x = (a_j)_{j\in\mathbf{Z}}$ and $y = (b_j)_{j\in\mathbf{Z}}$ in $X_A$, $d(x,y) = 0$ if $x = y$, and $d(x,y) = 1/(1+k)$ if $x \neq y$, where $k = \min\{|j| \mid j \in \mathbf{Z}, a_j \neq b_j\}$. The homeomorphism $\sigma_A : X_A \to X_A$ defined by

$$\sigma_A((a_j)_{j\in\mathbf{Z}}) = (a_{j+1})_{j\in\mathbf{Z}}, \quad (a_j)_{j\in\mathbf{Z}} \in X_A,$$

is called the *shift map.* The dynamical system $(X_A, \sigma_A)$ is called the *full shift* over $A$. Let $X$ be a closed subset of $X_A$ with $\sigma_A(X) = X$. Let $\sigma = \sigma_A|X$. Then we have a dynamical system $(X, \sigma)$, which is called a *subshift* over $A$.

Let $\tilde{X}_A = A^{\mathbf{N}}$. We define a metric $\tilde{d}$ on $\tilde{X}_A$ as follows: for $\tilde{x} = (a_j)_{j\in\mathbf{N}}$ and $\tilde{y} = (b_j)_{j\in\mathbf{N}}$ in $\tilde{X}_A$, $\tilde{d}(\tilde{x},\tilde{y}) = 0$ if $\tilde{x} = \tilde{y}$, and $\tilde{d}(\tilde{x},\tilde{y}) = 1/k$ if $\tilde{x} \neq \tilde{y}$, where $k = \min\{j \in \mathbf{N} \mid a_j \neq b_j\}$. The continuous map $\tilde{\sigma}_A : \tilde{X}_A \to \tilde{X}_A$ defined by

$$\tilde{\sigma}((a_j)_{j\in\mathbf{N}}) = (a_{j+1})_{j\in\mathbf{N}}, \quad (a_j)_{j\in\mathbf{N}} \in \tilde{X}_A,$$

is called the *one-sided shift map.* The dynamical system $(\tilde{X}_A, \tilde{\sigma}_A)$ is called the *one-sided full shift* over $A$. Let $\tilde{X}$ be a closed subset of $\tilde{X}_A$ with $\tilde{\sigma}_A(\tilde{X}) = \tilde{X}$. Let $\tilde{\sigma} = \tilde{\sigma}_A|X$. Then we have a dynamical system $(\tilde{X}, \tilde{\sigma})$ which is called an (onto) *one-sided subshift* over $A$.

**Theorem 1** (Hedlund [H], Reddy [R]). Let $X$ be a 0-dimensional compact, metric space. Then the following statements are valid.

(1) If $\varphi : X \to X$ is an expansive homeomorphism, then $(X, \varphi)$ is conjugate to a subshift.

(2) If $\varphi : X \to X$ is a positively expansive onto continuous map, then $(X, \varphi)$ is conjugate to a one-sided subshift.

Let $(X, \sigma)$ be a subshift over an alphabet $A$. Let $n \in \mathbf{N}$. Let $L_n(X)$ denote the set of all $n$-blocks or all words of length $n$ that appear on some bisequence in $X$, that is,

$$L_n(X) = \{a_0 \cdots a_{n-1} \mid (a_j)_{j\in\mathbf{Z}} \in X, a_j \in A\}.$$

Let $(X, \sigma_X)$ and $(Y, \sigma_Y)$ be subshifts. Let $m, n$ be nonnegative integers. A mapping $\phi : X \to Y$ is called a *block map of* $(m, n)$ *type* if it is given by a mapping $\Phi : L_{m+n+1}(X) \to L_1(Y)$ in such a way that if for $(a_j)_{j \in \mathbf{Z}} \in X$, $a_j \in L_1(X)$, then $\phi((a_j)_{j\in\mathbf{Z}}) = (b_j)_{j\in\mathbf{Z}}$ with

$$b_j = \Phi(a_{-m+j} \cdots a_{n+j}) \quad \text{for all} \quad j \in \mathbf{Z}.$$

In particular, a block map of $(0,0)$ type is called a *1-block map.*

**Theorem 2** (Curtis, Hedlund, and Lyndon [H]). Let $(X, \sigma_X)$ and $(Y, \sigma_Y)$ be subshifts. Let $\phi : X \to Y$ be a continuous map with $\phi\sigma_X = \sigma_Y\phi$. Then $\phi$ is a block map of $(m, n)$ type for some $m, n \geq 0$.

Let $(X, \sigma)$ be a subshift and let $k \in \mathbf{N}$. We define the *higher block system* of *order* $k$ of $(X, \sigma)$, which was defined by Adler and Marcus [AM], to be the subshift $(X^{[k]}, \sigma^{[k]})$ over the alphabet $L_k(X)$, where

$$X^{[k]} = \{(a_j \cdots a_{j+k-1})_{j\in\mathbf{Z}} | (a_j)_{j\in\mathbf{Z}} \in X\}.$$

Let $m, n \geq \mathbf{N}$ with $k = m + n + 1$. Define

$$\rho_X^{[m,n]} : (X, \sigma) \to (X^{[k]}, \sigma^{[k]})$$

to be the conjugacy such that

$$\rho_X^{[m,n]}((a_j)_{j\in\mathbf{Z}}) = (a_{j-m} \cdots a_{j+n})_{j\in\mathbf{Z}}, \quad (a_j)_{j\in\mathbf{Z}} \in X.$$

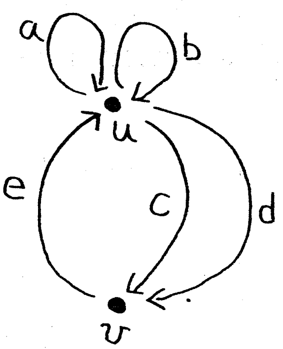
Let $(X, \sigma_X)$ and $(Y, \sigma_Y)$ be subshifts. Let $\phi : (X, \sigma_X) \to (Y, \sigma_Y)$ be of $(m, n)$ type with integers $m, n \geq 0$. Then there is a natural 1-block factor map $\bar{\phi} : (X^{[m+n+1]}, \sigma_{X^{[m+n+1]}}) \to (Y, \sigma_Y)$ such that the following diagram commutes:

$$\begin{tikzcd}
X^{[m+n+1]} \arrow[dr, "\bar{\phi}"] & \\
X \arrow[u, "\rho_X^{[m,n]}"] \arrow[r, "\phi"'] & Y
\end{tikzcd}.$$

It is often convenient to consider a factor map between subshifts as a 1-block map, passing through a higher block system in this way.

Let $(X, \sigma_X)$ be a subshift. Let $\varphi$ be an onto endomorphism of $(X, \sigma_X)$. Then the dynamical system $(X, \varphi)$ commutes with $(X, \sigma_X)$. More generally if $\varphi$ and $\tau$ are onto endomorphisms of $(X, \sigma_X)$ with $\varphi\tau = \tau\varphi$, then we call $(X,\varphi)$ and $(X,\tau)$ *commuting symbolic dynamical systems.* Thus in symbolic dynamics, commuting dynamical systems are dynamical systems of commuting block maps. We note that the "1-dimensional cellular automata" are the dynamical systems of block maps which commute with the full shifts.

Let $G$ be a graph. Here a graph means a directed graph which may have multiple arcs and loops. The following is an example of a graph.




Let $A_G$ and $V_G$ denote the arc-set of $G$ and the vertex-set of $G$, respectively. For the case of the example above,

$$A_G = \{a, b, c, d, e\} \quad \text{and} \quad V_G = \{u, v\}.$$

Let $i_G : A_G \to V_G$ and $t_G : A_G \to V_G$ be the mappings such that for $a \in A_G$, $i_G(a)$ is the initial vertex and $t_G(a)$ is the terminal vertex in $G$. Hence a graph $G$ is represented by

$$(*) \qquad V_G \xleftarrow{\ i_G\ } A_G \xrightarrow{\ t_G\ } V_G.$$

A graph $G$ is often given by its *adjacency matrix* $M_G$ or its *representation matrix* $\tilde{M}_G$: for the case of the example above,

$$M_G = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{M}_G = \begin{pmatrix} a+b, & c+d \\ e, & 0 \end{pmatrix}.$$

Let $X_G$ be the set of all points of $A_G^{\mathbf{Z}}$ compatible with the diagram $(*)$, i.e.,

$$X_G = \{(a_j)_{j\in\mathbf{Z}} \mid \forall j \in \mathbf{Z}, a_j \in A_G, t_G(a_j) = i_G(a_{j+1})\}.$$

Then we have a subshift $(X_G, \sigma_G)$, which is the *topological Markov shift* whose *defining graph [matrix]* is $G$ [$M_G$ or $\tilde{M}_G$]. If we let $\tilde{X}_G = \{(a_j)_{j\in\mathbf{N}} \mid \forall j \in \mathbf{N}, a_j \in A_G, t_G(a_j) = i_G(a_{j+1})\}$, then we have a *one-sided topological Markov shift* $(\tilde{X}_G, \tilde{\sigma}_G)$.

**Theorem 3** (Walters [W]). Let $f : X \to X$ be an onto continuous map of a 0-dimensional compact metric space with POTP. Then

(1) if $f$ is an expansive homeomorphism, then $(X, f)$ is conjugate to a topological Markov shift;

(2) if $f$ is positively expansive, then $(X, f)$ is conjugate to a one-sided topological Markov shift.

For graphs $\Gamma$ and $G$, a homomorphism from $\Gamma$ to $G$, written by $p : \Gamma \to G$, is a pair of mappings $p_A; A_\Gamma \to A_G$ (*arc-map*) and $p_V : V_\Gamma \to V_G$ (*vertex-map*) such that the following diagram commutes;

$$\begin{array}{ccccc} V_\Gamma & \xleftarrow{i_\Gamma} & A_\Gamma & \xrightarrow{t_\Gamma} & V_\Gamma \\ {\scriptstyle p_V}\downarrow & & {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p_V} \\ V_G & \xleftarrow[i_G]{} & A_G & \xrightarrow[t_G]{} & V_G \end{array} .$$

If $p_A$ and $p_V$ are onto, then we say that $p$ is *onto*.

A graph-homomorphism $h : \Gamma \to G$ defines a 1-block map. Any factor map $\phi : (X, \sigma_X) \to (Y, \sigma_Y)$ between topological Markov shifts is essentially given by a 1-block factor map $\phi_p$ defined by an onto graph-homomorphism $p : \Gamma \to G$, that is, there is a conjugacy $\psi$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y = X_G \\ {\scriptstyle \psi}\downarrow & \nearrow{\scriptstyle \phi_p} & \\ X_\Gamma & & \end{array} .$$

**Theorem 4** (Coven-Paul [CP]). Let $(X, \sigma_X)$ and $(Y, \sigma_Y)$ be topologically transitive topological Markov shifts. Let $\phi : X \to Y$ be an onto continuous map with $\phi\sigma_X = \sigma_Y\phi$. Then $h(\sigma_X) = h(\sigma_Y)$ if and only if $\phi$ is bounded-to-one. ($h$ denotes topological entropy.)

This is generalized to the following.

**Theorem 5** (Boyle [B]). Let $X$ and $Y$ be compact metric spaces. Let $f : X \to X$ and $g : Y \to Y$ be topologically transitive, expansive homeomorphisms with POTP. Let $\phi : X \to Y$ be an onto continuous map with $\phi f = g\phi$. Then $h(f) = h(g)$ if and only if $\phi$ is bounded-to-one.

We define a *textile system* $T$ over a graph $G$ to be an ordered pair of graph-homomorphisms $p : \Gamma \to G$ and $q : \Gamma \to G$ such that for $\alpha \in A_\Gamma$, the quadruple $(i_\Gamma(\alpha), t_\Gamma(\alpha), p_A(\alpha), q_A(\alpha))$ uniquely determines $\alpha$. We write $T = (p, q : \Gamma \to G)$. We have the following commutative diagram:

$$(**) \qquad \begin{array}{ccccc} V_G & \xleftarrow{i_G} & A_G & \xrightarrow{t_G} & V_G \\ {\scriptstyle p_V}\uparrow & & {\scriptstyle p_A}\uparrow & & \uparrow{\scriptstyle p_V} \\ V_\Gamma & \xleftarrow{i_\Gamma} & A_\Gamma & \xrightarrow{t_\Gamma} & V_\Gamma \\ {\scriptstyle q_V}\downarrow & & {\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle q_V} \\ V_G & \xleftarrow[i_G]{} & A_G & \xrightarrow[t_G]{} & V_G \end{array} .$$

We can define two graphs $G^T$ and $\Gamma^T$ by

$$\begin{aligned} G^T &: V_T \xleftarrow{p_V} V_\Gamma \xrightarrow{q_V} V_G \\ \Gamma^T &: A_G \xleftarrow{p_A} A_\Gamma \xrightarrow{q_A} A_G, \end{aligned}$$

and can define a textile system over $G^T$

$$T^* = (i, t : \Gamma^T \to G^T)$$

with graph-homomorphisms $i = (i_\Gamma, i_G)$ and $t = (t_\Gamma, t_G)$, which is called the *dual* of $T$.

Let $U_T$ be the set all points of $A_\Gamma^{\mathbf{Z}^2}$ (endowed with the product topology of the discrete topology on $A_\Gamma$) that are compatible with the diagram (**), i.e.,

$$\begin{aligned} U_T = \{(\alpha_{ij})_{i,j\in\mathbf{Z}} \mid \alpha_{ij} \in A_\Gamma, t_\Gamma(\alpha_{ij}) = i_\Gamma(\alpha_{i,j+1}) \\ \text{and} \quad q_A(\alpha_{ij}) = p_A(\alpha_{i+1,j}) \quad \text{for all} \quad i, j \in \mathbf{Z}\} \end{aligned}$$

Each point of $U_T$ is called a *textile* woven by $T$. Let

$$Z_T = \{(\alpha_{0j})_{j\in\mathbf{Z}} \mid (\alpha_{ij})_{i,j} \in U_T\}$$
$$X_T = \{(p_A(\alpha_{0j}))_{j\in\mathbf{Z}} \mid (\alpha_{ij})_{i,j\in\mathbf{Z}} \in U_T\}.$$

We have subshifts $(Z_T, \varsigma_T)$ and $(X_T, \sigma_T)$. We call $(X_T, \sigma_T)$ the *woof shift* (i.e., the subshifts of the horizontal threads) of $T$. The maps $p_A$ and $q_A$ naturally define onto 1-block maps

$$\xi_T : Z_T \to X_T \quad \text{and} \quad \eta_T : Z_T \to X_T.$$

For the dual $T^*$, we have $(Z_{T^*}, \varsigma_{T^*})$ and $(X_{T^*}, \sigma_{T^*})$ and 1-block maps $\xi_{T^*} : Z_{T^*} \to X_{T^*}$ and $\eta_{T^*} : Z_{T^*} \to X_{T^*}$. We call $(X_{T^*}, \sigma_{T^*})$ the *warp shift* (i.e., the subshifts of the vertical threads) of $T$. If $\xi_T$ is 1-1,then we define an onto endomorphism $\varphi_T$ of $(X_T, \sigma_T)$ by $\varphi_T = \eta_T \xi_T^{-1}$. If both $\xi_T$ and $\eta_T$ are 1-1, then $\varphi_T$ is an automorphism of $(X_T, \sigma_T)$. We also naturally define one-sided subshifts $(\tilde{Z}_T, \tilde{\varsigma}_T)$ and $(\tilde{X}_T, \tilde{\sigma}_T)$ and onto maps $\tilde{\xi}_T : \tilde{Z}_T \to \tilde{X}_T$ and $\tilde{\eta}_T : \tilde{Z}_T \to \tilde{X}_T$. We call $(\tilde{X}_T, \tilde{\sigma}_T)$ the *one-sided woof shift* of $T$, and $(\tilde{X}_{T^*}, \tilde{\sigma}_{T^*})$ is called the *one-sided warp shift* of $T$. If $\tilde{\xi}_T$ is 1-1, then we define an onto endomorphism $\tilde{\varphi}_T$ of $(\tilde{X}_T, \tilde{\sigma}_T)$ by $\tilde{\varphi}_T = \tilde{\eta}_T \tilde{\xi}_T^{-1}$. We say that $T$ is *nondegenerate* if $(X_T, \sigma_T) = (X_G, \sigma_G)$.

A textile system $T$ over $G$ also is defined by two graphs $G_1(= G)$ and $G_2(= G^T)$ with the same vertex-set and the finite set $Sq(T)$ of the *squares* of the form

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "b"'] & \bullet \arrow[d, "b'"] \\ \bullet \arrow[r, "a'"'] & \bullet \end{tikzcd}$$

where $a, a' \in A_{G_1}$, $b, b' \in A_{G_2}$ with

$$i_{G_1}(a) = i_{G_2}(b),\ t_{G_1}(a) = i_{G_2}(b'),\ t_{G_2}(b) = i_{G_1}(a'),$$
$$\text{and} \ \ t_{G_1}(a') = t_{G_2}(b').$$

Then a textile woven by $T$ is a Wang tiling generated by the Wang tiles in $Sq(T)$. (The tiles must be placed edge-to-edge without reflection and rotation so that edge colors match.)

Let $T$ be a textile system. For $(r, s) \in \mathbf{Z}^2$, let $\sigma_T^{(r,s)} : U_T \to U_T$ be defined by

$$\sigma_T^{(r,s)}((\alpha_{ij})_{i,j\in\mathbf{Z}}) = (\alpha_{i+r,j+s})_{i,j\in\mathbf{Z}}, \quad (\alpha_{ij})_{i,j\in\mathbf{Z}} \in U_T.$$

Define

$$\bar{U}_T = \{(\alpha_{ij})_{i\in\mathbf{N},j\in\mathbf{Z}} \mid (\alpha_{ij})_{i,j\in\mathbf{Z}} \in U_T\}$$

and

$$\hat{U}_T = \{(\alpha_{ij})_{i,j\in\mathbf{N}} \mid (\alpha_{ij})_{i,j\in\mathbf{Z}} \in U_T\}.$$

A point of $\bar{U}_T$ is called a *half textile* and that of $\hat{U}_T$ is called a *quarter textile*. For $r, s \in \mathbf{Z}$ with $r \geq 0$, we define $\bar{\sigma}_T^{(r,s)} : \bar{U}_T \to \bar{U}_T$ by

$$\bar{\sigma}_T^{(r,s)}((\alpha_{ij})_{i\in\mathbf{N},j\in\mathbf{Z}}) = (\alpha_{i+r,j+s})_{i\in\mathbf{N},j\in\mathbf{Z}}, \quad (\alpha_{ij})_{i\in\mathbf{N},j\in\mathbf{Z}} \in \bar{U}_T.$$

Similarly we define $\hat{\sigma}_T^{(r,s)} : \hat{U}_T \to \hat{U}_T$ for $r, s \geq 0$.

**Proposition 6** [N1]. Let $T$ be a textile system.

(1) If $\xi_T$ is 1-1, then $\varphi_T = \eta_T\xi_T^{-1}$ is expansive if and only if both $\xi_{T^*}$ and $\eta_{T^*}$ are 1-1; if all $\xi_T, \eta_T, \xi_{T^*}$ and $\eta_{T^*}$ are 1-1, then

$$(U_T, \sigma_T^{(1,0)}) \simeq (X_T, \varphi_T) \simeq (X_{T^*}, \sigma_{T^*})$$

and

$$(U_T, \sigma_T^{(0,1)}) \simeq (X_{T^*}, \varphi_{T^*}) \simeq (X_T, \sigma_T).$$

(2) If $\xi_T$ is 1-1, then $\varphi_T = \eta_T\xi_T^{-1}$ is positively expansive if and only if $\tilde{\xi}_{T^*}$ and $\tilde{\eta}_{T^*}$ are 1-1; if $\tilde{\xi}_{T^*}$ is 1-1, then $\tilde{\varphi}_{T^*} = \tilde{\eta}_{T^*}\tilde{\xi}_{T^*}^{-1}$ is expansive if and only if $\xi_T$ is 1-1; if all $\xi_T$, $\tilde{\xi}_{T^*}$ and $\tilde{\eta}_{T^*}$ are 1-1, then

$$(\bar{U}_T, \bar{\sigma}_T^{(1,0)}) \simeq (X_T, \varphi_T) \simeq (\tilde{X}_{T^*}, \tilde{\sigma}_{T^*})$$

and

$$(\bar{U}_T, \bar{\sigma}_T^{(0,1)}) \simeq (\tilde{X}_{T^*}, \tilde{\varphi}_{T^*}) \simeq (X_T, \sigma_T).$$

(3) If $\tilde{\xi}_T$ is 1-1, then $\tilde{\varphi}_T = \tilde{\eta}_T\tilde{\xi}_T^{-1}$ is positively expansive if and only if $\tilde{\xi}_{T^*}$ is 1-1; if both $\tilde{\xi}_T$ and $\tilde{\xi}_{T^*}$ are 1-1, then

$$(\hat{U}_T, \hat{\sigma}_T^{(1,0)}) \simeq (\tilde{X}_T, \tilde{\varphi}_T) \simeq (\tilde{X}_{T^*}, \tilde{\sigma}_{T^*}).$$

**Theorem 7** ([N1]). Let $(X, f)$ and $(X, g)$ are commuting dynamical systems which are conjugate to topological Markov shifts. Then there are a nondegenerate textile system $T$ with $T^*$ nondegenerate and with all $\xi_T, \eta_T, \xi_{T^*}$ and $\eta_{T^*}$ 1-1, and a homeomorphism $\psi : X \to X_T$ such that the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ X_T \arrow[r, "\sigma_T"] & X_T \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ X_T \arrow[r, "\varphi_T"] & X_T \end{tikzcd} .$$

Let $p : \Gamma \to G$ be a graph homomorphism with $p = (p_A, p_V)$. We say that $p$ is *right [left] resolving* if for any $\alpha_1, \alpha_2 \in A_\Gamma$, $i_\Gamma(\alpha_1) = i_\Gamma(\alpha_2)[t_\Gamma(\alpha_1) = t_\Gamma(\alpha_2)]$ and $p_A(\alpha_1) = p_A(\alpha_2)$ imply $\alpha_1 = \alpha_2$. We say that $p$ is *right complete [left complete]* if for any $u \in V_\Gamma$ and $a \in A_G$ with $i_G(a) = p_V(u)[t_G(a) = p_V(u)]$, there is $\alpha \in A_\Gamma$ such that $p_A(\alpha) = a$.

The following is well known (see [LM]).

**Proposition 8.** Let $p : \Gamma \to G$ be a right [left] resolving graph-homomorphism. Let $\phi_p : X_\Gamma \to X_G$ be the 1-block map defined by $p$. Then

(1) if $\phi_p$ is a conjugacy of $(X_\Gamma, \sigma_\Gamma)$ onto $(X_G, \sigma_G)$, then $p$ is right [left] complete;

(2) if $\sigma_\Gamma$ and $\sigma_G$ are topologically transitive and $h(\sigma_\Gamma) = h(\sigma_G)$, then $p$ is right [left] complete.

A textile system $T = (p, q : \Gamma \to G)$ is said to be *LR* if $p$ is right resolving and right complete and $q$ is left resolving and left complete. $T$ is said to be *LL* if both $p$ and $q$ are left resolving and left complete.

Let $G_1$ and $G_2$ be graphs with the same vertex-set. Let us call a "path" of length 2 of the form $ab$ with $t_{G_1}(a) = i_{G_2}(b)$, a *$G_1$-$G_2$ path*, which is actually an arc of the graph $G_1G_2$ with adjacency matrix $M_{G_1}M_{G_2}$. Then for a textile system $T$, $Sq(T) \subset A_{G_1G_2} \times A_{G_2G_1}$ is a relation such that each element in $Sq(T)$ is the ordered pair $(ab', ba')$ of a $G_1$-$G_2$ path $ab'$ and a $G_2$-$G_1$ path $ba'$ with the same initial vertex and the same terminal vertex.

$$\begin{tikzcd} \cdot \arrow[r, dash, "a"] \arrow[d, dash, "b"'] & \cdot \arrow[d, "b'"] \\ \cdot \arrow[r, "a'"'] & \cdot \end{tikzcd}$$

An LR textile system $T$ over $G$ is also defined to be a textile system $T$ such that the relation $Sq(T)$ is a one-to-one correspondence between the set of all $G_1$-$G_2$ paths and that of all $G_2$-$G_1$ paths with $G_1 = G$ and $G_2 = G^T$. Hence it is given by commuting two nonnegative integral matrices $M, N$ and a legal one-to-one correspondence $k$ between the set of all $G_1$-$G_2$ paths and that of all $G_2$-$G_1$ paths with $M_{G_1} = M$ and $M_{G_2} = N$. This is given by a *specified equivalence*

$$\tilde{M}\tilde{N} \overset{k}{\simeq} \tilde{N}\tilde{M}.$$

**Proposition 9** [N1]. Let $T$ be an LR textile system given by a specified equivalence $\tilde{M}\tilde{N} \overset{k}{\simeq} \tilde{N}\tilde{M}$.

(1) For $r, s \in \mathbf{N}$, $(U_T, \sigma_T^{(r,s)})$ is conjugate to the topological Markov shift whose defining matrix is $M^s N^r$.

(2) For $r, s \in \mathbf{N}$, $(\hat{U}_T, \hat{\sigma}_T^{(r,s)})$ is conjugate to the one-sided topological Markov shift whose defining matrix is $M^s N^r$.

Let $(X_1, \sigma_1)$ and $(X_2, \sigma_2)$ be subshifts and let $\phi : X_1 \to X_2$ be a continuous map with $\phi\sigma_1 = \sigma_2\phi$. We say that $\phi$ is *right closing* if for any $x, y \in X_1$, $\lim_{n\to\infty} d_{X_1}(\sigma_1^{-n}(x), \sigma_1^{-n}(y)) = 0$ and $\phi(x) = \phi(y)$ imply $x = y$.

**Proposition 10.** Let $(X, \sigma)$ be a subshift and $\varphi$ an onto endomorphism of $(X, \sigma)$. Then $\widetilde{\varphi\sigma^n}$ is positively expansive for some $n \in \mathbf{N}$ if and only if $\varphi$ is right closing. (Here, $(\tilde{X}, \tilde{\sigma})$ is the-one-sided subshift naturally induced from $(X, \sigma)$ and $\widetilde{\varphi\sigma^n}$ is the endomorphism of $(\tilde{X}, \tilde{\sigma})$ induced from $\varphi\sigma^n$ for sufficiently large $n$.)

According to the following proposition, we can say that all right closing endomorphisms of topological Markov shifts stem from LR textile systems. In particular, so do all automorphisms because they are right closing.

**Proposition 11** [N1]. Let $(X, \sigma)$ be a topological Markov shift and $\varphi$ an endomorphism of $(X, \sigma)$. Then $\varphi$ is right closing if and only if there are an integer $n \geq 0$ and an LR textile system $T$ with $\xi_T$ 1-1, $(X_T, \sigma_T) = (X, \sigma)$ and $\varphi = \varphi_T \sigma^{-n}$.

The following theorem was independently given in [N2] and in [Ku] (see also [BM] and [BFF]).

**Theorem 12** ([N2], Kůrka [Ku]). Let $(X, f)$ and $(X, g)$ be commuting positively expansive dynamical systems with $(X, f)$ conjugate to a topologically transitive, one-sided topological Markov shift. Then $(X, g)$ is conjugate to a one-sided topological Markov shift.

**Theorem 13** ([N1]). Let $(X, f)$ and $(X, g)$ be commuting dynamical systems which are conjugate to one-sided topological Markov shifts. Then there are an LR textile system $T$ with both $\tilde{\xi}_T$ and $\tilde{\xi}_{T^*}$ 1-1, and a homeomorphism $\psi : X \to \tilde{X}_T$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \psi\downarrow & & \downarrow\psi \\ \tilde{X}_T & \xrightarrow{\tilde{\sigma}_T} & \tilde{X}_T \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{g} & X \\ \psi\downarrow & & \downarrow\psi \\ \tilde{X}_T & \xrightarrow{\tilde{\varphi}_T} & \tilde{X}_T \end{array}$$

with $\tilde{\varphi}_T = \tilde{\eta}_T \tilde{\xi}_T^{-1}$. (Hence, if $T$ is defined by a specified equivalence $\tilde{M}\tilde{N} \overset{k}{\simeq} \tilde{N}\tilde{M}$, then for all $r, s \geq 0$ with $(r, s) \neq (0, 0)$, $(X, f^r g^s)$ is conjugate to a one-sided topological Markov shift whose defining matrix $M^r N^s$)

**Theorem 14.** Let $n \in \mathbf{N}$. If $\tau_1, \cdots, \tau_n$ are pairwise commuting, positively expansive, onto continuous maps with POTP, of a 0-dimensional compact metric space $X$, then there are pairwise commuting, nonnegative integral matrices $M_1, \cdots, M_n$ such that $(X, \tau_1^{k_1} \cdots \tau_n^{k_n})$ is conjugate to the one-sided topological Markov shift whose defining matrix is $M_1^{k_1} \cdots M_n^{k_n}$ for all integers $k_1, \cdots, k_n \geq 0$ with $(k_1, \cdots, k_n) \neq (0, \cdots, 0)$.

The following theorem was independently given in [N2] and in [Ku].

**Theorem 15** ([N2], Kůrka [Ku]). Let $(X, f)$ and $(X, g)$ be commuting dynamical systems with $f$ expansive and $g$ positively expansive. If $(X, f)$ is conjugate to a topologically transitive topological Markov shift, then $(X, g)$ is conjugate to a one-sided topological Markov shift.

**Theorem 16** [N1]. Let $(X, f)$ and $(X, g)$ are commuting dynamical systems with $(X, f)$ conjugate to topologically mixing topological Markov shift. Then there are a textile system such that $T^*$ is LL, $\xi_T, \tilde{\xi}_{T^*}, \tilde{\eta}_{T^*}$ are 1-1, and $(\tilde{X}_{T^*}, \tilde{\sigma}_{T^*})$ is conjugate to a one-sided full shift, and a homeomorphism $\psi : X \to X_T$ such that the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ X_T \arrow[r, "\sigma_T"] & X_T \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ X_T \arrow[r, "\varphi_T"] & X_T \end{tikzcd}.$$

(Hence $(X, g)$ is conjugate to a one-sided full shift.)

Let $f : X \to X$ be a continuous map. Let $\mathcal{O}_f$ be the metric space of all orbits of $f$ endowed with the metric $d_{\mathcal{O}_f}$ defined as follows: for $\boldsymbol{x} = (x_j)_{j\in\mathbf{Z}}$ and $\boldsymbol{y} = (y_j)_{j\in\mathbf{Z}}$ in $\mathcal{O}_f$,

$$d_{\mathcal{O}_f}(\boldsymbol{x}, \boldsymbol{y}) = \sup\{2^{-|j|} d_X(x_j, y_j) \mid j \in \mathbf{Z}\}.$$

We define $\sigma_f : \mathcal{O}_f \to \mathcal{O}_f$ by

$$\sigma_f((x_j)_{j\in\mathbf{Z}}) = (f(x_j))_{j\in\mathbf{Z}} = (x_{j+1})_{j\in\mathbf{Z}}, \quad (x_j)_{j\in\mathbf{Z}} \in \mathcal{O}_f.$$

For $\epsilon > 0$ and $\boldsymbol{x} \in \mathcal{O}_f$, we have

$$W^u_\epsilon(\boldsymbol{x}, \sigma_f) = \{(y_j)_{j\in\mathbf{Z}} \in \mathcal{O}_f \mid d_X(x_j, y_j) \leq \epsilon \text{ for } j \leq 0\}$$

and

$$W^s_\epsilon(\boldsymbol{x}, \sigma_f) = \{(y_j)_{j\in\mathbf{Z}} \in \mathcal{O}_f \mid d_X(x_j, y_j) \leq \epsilon \text{ for } j \geq 0\}.$$

We say that $f$ has *canonical coordinates* (CC) if for any $\epsilon > 0$, there is $\delta > 0$ such that for any $\boldsymbol{x} = (x_j)_{j\in\mathbf{Z}}$ and $\boldsymbol{y} = (y_j)_{j\in\mathbf{Z}}$ in $\mathcal{O}_f$,

$$d_X(x_0, y_0) \leq \delta \Rightarrow W^u_\epsilon(\boldsymbol{x}, \sigma_f) \cap W^s_\epsilon(\boldsymbol{y}, \sigma_f) \neq \emptyset.$$

In line with the definition of 'coding' given by Bolye and Lind [BL], we define the following. Let $A = \{\phi_i : X \to X_i \mid i \in I\}$ and $B = \{\phi'_j : X \to X'_j \mid j \in J\}$ be two sets of continuous maps between compact metric spaces, where $I$ and $J$ are indexing sets. Let $\epsilon, \epsilon' > 0$. We say that $A$ *$(\epsilon, \epsilon')$-codes* $B$ if for any $x, y \in X$,

$$\begin{aligned} &\forall i \in I, d_{X_i}(\phi_i(x), \phi_i(y)) \leq \epsilon \\ &\Rightarrow \forall j \in J, d_{X_j}(\phi'_j(x), \phi'_j(y)) \leq \epsilon'. \end{aligned}$$

If $A$ $(\epsilon, \epsilon)$-codes $B$, then we say that $A$ *$\epsilon$-codes* $B$.

The following is well known.

**Lemma 17.** Let $f : X \to X$ be a continuous map of a compact metric space. Let $\epsilon > 0$.

(1) If $f$ is $\epsilon$-expansive (i.e., $\epsilon$ is an expansive constant), then for any $\epsilon' > 0$ there is an integer $m \geq 0$ such that $\{f^j \mid 0 \leq j \leq 2m\}$ $(\epsilon, \epsilon')$-codes $\{f^m\}$ (i.e., for any $x, x' \in X$, if $d_X(f^j(x), f^j(x')) \leq \epsilon$ for $0 \leq j \leq 2m$, then $d_X(f^m(x), f^m(x')) \leq \epsilon'$)

(2) If $f$ is positively $\epsilon$-expansive, then for any $\epsilon' > 0$, there is an integer $m \geq 0$ such that $\{f^j \mid 0 \leq j \leq m\}$ $(\epsilon, \epsilon')$-codes $\{\mathrm{id}_X\}$.

Let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact metric spaces. Let $\epsilon > 0$. Let $m, n$ be nonnegative integers. Then a continuous map $\phi : X \to Y$ with $\phi f = g\phi$ is said to be *of* $(m, n)$ *type with respect to* $\epsilon$ if $\{f^j \mid j = 0, \cdots, m + n\}$ $\epsilon$-codes $\{g^m\phi\}$.

Boyle and Lind [BL] introduced a 'finite' version of expansiveness which is an analogue of the block maps. The following is a version of it and a direct corollary to the lemma above.

**Corollary 18** (Boyle and Lind). Let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact metric spaces. Let $\phi : X \to Y$ be a continuous map with $\phi f = g\phi$. Let $\epsilon > 0$.

(1) If $f$ is $\epsilon$-expansive, then there are integers $m, n \geq 0$ such that $\phi$ is of $(m, n)$ type with respect to $\epsilon$.

(2) If $f$ is positively $\epsilon$-expansive, then there is an integer $n \geq 0$ such that $\phi$ is of $(0, n)$ type with respect to $\epsilon$.

Let $f : X \to X$ and $g : Y \to Y$ be onto continuous maps of compact metric spaces. Let $\epsilon > 0$. Let $\phi : (X, f) \to (Y, g)$ be a factor map. We say that $\phi$ is $\epsilon$-*preserving* if $\{\mathrm{id}_X\}$ $\epsilon$-codes $\{\phi\}$. The $\epsilon$-preserving factor maps are an analogue of the 1-block maps in symbolic dynamics. If $f$ is $\epsilon$-expansive, then $\phi$ becomes $\epsilon$-preserving passing through a 'higher block system' of sufficiently large order of $(X, f)$.

Let $\phi$ be $\epsilon$-preserving. We say that $\phi$ is *right* $\epsilon$-*resolving* if $\{\mathrm{id}_X, g\phi\}$ $\epsilon$-codes $\{f\}$ (i.e., for any $x, x' \in X$, if $d_X(x, x') \leq \epsilon$ and $d_Y(g\phi(x), g\phi(x')) \leq \epsilon$, then $d_X(f(x), f(x')) \leq \epsilon$ ). We say that $\phi$ is *left* $\epsilon$-*resolving* if $\{\phi, f\}$ $\epsilon$-codes $\{\mathrm{id}_X\}$.

Let $\tilde{\phi} : (\mathcal{O}_f, \sigma_f) \to (\mathcal{O}_g, \sigma_g)$ be the factor map induced by $\phi$, i.e., for

$\boldsymbol{x} = (x_j)_{j\in\mathbf{Z}}$, $\tilde{\phi}(\boldsymbol{x}) = (\phi(x_j))_{j\in\mathbf{Z}}$. Since $\phi$ is $\epsilon$-preserving, for every $\boldsymbol{x} \in \mathcal{O}_f$,

$$\begin{aligned} W^u_\epsilon(\tilde{\phi}(\boldsymbol{x}), \sigma_g) &\supset \tilde{\phi}(W^u_\epsilon(\boldsymbol{x}, \sigma_f)) \quad \text{and} \\ W^s_\epsilon(\tilde{\phi}(\boldsymbol{x}), \sigma_g) &\supset \tilde{\phi}(W^s_\epsilon(\boldsymbol{x}, \sigma_f)). \end{aligned}$$

We say that $\phi$ is *right $\epsilon$-complete*, if for all $\boldsymbol{x} \in \mathcal{O}_f$,

$$W^u_\epsilon(\tilde{\phi}(\boldsymbol{x}), \sigma_g) = \tilde{\phi}(W^u_\epsilon(\boldsymbol{x}, \sigma_f)).$$

We also say that $\phi$ is *left $\epsilon$-complete*, if for all $\boldsymbol{x} \in \mathcal{O}_f$,

$$W^s_\epsilon(\tilde{\phi}(\boldsymbol{x}), \sigma_g) = \tilde{\phi}(W^s_\epsilon(\boldsymbol{x}, \sigma_f)).$$

The following proposition is a generalization of (1) of Proposition 8.

**Proposition 19.** Let $f : X \to X$ and $g : Y \to Y$ be onto continuous maps of compact metric spaces. Let $\epsilon > 0$. Let $g$ be $\epsilon$-expansive. Let $\phi : (X, f) \to (Y, g)$ be an $\epsilon$-preserving conjugacy.
(1) If $\phi$ is right $\epsilon$-resolving, then it is right $\epsilon$-complete.
(2) If $\phi$ is left $\epsilon$-resolving, then it is left $\epsilon$-complete.

The following theorem is a generalization of (2) of Proposition 8.

**Theorem 20.** Let $X$ and $Y$ be compact metric spaces. Let $f : X \to X$ and $g : Y \to Y$ be topologically transitive, onto continuous maps with $h(f) = h(g)$ and both having POTP. Let $\phi : (X, f) \to (Y, g)$ be an onto factor map. Then for all sufficiently small $\epsilon > 0$, the following are valid:
(1) if $\phi$ is $\epsilon$-preserving and right $\epsilon$-resolving, then $\phi$ is right $\epsilon$-complete.
(2) if $\phi$ is $\epsilon$-preserving and left $\epsilon$-resolving, then $\phi$ is left $\epsilon$-complete.

Let $\epsilon > 0$. We define an *$\epsilon$-textile-orbit-system* $T$ to be a quadruple of $\epsilon$-preserving onto continuous maps

$$i_X : X \to W,\ t_X : X \to W,\ p_Y : Y \to W,\ q_Y : Y \to W$$

between compact metric spaces such that $i_X$ and $p_Y$ are homeomorphisms and the following diagram commutes:

$$\begin{array}{ccccc}
W & \xrightarrow{i_X^{-1}} & X & \xrightarrow{t_X} & W \\
{\scriptstyle p_Y^{-1}}\downarrow & & & & \downarrow{\scriptstyle p_Y^{-1}} \\
Y & & & & Y \\
{\scriptstyle q_Y}\downarrow & & & & \downarrow{\scriptstyle q_Y} \\
W & \xrightarrow{i_X^{-1}} & X & \xrightarrow{t_X} & W
\end{array} .$$

We write $T = (i_X, t_X : X \to W;\ p_Y, q_Y : Y \to W)$. We call $z = (y, y', x, x') \in Y \times Y \times X \times X$ a *square* of $T$ if $i_X(x) = p_Y(y), q_Y(y) = i_X(x'), t_X(x) = p_Y(y')$ and $q_Y(y') = t_X(x')$. Let $Z$ be the set of all squares of $T$ and let it be endowed with the max metric. We call $Z$ the *square space* of $T$. We define projections $i_Z, t_Z, p_Z, q_Z$ as follows: for $z = (y, y', x, x') \in \mathbf{Z}$, $i_Z(z) = y, t_Z(z) = y', p_Z(z) = x$, and $q_Z(z) = x'$. Then all the projections are $\epsilon$-preserving, $i_Z$ and $p_Z$ are homeomorphisms, and the following diagram commutes:

$$\begin{array}{ccccc}
W & \xleftarrow{i_X} & X & \xrightarrow{t_X} & W \\
{\scriptstyle p_Y}\uparrow & & \uparrow{\scriptstyle p_Z} & & \uparrow{\scriptstyle p_Y} \\
Y & \xleftarrow{i_Z} & Z & \xrightarrow{t_Z} & Y \\
{\scriptstyle q_Y}\downarrow & & \downarrow{\scriptstyle q_Z} & & \downarrow{\scriptstyle q_Y} \\
W & \xleftarrow[i_X]{} & X & \xrightarrow[t_X]{} & W
\end{array} .$$

A two-dimensional configuration $(z_{ij})_{i,j \in \mathbf{Z}}$ of squares is called a *textile orbit* of $T$ if for all $i, j \in \mathbf{Z}$,

$$t_Z(z_{i,j-1}) = i_Z(z_{ij}) \quad \text{and} \quad q_Z(z_{i-1,j}) = p_Z(z_{ij}).$$

We say that $T$ is *LL* if $\{t_Z, p_Z\}$ $\epsilon$-codes $\{\mathrm{id}_Z\}$ and if $\{t_Z, q_Z\}$ $\epsilon$-codes $\{\mathrm{id}_Z\}$. We also say that $T$ is *LR* if $\{t_Z, p_Z\}$ and $\{i_Z, q_Z\}$ $\epsilon$-code each other.

Using Theorem 20, one can prove the following lemma.

**Lemma 21.** Let $\epsilon > 0$. Let $T = (i_X, t_X : X \to W; p_Y, q_Y : Y \to W)$ be an $\epsilon$-textile-orbit-system. Let $Z$ be the square space of $T$. Let $f = i_Z^{-1} t_Z$, let $f^* = p_Z^{-1} q_Z$, and let $g^* = p_Y^{-1} q_Y$. Assume that $g^*$ is topologically transitive and expansive, and has POTP (and hence $f^*$ has the same properties because $i_Z : (Z, f^*) \to (Y, g^*)$ is a conjugacy) and that there are $c, \epsilon_0, \delta_0 > 0$ with $\epsilon < \min\{\delta_0/3, c/5\}$ and $\delta_0 \leq \epsilon_0$ such that both $f^*$ and $g^*$ are $c$-expansive, $\{\mathrm{id}_Z\}$ $(\epsilon_0, c/2)$-codes $\{t_Z\}$ and any $\delta_0$-pseudo-orbit of $f^*$ is $\epsilon_0$-traced by some orbit of $f^*$. Then the following statements are valid.

(1) If $T$ is LL, then $f$ is constant-to-one and open, and moreover for any textile orbits $(z_{ij})_{i,j \in \mathbf{Z}}$ and $(z'_{ij})_{i,j \in \mathbf{Z}}$ of $T$,

$$\begin{aligned} &W^u_\epsilon((z_{i0})_{i \in \mathbf{Z}}, \sigma_{f^*}) \cap W^s_\epsilon((z'_{i0})_{i \in \mathbf{Z}}, \sigma_{f^*}) \neq \emptyset \\ \Rightarrow\ &W^u_{2\epsilon}((z_{0j})_{j \in \mathbf{Z}}, \sigma_f) \cap W^s_0((z'_{0j})_{j \in \mathbf{Z}}, \sigma_f) \neq \emptyset. \end{aligned}$$

(2) If $T$ is LR, then for any textile orbits $(z_{ij})_{i,j \in \mathbf{Z}}$ and $(z'_{ij})_{i,j \in \mathbf{Z}}$ of $T$ if

$$W^u_\epsilon((z_{i0})_{i \in \mathbf{Z}}, \sigma_{f^*}) \cap W^s_\epsilon((z'_{i0})_{i \in \mathbf{Z}}, \sigma_{f^*}) \ni \boldsymbol{z},$$

then there is a textile orbit $(\bar{z}_{ij})_{i,j \in \mathbf{Z}}$, of $T$ such that $(\bar{z}_{i0})_{i \in \mathbf{Z}} = \boldsymbol{z}$,

$$d_Z(\bar{z}_{ij}, z_{ij}) \leq \epsilon \quad \forall i, j \leq 0 \quad \text{and} \quad d_Z(\bar{z}_{ij}, z'_{ij}) \leq \epsilon \quad \forall i, j \geq 0.$$

Using this lemma, one can prove the following two theorems, which are generalizations of Theorems 15 and 12, respectively.

**Theorem 22.** Let $X$ be a compact metric space. Let $\varphi : X \to X$ be a topologically transitive, expansive homeomorphism and $\tau : X \to X$ positively expansive onto continuous map. If $\varphi\tau = \tau\varphi$ and $\varphi$ has POTP, then $\tau$ has POTP and is constant-to-one.

**Theorem 23.** Let $X$ be an compact metric space. Let $\varphi : X \to X$ and $\tau : X \to X$ be positively expansive, onto continuous maps with $\varphi\tau = \tau\varphi$. If $\varphi$ is topologically transitive and has POTP, then $\tau$ has POTP.

On the other hand, the following lemma can be proved by using Proposition 19.

**Lemma 24.** Let $\epsilon > 0$. Let $T = (i_X, t_X; X \to W; p_Y, q_Y : Y \to W)$ be an LR $\epsilon$-textile-orbit-system with $t_X$ 1-1. Let $Z$ be the square space of $T$.

Let $f = i_Z^{-1}t_Z$, let $f^* = p_Z^{-1}q_Z$, let $g^* = p_Y^{-1}q_Y$, and let $g^*$ be $\epsilon$-expansive. Then for any textile orbits $(z_{ij})_{i,j\in\mathbf{Z}}$ and $(z'_{ij})_{i,j\in\mathbf{Z}}$ of $T$ if

$$W_\epsilon^u((z_{i0})_{i\in\mathbf{Z}}, \sigma_{f^*}) \cap W_\epsilon^s((z'_{i0})_{i\in\mathbf{Z}}, \sigma_{f^*}) \ni \boldsymbol{z},$$

then there is a textile orbit $(\bar{z}_{ij})_{i,j\in\mathbf{Z}}$ of $T$ such that $(\bar{z}_{i0})_{i\in\mathbf{Z}} = \boldsymbol{z}$,

$$d_Z(\bar{z}_{ij}, z_{ij}) \leq \epsilon \ \ \forall i,j \leq 0 \quad \text{and} \quad d_Z(\bar{z}_{ij}, z'_{ij}) \leq \epsilon \ \ \forall i,j \geq 0.$$

Let $\tau : X \to X$ be an expansive onto continuous map of a compact metric space. An onto continuous map $\varphi : X \to X$ is called an *LR endomorphism* of $(X,\tau)$ if there are $\epsilon > 0$ and an LR $\epsilon$-textile- orbit-system $T = (i_X, t_X : X \to W; p_Y, q_Y : Y \to W)$ such that $\tau = i_X^{-1}t_X$ and $\varphi = q_Z p_Z^{-1}$ with $\tau$ $\epsilon$-expansive, where $Z$ is the square space of $T$. We define a *positively LR endomorphism*of $(X,\tau)$ by replacing 'with $\tau$ $\epsilon$-expansive' by 'with $\tau$ positively $\epsilon$-expansive' in this defnition. We say that $\varphi$ is an *essentially LR endomorphism* of $(X,\tau)$ if there are an onto continuous map $\tau' : X' \to X'$, an LR endomorphism $\varphi'$ of $(X',\tau')$, and a conjugacy $\psi : (X,\tau) \to (X',\tau')$ such that $\varphi = \psi^{-1}\varphi'\psi$ and $\tau = \psi^{-1}\tau'\psi$. We define an *essentially positively LR endomorphism* similarly.

By using Lemma 24, the following theorem can be proved.

**Theorem 25.** Let $\tau : X \to X$ be an expansive onto continuous map of a compact metric space and $\varphi$ an essentially LR automorphism of $(X,\tau)$. If $\tau$ has CC, then so does $\varphi$.

This theorem is closely related with the result of Boyle and Lind [BL] that if one direction in a 'expansive component' of directions for a $\mathbf{Z}^d$ action on a compact metric space, is Markov, then so are all the directions of the component. For a 'Markov direction' is a generalized notion of an expansive $\mathbf{Z}^d$-action with canonical coordinates and one can see that if $\tau$ and $\varphi$ are expansive homeomorphisms of a compact metric space $X$, then there are $m$ and $n$ in $\mathbf{N}$ such that $\varphi^m$ is an essentially LR automorphism of $(X,\tau^n)$ if and only if the directions of $\tau$ and $\varphi$ are in the same expansive component of directions for the $\mathbf{Z}^2$-action generated by $\tau$ and $\varphi$.

**Proposition 26.** Let $\tau : X \to X$ be an expansive onto continuous map of a compact metric space and $\varphi$ an automorphism of $(X,\tau)$. Then for all sufficiently large $n, \varphi\tau^n$ is an essentially LR endomorphism of $(X,\tau)$.

**Proposition 27.** Let $\tau : X \to X$ be an expansive onto continuous map of a compact metric space and $\varphi$ an essentially LR endomorphism of $(X, \tau)$. Then the following statements are valid:

(1) for all integers $m, n \geq 0, \varphi^m \tau^n$ is an essentially LR endomorphism of $(X, \tau)$;

(2) for all integers $m \geq 0$ and $n \geq 1, \varphi^m \tau^n$ is expansive.

**Propositon 28.** Let $\tau : X \to X$ be a positively expansive onto continuous map of a compact metric space and $\varphi$ an onto endomorphism of $(X, \tau)$. Then the following statements are valid.

(1) If $\varphi$ is an essentially positively LR endomorphism of $(X, \tau)$, then $\varphi^m \tau^n$ is an essentially positively LR endomorphism of $(X, \tau)$ for all integers $m, n \geq 0$ and positively expansive for all integers $m \geq 0$ and $n \geq 1$.

(2) If $\varphi$ is positively expansive, then $\varphi$ is an essentially positively LR endomorphism of $(X, \tau)$.

**Corollary 29.** If $\tau : X \to X$ and $\varphi : X \to X$ are commuting positively expansive onto continuous maps of a compact metric space, then $\tau\varphi$ is positively expansive.

Let $Y$ be a compact metric space. Let $f : Y \to Y$ be an onto continuous map and $g : Y \to Y$ be an expansive onto continuous map with $fg = gf$. We say that $f$ is a *directionally LR endomorphism* of $(Y, g)$ if there are $m, n \in \mathbf{N}$ such that $f^m$ is an essentially LR endomorphism of $(Y, g^n)$.

**Theorem 30.** Let $Y$ be a compact metric space. Let $H(Y)$ denote the group of homeomorphisms of $Y$ onto itself, and $E(Y)$ the set of expansive homeomorphisms in $H(Y)$. If we write $f_1 \sim f_2$ for $f_1 \in H(Y)$ and $f_2 \in E(Y)$ to mean that $f_1$ is a directionally LR automorphism of $(Y, f_2)$, then the following statements are valid.

(1) If $f \in E(Y)$, then $f \sim f$.

(2) If $f_1, f_2 \in E(Y)$ and $f_1 \sim f_2$, then $f_2 \sim f_1$.

(3) If $f_1 \in H(Y)$ and $f_2, f_3 \in E(Y)$ with $f_1 f_3 = f_3 f_1$, then $f_1 \sim f_2$ and $f_2 \sim f_3$ imply $f_1 \sim f_3$.

(4) If $f \in E(Y)$ and $g \in H(Y)$ with $fg = gf$, then there is $m \in \mathbf{N}$ such that $f^n g \sim f$ for all $n \geq m$.

(5) If $f_1 \sim g$ and $f_2 \sim g$ with $f_1, f_2 \in H(Y)$, $g \in E(Y)$ and $f_1 f_2 = f_2 f_1$, then if there are integers $m, n \geq 0$ such that $f_1^m f_2^n \in E(Y)$, then $f_1 f_2 \in E(Y)$ and $f_1 f_2 \sim g$.

Thus for $f \in E(Y)$ and a commutative subgroup $K$ of $H(Y)$ containing $f$, we define

$$C_K(f) = \{g \in K \mid g \sim f\}$$

and call this the *DLR cone containing $f$ in $K$*.

This notion is closely related with that of an expansive component of 1-frames for a $\mathbf{Z}^d$-action in the theory of Boyle and Lind [BL].

**Theorem 31.** An expansive directionally LR automorphism of a topological Markov shift is an essentially LR automorphism of the shift.

**Theorem 32.** Let $n \in \mathbf{N}$. If $\tau_1, \cdots, \tau_n$ are pairwise commuting, expansive homeomorphisms with POTP, of a 0-dimensional compact metric space $X$, and if they belong to the same ELR cone, then there are $m \in \mathbf{N}$ and pairwise commuting nonnegative integral matrices $M_1, \cdots, M_n$ such that $(X, \tau_1^{mk_1} \cdots \tau_n^{mk_n})$ is conjugate to the topological Markov shift whose defining matrix is $M_1^{k_1} \cdots M_n^{k_n}$ for all integers $k_1, \cdots, k_n \geq 0$ with $(k_1, \cdots, k_n) \neq (0, \cdots, 0)$.

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