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On the solution of the Time Dependent Ginzburg-Landau equations

BY

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d=2,3)$ with smooth boundary $\partial\Omega$. We consider the Time Dependent Ginzburg-Landau equations:

$$\psi_t = \mathbf{D}_A^2\psi + \kappa(1-|\psi|^2)\psi + i\Phi\psi, \qquad \text{in } (0,\infty)\times\Omega, \tag{1}$$

$$\mathbf{A}_t = -\mathrm{rot}^2\mathbf{A} - \frac{i}{2}\{\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\} + \nabla\Phi, \qquad \text{in } (0,\infty)\times\Omega, \tag{2}$$

where $\mathbf{D}_A\psi = (\nabla\psi - i\mathbf{A}\psi)$, ψ is the complex order parameter, $\mathbf{A}$ is the magnetic vector potential, Φ is the scalar electric potential and κ is a real positive constant called the Ginzburg-Landau parameter of the substance. The system (1)-(2), which we call the TDGL equations below, was proposed by A.Schmid [8] or L.P.Gor'kov and G.M. Eliashberg [6]. The TDGL equations have an important property, namely, that of gauge invariance. Throughout this paper we consider the problem in the Coulomb gauge, namely,

$$\mathrm{div}\mathbf{A} = 0, \qquad \text{in } \Omega. \tag{3}$$

Then, the boundary conditions are prescribed as

$$\frac{\partial\psi}{\partial\mathbf{n}} = 0, \qquad \text{on } \partial\Omega, \tag{4}$$

$$\mathbf{A}\cdot\mathbf{n} = 0, \qquad \text{on } \partial\Omega, \tag{5}$$

$$\mathrm{rot}\mathbf{A}\times\mathbf{n} = 0, \qquad \text{on } \partial\Omega, \tag{6}$$

$$\frac{\partial\Phi}{\partial\mathbf{n}} = 0, \qquad \text{on } \partial\Omega, \tag{7}$$

where $\mathbf{n}$ denotes the unit outer normal vector to the boundary $\partial\Omega$. The initial conditions are

$$\psi(0,x) = \psi_0, \qquad \text{in } \Omega, \tag{8}$$
$$\mathbf{A}(0,x) = \mathbf{A}_0, \qquad \text{in } \Omega. \tag{9}$$

In the following, in order to get the uniqueness of solutions, we assume that

$$\int_\Omega \Phi(t,x)dx = 0, \qquad \forall t \geq 0. \tag{10}$$

Let $W^{m,p}(\Omega)$ be the standard Sobolev space of complex-valued functions, and as usual, $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$. $\mathbf{W}^{m,p}(\Omega)$ and $\mathbf{H}^m(\Omega)$ denote the Sobolev spaces of real vector-valued functions. As usual, $L^2(\Omega) = H^0(\Omega)$ and $\mathbf{L}^2(\Omega) = \mathbf{H}^0(\Omega)$. We introduce the following function spaces :

$$\begin{aligned}
C_n^\infty(\Omega) &= \{\psi \in C^\infty(\Omega);\ \frac{\partial\psi}{\partial\mathbf{n}} = 0 \ \text{ on } \partial\Omega\},\\
\mathbf{H}_\sigma(\Omega) &= \{\mathbf{A} \in \mathbf{L}^2(\Omega);\ \mathrm{div}\mathbf{A} = 0 \ \text{ in } \Omega,\ \mathbf{A}\cdot\mathbf{n} = 0 \ \text{ on } \partial\Omega\},\\
\mathbf{H}_\sigma^1(\Omega) &= \{\mathbf{A} \in \mathbf{H}^1(\Omega); \mathrm{div}\mathbf{A} = 0 \ \text{ in } \Omega,\\
&\qquad\qquad \mathrm{rot}\mathbf{A} \times \mathbf{n} = 0,\ \mathbf{A}\cdot\mathbf{n} = 0 \ \text{ on } \partial\Omega\},\\
\mathbf{C}_\sigma^\infty(\Omega) &= \{\mathbf{A} \in \{C_0^\infty(\Omega)\}^d;\ \mathrm{div}\mathbf{A} = 0 \ \text{ in } \Omega,\}.
\end{aligned}$$

Clearly, $C_n^\infty(\Omega)$ is dense in $L^2(\Omega)$ and $\mathbf{C}_\sigma^\infty(\Omega)$ is dense in $\mathbf{H}_\sigma(\Omega)$ (See [11]).

The scalar products and the norms of $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ are denoted by

$$\begin{aligned}
(\psi,\varphi)_{L^2} &= \int_\Omega \psi(x)\overline{\varphi(x)}dx, & \|\psi\|_{L^2} &= (\psi,\psi)_{L^2}^{1/2},\\
(\mathbf{A},\mathbf{B})_{\mathbf{L}^2} &= \int_\Omega \mathbf{A}(x)\cdot\mathbf{B}(x)dx, & \|\mathbf{A}\|_{\mathbf{L}^2} &= (\mathbf{A},\mathbf{A})_{\mathbf{L}^2}^{1/2}.
\end{aligned}$$

Also, the norms of $L^\infty(\Omega)$ and $\mathbf{L}^\infty(\Omega)$ are denoted by

$$\|\psi\|_{L^\infty} = \operatorname*{ess\ sup}_{x\in\Omega} |\psi(x)|, \qquad \|\mathbf{A}\|_{\mathbf{L}^\infty} = \operatorname*{ess\ sup}_{x\in\Omega} |\mathbf{A}(x)|.$$

$\mathcal{L}^2(\Omega)$ is defined by $L^2(\Omega) \oplus \mathbf{H}_\sigma(\Omega)$ where $\oplus$ implies direct product. Clearly $\mathcal{L}^2(\Omega)$ becomes a Hilbert space. The scalar product and the norm of $\mathcal{L}^2(\Omega)$ are defined by

$$\begin{aligned}
(U,V)_{\mathcal{L}^2} &= \int_\Omega \psi(x)\overline{\varphi(x)}dx + \int_\Omega \mathbf{A}(x)\cdot\mathbf{B}(x)dx,\\
\|U\|_{\mathcal{L}^2}^2 &= \|\psi\|_{L^2}^2 + \|\mathbf{A}\|_{\mathbf{L}^2}^2 = \|\psi\|_{L^2}^2 + \|\mathbf{A}\|_{\mathbf{H}_\sigma}^2,
\end{aligned}$$

where $U = (\psi,\mathbf{A})^t$ and $V = (\varphi,\mathbf{B})^t$. For all $1 \leq p < \infty$, $\mathcal{L}^p(\Omega)$ is defined by $L^p(\Omega) \oplus \mathbf{L}^p(\Omega)$. We denote the norm in $\mathcal{L}^p(\Omega)$ by

$$\|U\|_{\mathcal{L}^p}^p = \|\psi\|_{L^p}^p + \|\mathbf{A}\|_{\mathbf{L}^p}^p. \tag{11}$$

Analogously, $\mathcal{H}^m(\Omega)$, $\mathcal{W}^{m,p}(\Omega)$ and $\mathcal{L}^\infty(\Omega)$ are defined by $H^m(\Omega)\oplus\mathbf{H}^m(\Omega)$, $W^{m,p}(\Omega)\oplus\mathbf{W}^{m,p}(\Omega)$ and $L^\infty(\Omega)\oplus\mathbf{L}^\infty(\Omega)$, respectively. We introduce

$$\mathcal{H}^\alpha_\sigma(\Omega) = H^\alpha(\Omega)\oplus(\mathbf{H}^\alpha(\Omega)\cap\mathbf{H}^1_\sigma(\Omega)),$$

where $\alpha \geq 1$.

Since $\mathrm{div}\mathbf{A} = 0$, the TDGL equations (1)-(2) are rewritten as

$$\psi_t = \Delta\psi - 2i(\mathbf{A}\cdot\nabla\psi) - |\mathbf{A}|^2\psi + \kappa(1-|\psi|^2)\psi + i\Phi\psi, \tag{12}$$

$$\mathbf{A}_t = \Delta\mathbf{A} - P[\frac{i}{2}\{\overline{\psi}(\nabla\psi) - \psi(\overline{\nabla\psi})\} + |\psi|^2\mathbf{A}], \tag{13}$$

$$\Delta\Phi = \mathrm{div}\left\{\frac{i}{2}\{\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\}\right\}, \tag{14}$$

where P is the orthogonal projection $P : \mathbf{L}^2(\Omega) \to \mathbf{H}_\sigma(\Omega)$.

First, we discuss (14). We recall the Gagliardo-Nirenberg inequality

$$\|\partial^j f\|_{L^p} \leq C\|\partial^k f\|^s_{L^q}\ \|f\|^{1-s}_{L^r}, \tag{15}$$

where $\frac{1}{p} - \frac{j}{d} = s(\frac{1}{q} - \frac{k}{d}) + (1-s)\frac{1}{r}$, $j/k \leq s \leq 1$ (if $k - j - \frac{d}{q}$ is a non-negative integer, only $s < 1$ is allowed.) where d is a number of dimension. Hereafter, C denotes various positive constants which may change from line to line. Calculating the right-hand side of (14), we have

$$\mathrm{div}\left\{\frac{i}{2}\{\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\}\right\} = \frac{i}{2}\left\{\overline{\psi}\Delta\psi - \psi\overline{\Delta\psi} - 2i(\nabla|\psi|^2)\mathbf{A}\right\}. \tag{16}$$

Using the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}&\|\mathrm{div}\left\{\frac{i}{2}\{\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\}\right\}\|_{L^2}\\ &\leq C\left\{\|\psi\|_{L^\infty}\ \|\Delta\psi\|_{L^2} + \|\mathbf{A}\|_{\mathbf{L}^4}\ \|\psi\|_{L^\infty}\ \|\nabla\psi\|_{L^4}\right\}\\ &\leq C\left\{\|\psi\|^{7/4}_{H^2}\ \|\psi\|^{1/4}_{L^2} + \|\mathbf{A}\|_{\mathbf{L}^4}\ \|\psi\|^{13/8}_{H^2}\ \|\psi\|^{3/8}_{L^2}\right\}, \qquad \text{for } d = 3,\end{aligned} \tag{17}$$

$$\begin{aligned}&\|\mathrm{div}\left\{\frac{i}{2}\{\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\}\right\}\|_{L^2}\\ &\leq C\left\{\|\psi\|^{3/2}_{H^2}\ \|\psi\|^{1/2}_{L^2} + \|\mathbf{A}\|_{\mathbf{L}^3}\ \|\psi\|^{4/3}_{H^2}\ \|\psi\|^{2/3}_{L^2}\right\}, \qquad \text{for } d = 2,\end{aligned} \tag{18}$$

where C is a constant depending on Ω. By the standard elliptic theorems, we have

Lemma 1 *We suppose that* $(\psi, \mathbf{A}) \in \mathcal{H}^2_\sigma(\Omega)$ *is given. Then, the boundary value problem*

$$\Delta\Phi = \mathrm{div}\left\{\frac{i}{2}\{\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\}\right\}, \qquad in\ \Omega \tag{19}$$

$$\frac{\partial\Phi}{\partial n} = 0, \qquad on\ \partial\Omega \tag{20}$$

$$\int_\Omega \Phi(x)dx = 0, \tag{21}$$

has a unique solution $\Phi(\psi, \mathbf{A}) \in H^2(\Omega)$.

We employ the semi-group theory. (12) and (13) with the initial conditions (8), (9) and the boundary conditions (4)-(7) are expressed as an abstract evolution equation of the form

$$\frac{d}{dt}U + \mathcal{A}U = \mathcal{B}(U), \tag{22}$$

$$U(0) = U_0 = (\psi_0,\ \mathbf{A}_0)^t, \tag{23}$$

where $U = (\psi, \mathbf{A})^t$ is a pair of functions, $\mathcal{A} = \begin{pmatrix} -\Delta + \beta I & 0 \\ 0 & -\Delta + \beta I \end{pmatrix}$, $D(\mathcal{A}) \equiv \mathcal{H}^2_\sigma(\Omega)$ and

$$\mathcal{B}(U) = \begin{pmatrix} (\beta + \kappa)\psi - 2i(\mathbf{A} \cdot \nabla\psi) - |\mathbf{A}|^2\psi - \kappa|\psi|^2\psi + i\Phi(\psi, \mathbf{A})\psi \\ \beta\mathbf{A} - P\left[\frac{i}{2}\left\{\overline{\psi}(\nabla\psi) - \psi(\overline{\nabla\psi})\right\} + |\psi|^2\mathbf{A}\right] \end{pmatrix},$$

where $D(\mathcal{B}) \supset D(\mathcal{A}^\alpha)$ ($\alpha \geq \frac{5}{8}$ $(d = 3)$, $\alpha \geq \frac{1}{2}$ $(d = 2)$) (See below proposition 2, proposition 3).

If $-\mathcal{A}$ is invertible and the infinitesimal generator of an analytic semigroup, $\mathcal{A}^\alpha$ can be defined for $0 \leq \alpha \leq 1$ so that $\mathcal{A}^\alpha$ is a closed linear invertible operator with domain $D(\mathcal{A}^\alpha)$ dense in X. The closedness of $\mathcal{A}^\alpha$ implies that $D(\mathcal{A}^\alpha)$ endowed with the graph norm of $\mathcal{A}^\alpha$, (*i.e.* the norm $|||U||| = \|U\|_X + \|\mathcal{A}^\alpha U\|_X$), is a Banach space. Since $\mathcal{A}^\alpha$ is invertible, its graph norm $|||\cdot|||$ is equivalent to the norm $\|U\|_\alpha = \|\mathcal{A}^\alpha U\|_X$. Thus, $D(\mathcal{A}^\alpha)$ equipped with the norm $\|\cdot\|_\alpha$ is a Banach space, which we denote by X_α. From this definition it is clear that $0 < \alpha < \beta$ implies $X_\alpha \supset X_\beta$ and that the imbedding of X_β into X_α is continuous. The following theorem is well-known.

Theorem 1 (H. Fujita and T. Kato [5]) *We consider that*

$$\frac{d}{dt}u(t) + \mathcal{A}u(t) = B(u) \qquad (t > 0), \tag{24}$$

$$u(0) = u_0. \tag{25}$$

Let $-\mathcal{A}$ be the infinitesimal generator of an analytic semigroup $T(t)$ satisfying $\|T(t)\| \leq M$ and assume further that $0 \in \rho(-\mathcal{A})$. Let $\mathcal{U}$ be an open subset of X_α. Let the function $B : \mathcal{U} \to X$ satisfy that for every $u \in \mathcal{U}$, there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ and constant $L > 0$, such that

$$\|B(u_1) - B(u_2)\|_X \leq L\|u_1 - u_2\|_\alpha, \qquad \textit{for all } u_i \in \mathcal{V}\ (i = 1, 2).$$

Then, for every initial data $u_0 \in \mathcal{U}$, the initial value problem (24), (25) has a unique local solution $u \in C([0, T_0) : X) \cap C((0, T_0) : D(\mathcal{A})) \cap C^1((0, T_0) : X)$ where $T_0 = T_0(\|u_0\|_\alpha) > 0$.

Applying Theorem 1, we have our main results.

Theorem 2 (Existence of local strong solutions.) *Let* $U_0 \in D(\mathcal{A}^\alpha)$ *with* $\alpha \geq \frac{5}{8}(d=3)$, $\frac{1}{2}$ $(d=2)$. *The initial value problem of the TDGL equations* (12)-(14) *with the conditions* (3)-(10), *has a unique local strong solution* $U \in C([0,T_0) : \mathcal{L}^2(\Omega)) \cap C((0,T_0) : D(\mathcal{A})) \cap C^1((0,T_0) : \mathcal{L}^2(\Omega))$ *where* $T_0 = T_0(\|U_0\|_\alpha) > 0$.

For $d = 2$, we have next theorem.

Theorem 3 (Global strong solution) *If* $d = 2$, *then the solution in Theorem* 2 *is global. i.e. we may take* $T_0 = \infty$.

2 Proof of Theorem 2.

In order to prove Theorem 2, we prepare three propositions.

Proposition 1 $-\mathcal{A}$ *is the infinitesimal generator of* C_0 *semigroup of contraction on* $\mathcal{L}^2(\Omega)$.

Proof of Proposition 1. Since $\mathrm{div}\mathbf{A} = 0$, we have

$$|\nabla \mathbf{A}|^2 - |\mathrm{rot}\mathbf{A}|^2 = \mathrm{div}\,\{(\mathbf{A}\cdot\nabla)\mathbf{A}\}. \tag{26}$$

Therefore,

$$\int_\Omega |\nabla \mathbf{A}|^2 dx = \int_\Omega |\mathrm{rot}\mathbf{A}|^2 dx + \int_{\partial\Omega} \{(\mathbf{A}\cdot\nabla)\mathbf{A}\}\cdot \mathbf{n} dS. \tag{27}$$

We estimate the second term on the right-hand side of (27) as

$$\begin{aligned} |\sum_{j,k=1}^{d} \int_{\partial\Omega} A_j \frac{\partial A_k}{\partial x_j}\cdot \mathbf{n}_k dS| &= |\sum_{j,k=1}^{d} \int_{\partial\Omega} A_j A_k \frac{\partial \mathbf{n}_k}{\partial x_j} dS| \leq C\int_{\partial\Omega} |\mathbf{A}|^2 dS \\ &\leq \epsilon\|\nabla\mathbf{A}\|_{\mathbf{L}^2}^2 + C_\epsilon \|\mathbf{A}\|_{\mathbf{L}^2}^2, \end{aligned} \tag{28}$$

for any $\epsilon(<< 1)$. Here $\mathbf{A} = (A_1, A_2, A_3)$ $(d = 3)$, $\mathbf{A} = (A_1, A_2)$ $(d = 2)$ and $C_\epsilon(\geq \epsilon)$ is a positive constant depending on ϵ. Since (27) and (28), we have

$$(1-\epsilon)\|\mathbf{A}\|_{\mathbf{H}^1}^2 \leq \int_\Omega |\mathrm{rot}\mathbf{A}|^2 dx + \beta\|\mathbf{A}\|_{\mathbf{L}^2}^2, \tag{29}$$

where $\beta = 1 - \epsilon + C_\epsilon(\geq 1)$.

We next consider the operator $\mathcal{A}$. Clearly $\mathcal{A}$ is strongly elliptic. Since $\Delta =$

$-\mathrm{rot}^2$+grad div, using (4)-(6) and (29), we have for $U(=(\psi,\mathbf{A})^t)\in C_n^\infty(\Omega)\oplus\mathbf{C}_\sigma^\infty(\Omega)$,

$$\begin{aligned}
&(-\mathcal{A}U,U)_{\mathcal{L}^2}\\
&=\int_\Omega \Delta\psi(x)\overline{\psi(x)}dx+\int_\Omega \Delta\mathbf{A}(x)\cdot\mathbf{A}(x)dx-\beta(\psi,\psi)_{L^2}-\beta(\mathbf{A},\mathbf{A})_{\mathbf{L}^2}\\
&=-\int_\Omega \nabla\psi(x)\cdot\overline{\nabla\psi(x)}dx-\int_\Omega \mathrm{rot}\mathbf{A}(x)\cdot\mathrm{rot}\mathbf{A}(x)dx-\beta(\psi,\psi)_{L^2}-\beta(\mathbf{A},\mathbf{A})_{\mathbf{L}^2}\\
&\le -\|\psi\|_{H^1}^2+\|\psi\|_{L^2}^2-(1-\epsilon)\|\mathbf{A}\|_{\mathbf{H}^1}^2-\beta\|\psi\|_{L^2}^2\\
&\le -(1-\epsilon)\|U\|_{\mathcal{H}^1}^2\\
&\le 0.
\end{aligned}\tag{30}$$

which holds valid for every $U\in\mathcal{H}_\sigma^2(\Omega)$. Therefore, $-\mathcal{A}$ is dissipative.

We define a continuous sesquilinear form on $\mathcal{H}_\sigma^1(\Omega)\times\mathcal{H}_\sigma^1(\Omega)$ by

$$a(U,V)=\int_\Omega\nabla\psi\cdot\overline{\nabla\varphi}dx+\int_\Omega \mathrm{rot}\mathbf{A}\cdot\mathrm{rot}\mathbf{B}\,dx+\beta(\psi,\varphi)_{L^2}+\beta(\mathbf{A},\mathbf{B})_{\mathbf{L}^2}+\lambda(U,V)_{\mathcal{L}^2}\tag{31}$$

where $U=(\psi,\mathbf{A})^t$, $V=(\varphi,\mathbf{B})^t$ and λ is a complex number. We have

$$\begin{aligned}
&|a(U,V)|\\
&\le\left\{\int_\Omega|\nabla\psi|^2dx\right\}^{1/2}\left\{\int_\Omega|\nabla\varphi|^2dx\right\}^{1/2}+\left\{\int_\Omega|\mathrm{rot}\mathbf{A}|^2dx\right\}^{1/2}\left\{\int_\Omega|\mathrm{rot}\mathbf{B}|^2dx\right\}^{1/2}\\
&\quad+\beta\|\psi\|_{L^2}\ \|\varphi\|_{L^2}+\beta\|\mathbf{A}\|_{\mathbf{L}^2}\ \|\mathbf{B}\|_{\mathbf{L}^2}+|\lambda|\|U\|_{\mathcal{L}^2}\ \|V\|_{\mathcal{L}^2}\\
&\le(C_\beta+|\lambda|)\|U\|_{\mathcal{H}^1}\ \|V\|_{\mathcal{H}^1}
\end{aligned}\tag{32}$$

where $C_\beta=\max\{2,\beta\}$. For $0\le|\lambda|<1-\epsilon$, we obtain

$$\begin{aligned}
|a(U,U)| &\ge \left|\int_\Omega\nabla\psi(x)\cdot\overline{\nabla\psi(x)}dx+\int_\Omega\mathrm{rot}\mathbf{A}(x)\cdot\mathrm{rot}\mathbf{A}(x)dx\right.\\
&\quad\left.+\beta(\psi,\psi)_{L^2}+\beta(\mathbf{A},\mathbf{A})_{\mathbf{L}^2}\right|-|\lambda|\|U\|_{\mathcal{L}^2}^2\\
&\ge (1-\epsilon-|\lambda|)\|U\|_{\mathcal{H}^1}^2.
\end{aligned}\tag{33}$$

If $0\le|\lambda|<1-\epsilon$, then $a(U,V)$ is coercive. We may apply the classical Lax-Milgram theorem to obtain a unique weak solution $U\in\mathcal{H}_\sigma^1(\Omega)$ to the boundary value problem

$$\mathcal{A}U+\lambda U=f\tag{34}$$

for all $f\in\mathcal{L}^2(\Omega)$ and $0\le|\lambda|<1-\epsilon$. Especially, this solution belongs to $H^2(\Omega)\oplus\mathbf{H}^2(\Omega)$. This implies that the range of $\lambda I+\mathcal{A}$ is $\mathcal{L}^2(\Omega)$ for all $0<|\lambda|<1-\epsilon$.

Clearly, $D(-\mathcal{A})\supset C_n^\infty(\Omega)\oplus\mathbf{C}_\sigma^\infty(\Omega)$. Since $C_n^\infty(\Omega)\oplus\mathbf{C}_\sigma^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$, it follows that $D(-\mathcal{A})$ is dense in $\mathcal{L}^2(\Omega)$.

From the Lumer-Phillips theorem, it follows that $-\mathcal{A}$ is the infinitesimal generator of C_0 semigroup of contraction on $\mathcal{L}^2(\Omega)$. Q.E.D.

Since $\mathcal{A}$ is self-adjoint and non-negative, operator $-\mathcal{A}$ is infinitesimal of an analytic semigroup on $\mathcal{L}^2(\Omega)$. Therefore we may define fractional powers of $\mathcal{A}$. For $0 < \alpha < 1$, we define the path of integration $\mathcal{C}$ into the upper and lower sides of negative real axis and obtain

$$\mathcal{A}^{-\alpha}U \equiv \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{-\alpha}(tI+\mathcal{A})^{-1}U\,dt, \qquad 0<\alpha<1. \tag{35}$$

If $U \in D(\mathcal{A})$, then we define

$$\mathcal{A}^{\alpha}U \equiv \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{\alpha-1}\mathcal{A}(tI+\mathcal{A})^{-1}U\,dt, \qquad 0<\alpha<1. \tag{36}$$

Using (30) and Schwarz's inequality, we have

$$\begin{aligned} \|\mathcal{A}U\|_{\mathcal{L}^2}\,\|U\|_{\mathcal{L}^2} &\geq (\mathcal{A}U,U)_{\mathcal{L}^2} \\ &\geq (1-\epsilon)\|U\|^2_{\mathcal{H}^1} \geq (1-\epsilon)\|U\|^2_{\mathcal{L}^2} \\ &\geq \delta\|U\|^2_{\mathcal{L}^2}, \end{aligned} \tag{37}$$

for some $0<\delta\leq 1-\epsilon$. Therefore,

$$\|\mathcal{A}^{-1}U\|_{\mathcal{L}^2} \leq \delta^{-1}\|U\|_{\mathcal{L}^2}. \tag{38}$$

$D(\mathcal{A}^\alpha)$ endowed with the graph norm of $\mathcal{A}^\alpha$, is a Banach space. Since $\mathcal{A}^\alpha$ is invertible, its graph norm $|||\cdot|||$ is equivalent to the norm $\|U\|_\alpha = \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}$. Thus, $D(\mathcal{A}^\alpha)$ equipped with the norm $\|\cdot\|_\alpha$ is a Banach space, which we also denote by $D(\mathcal{A}^\alpha)$. It is clearly that $D(\mathcal{A}^\alpha)\simeq \mathcal{H}^{2\alpha}_\sigma(\Omega)$ and

$$\|U\|_{\mathcal{H}^{2\alpha}_\sigma} \leq C\|U\|_\alpha = C\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}, \tag{39}$$

(See [12]). When $d=3$, using Sobolev's inequality and Hölder's inequality, we have

$$\|U\|_{\mathcal{W}^{1,12/5}} \leq C\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}, \quad \text{for all } \alpha\geq\frac{5}{8}, \tag{40}$$

$$\|U\|_{\mathcal{L}^p} \leq C\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}, \quad \text{for all } \alpha\geq\frac{5}{8}, \tag{41}$$

for any $1\leq p\leq 12$. When $d=2$, we have

$$\|U\|_{\mathcal{L}^p} \leq C\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}, \quad \text{for all } \alpha\geq\frac{1}{2}, \tag{42}$$

$$\|U\|_{\mathcal{H}^1} \leq C\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}, \quad \text{for all } \alpha\geq\frac{1}{2} \tag{43}$$

for any $1\leq p\leq +\infty$.

We now turn to the nonlinear term of (22). We have

Proposition 2 *If $d = 3$, $\alpha \geq \frac{5}{8}$ and $U \in D(\mathcal{A})$, then $\mathcal{B}(U)$ is well-defined and*

$$\|\mathcal{B}(U)\|_{\mathcal{L}^2} \leq C\left\{\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2} + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^2 + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^3 + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^4\right\}, \tag{44}$$

where C is a constant depending on β, κ, and Ω. Moreover, if $U, V \in D(\mathcal{A})$, then

$$\|\mathcal{B}(U) - \mathcal{B}(V)\|_{\mathcal{L}^2} \leq C\|\mathcal{A}^\alpha U - \mathcal{A}^\alpha V\|_{\mathcal{L}^2}, \tag{45}$$

where C is a constant depending on β, κ, $\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}$, $\|\mathcal{A}^\alpha V\|_{\mathcal{L}^2}$ and Ω.

Proof of Proposition 2. We note that $D(\mathcal{A}) \subset \mathcal{H}^2(\Omega)$. From Sobolev's theorem, it follows that $U \in \mathcal{L}^\infty(\Omega)$. Therefore, $\mathcal{B}(U) \in \mathcal{L}^2(\Omega)$ and is well-defined. Let $U = (\psi, \mathbf{A})^t$ and $V = (\varphi, \mathbf{B})^t$. Note that, from Lemma 1, Φ is uniquely defined by $(\psi, \mathbf{A})$. Let

$$\Phi_1 = \Phi(\psi, \mathbf{A}), \qquad \Phi_2 = \Phi(\varphi, \mathbf{B}).$$

From (14), we have

$$\begin{aligned}
\|\nabla\Phi_1\|_{\mathbf{L}^2}^2 &= -\int_\Omega \Phi_1 \operatorname{div}\left\{\frac{i}{2}\{\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\}\right\} dx \\
&= \int_\Omega \nabla\Phi_1 \cdot \left\{\frac{i}{2}\{\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\}\right\} dx \\
&\leq \|\nabla\Phi_1\|_{\mathbf{L}^2} \|\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\|_{\mathbf{L}^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\nabla\Phi_1\|_{\mathbf{L}^2} &\leq \|\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\|_{\mathbf{L}^2}, &(46)\\
\|\Phi_1\|_{L^2} &\leq C\|\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\|_{\mathbf{L}^2}. &(47)
\end{aligned}$$

From Gagliardo-Nirenberg's inequality, we have

$$\begin{aligned}
\|\Phi_1\|_{L^3} &\leq \|\nabla\Phi_1\|_{\mathbf{L}^2}^{1/2} \|\Phi_1\|_{L^2}^{1/2} \leq C\|\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\|_{\mathbf{L}^2} \\
&\leq C\left\{2\|\psi\|_{L^{12}} \|\psi\|_{W^{1,12/5}} + \|\psi\|_{L^6}^2 \|\mathbf{A}\|_{\mathbf{L}^6}\right\}. &(48)
\end{aligned}$$

We now estimate

$$\begin{aligned}
\|\mathcal{B}(U)\|_{\mathcal{L}^2} \leq C\Big\{&\|\psi\|_{L^2} + \|\mathbf{A}\|_{\mathbf{L}^2} + \|\mathbf{A}\cdot\nabla\psi\|_{L^2} + \||\mathbf{A}|^2\psi\|_{L^2} + \||\psi|^2\psi\|_{L^2} \\
&+ \|\Phi_1\psi\|_{L^2} + 2\|\psi\overline{\nabla\psi}\|_{\mathbf{L}^2} + \||\psi|^2\mathbf{A}\|_{\mathbf{L}^2}\Big\}. \qquad (49)
\end{aligned}$$

Using (40) and (41), we have

$$\begin{aligned}
\|\mathcal{B}(U)\|_{\mathcal{L}^2} \le C\Big\{ &\|\psi\|_{L^2} + \|\mathbf{A}\|_{\mathbf{L}^2} + \|\mathbf{A}\|_{\mathbf{L}^{12}}\ \|\psi\|_{W^{1,12/5}} + \|\mathbf{A}\|^2_{\mathbf{L}^6}\ \|\psi\|_{L^6} \\
&+\|\psi\|^3_{L^6} + (2\|\psi\|_{L^{12}}\ \|\psi\|_{W^{1,12/5}} + \|\psi\|^2_{L^6}\ \|\mathbf{A}\|_{\mathbf{L}^6})\ \|\psi\|_{L^6} \\
&+2\|\psi\|_{L^{12}}\ \|\psi\|_{W^{1,12/5}} + \|\psi\|^2_{L^6}\ \|\mathbf{A}\|_{\mathbf{L}^6}\Big\} \\
\le C\Big\{ &\|U\|_{\mathcal{L}^2} + \|U\|_{\mathcal{L}^2} + \|U\|_{\mathcal{L}^{12}}\ \|U\|_{\mathcal{W}^{1,12/5}} + \|U\|^2_{\mathcal{L}^6}\ \|U\|_{\mathcal{L}^6} \\
&+\|U\|^3_{\mathcal{L}^6} + (2\|U\|_{\mathcal{L}^{12}}\ \|U\|_{\mathcal{W}^{1,12/5}} + \|U\|^3_{\mathcal{L}^6})\ \|U\|_{\mathcal{L}^6} \\
&+2\|U\|_{\mathcal{L}^{12}}\ \|U\|_{\mathcal{W}^{1,12/5}} + \|U\|^3_{\mathcal{L}^6}\Big\} \\
\le C'\Big\{ &\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2} + \|\mathcal{A}^\alpha U\|^2_{\mathcal{L}^2} + \|\mathcal{A}^\alpha U\|^3_{\mathcal{L}^2} + \|\mathcal{A}^\alpha U\|^4_{\mathcal{L}^2}\Big\},
\end{aligned} \tag{50}$$

for all $\alpha \ge \frac{5}{8}$. Thus we have the first assertion.

Next, we show (45). Suppose that

$$\begin{aligned}
\|\mathcal{B}(U) - \mathcal{B}(V)\|_{\mathcal{L}^2} \ \le\ & C\Big\{\|\psi - \varphi\|_{L^2} + \|\mathbf{A} - \mathbf{B}\|_{\mathbf{L}^2} \\
&+\|\mathbf{A}\cdot\nabla\psi - \mathbf{B}\cdot\nabla\varphi\|_{L^2} + \||\mathbf{A}|^2\psi - |\mathbf{B}|^2\varphi\|_{L^2} \\
&+\||\varphi|^2\varphi - |\psi|^2\psi\|_{L^2} + \|\Phi_1\psi - \Phi_2\varphi\|_{L^2} \\
&+2\|\psi(\overline{\nabla\psi}) - \varphi(\overline{\nabla\varphi})\|_{\mathbf{L}^2} + \||\psi|^2\mathbf{A} - |\varphi|^2\mathbf{B}\|_{\mathbf{L}^2}\Big\} \\
=\ & C\Big\{I_0 + I_1 + I_2 + I_3 + I_4 + 2I_5 + I_6\Big\},
\end{aligned} \tag{51}$$

where C is a constant depending on β and λ,

$$\begin{aligned}
I_0 &= \|\psi - \varphi\|_{L^2} + \|\mathbf{A} - \mathbf{B}\|_{\mathbf{L}^2}, \\
I_1 &= \|\mathbf{A}\cdot\nabla\psi - \mathbf{B}\cdot\nabla\varphi\|_{L^2}, \\
I_2 &= \||\mathbf{A}|^2\psi - |\mathbf{B}|^2\varphi\|_{L^2}, \\
I_3 &= \||\varphi|^2\varphi - |\psi|^2\psi\|_{L^2}, \\
I_4 &= \|\Phi_1\psi - \Phi_2\varphi\|_{L^2}, \\
I_5 &= \|\psi(\overline{\nabla\psi}) - \varphi(\overline{\nabla\varphi})\|_{\mathbf{L}^2}, \\
I_6 &= \||\psi|^2\mathbf{A} - |\varphi|^2\mathbf{B}\|_{\mathbf{L}^2}.
\end{aligned}$$

I_0 is estimated as follows:

$$I_0 = \|U - V\|_{\mathcal{L}^2} \le \|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}, \qquad \text{for all } \gamma \ge \frac{5}{8}. \tag{52}$$

I_1 and I_5 are estimated as follows:

$$
\begin{aligned}
I_1 &\le \|\mathbf{A}\cdot\nabla\psi - \mathbf{A}\cdot\nabla\varphi\|_{L^2} + \|\mathbf{A}\cdot\nabla\varphi - \mathbf{B}\cdot\nabla\varphi\|_{L^2} \\
&= \|\mathbf{A}\cdot(\nabla\psi - \nabla\varphi)\|_{L^2} + \|(\mathbf{A}-\mathbf{B})\cdot\nabla\varphi\|_{L^2} \\
&\le \|\mathbf{A}\|_{\mathbf{L}^{12}}\ \|\nabla(\psi-\varphi)\|_{\mathbf{L}^{12/5}} + \|\mathbf{A}-\mathbf{B}\|_{\mathbf{L}^{12}}\ \|\nabla\varphi\|_{\mathbf{L}^{12/5}} \\
&\le \|U\|_{\mathcal{L}^{12}}\ \|\nabla(U-V)\|_{\mathcal{L}^{12/5}} + \|U-V\|_{\mathcal{L}^{12}}\ \|\nabla V\|_{\mathcal{L}^{12/5}}, \\
&\le \|U\|_{\mathcal{L}^{12}}\ \|U-V\|_{\mathcal{W}^{1,12/5}} + \|U-V\|_{\mathcal{L}^{12}}\ \|V\|_{\mathcal{W}^{1,12/5}}.
\end{aligned}
\tag{53}
$$

Using (40) and (41), we have

$$
\begin{aligned}
I_1 &\le C\Big\{\|\mathcal{A}^\gamma U\|_{\mathcal{L}^2}\ \|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2} + \|\mathcal{A}^\gamma V\|_{\mathcal{L}^2}\ \|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}\Big\} \\
&\le C_1\|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}, \qquad \text{for all } \gamma \ge \frac{5}{8}.
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
I_5 &\le \|\psi(\overline{\nabla\psi}) - \psi(\overline{\nabla\varphi})\|_{\mathbf{L}^2} + \|\psi(\overline{\nabla\varphi}) - \varphi(\overline{\nabla\varphi})\|_{\mathbf{L}^2} \\
&\le \|\psi\|_{L^{12}}\ \|\overline{\nabla(\psi-\varphi)}\|_{\mathbf{L}^{12/5}} + \|\overline{\nabla\varphi}\|_{\mathbf{L}^{12/5}}\ \|\psi-\varphi\|_{L^{12}} \\
&\le \|U\|_{\mathcal{L}^{12}}\ \|\overline{\nabla(U-V)}\|_{\mathcal{L}^{12/5}} + \|\overline{\nabla U}\|_{\mathcal{L}^{12/5}}\ \|U-V\|_{\mathcal{L}^{12}} \\
&\le \|U\|_{\mathcal{L}^{12}}\ \|\overline{U-V}\|_{\mathcal{W}^{1,12/5}} + \|\overline{U}\|_{\mathcal{W}^{1,12/5}}\ \|U-V\|_{\mathcal{L}^{12}} \\
&\le C\Big\{\|\mathcal{A}^\gamma U\|_{\mathcal{L}^2}\ \|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2} + \|\mathcal{A}^\gamma V\|_{\mathcal{L}^2}\ \|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}\Big\} \\
&\le C_5\|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}, \qquad \text{for all } \gamma \ge \frac{5}{8}.
\end{aligned}
\tag{55}
$$

Using (41), we obtain the following estimates of I_2 and I_6:

$$
\begin{aligned}
I_2 &\le \||\mathbf{A}|^2\psi - |\mathbf{B}|^2\psi\|_{L^2} + \||\mathbf{B}|^2\psi - |\mathbf{B}|^2\varphi\|_{L^2} \\
&\le \||\mathbf{A}|^2 - |\mathbf{B}|^2\|_{\mathbf{L}^3}\ \|\psi\|_{L^6} + \||\mathbf{B}|^2\|_{\mathbf{L}^3}\ \|\psi-\varphi\|_{L^6} \\
&\le (\|\mathbf{A}\|_{\mathbf{L}^6} + \|\mathbf{B}\|_{\mathbf{L}^6})\ \|\mathbf{A}-\mathbf{B}\|_{\mathbf{L}^6}\ \|\psi\|_{L^6} + \|\mathbf{B}\|^2_{\mathbf{L}^6}\ \|\psi-\varphi\|_{L^6} \\
&\le C\|U-V\|_{\mathcal{L}^6} \le C_2\|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}, \qquad \text{for all } \gamma \ge \frac{5}{8},
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
I_6 &\le \|(|\psi|^2 - |\varphi|^2)\mathbf{A}\|_{\mathbf{L}^2} + \||\varphi|^2(\mathbf{A}-\mathbf{B})\|_{\mathbf{L}^2} \\
&\le \||\psi|^2 - |\varphi|^2\|_{L^3}\ \|\mathbf{A}\|_{\mathbf{L}^6} + \||\varphi|^2\|_{L^3}\ \|\mathbf{A}-\mathbf{B}\|_{\mathbf{L}^6} \\
&\le \||\psi| - |\varphi|\|_{L^6}\ \||\psi| + |\varphi|\|_{L^6}\ \|\mathbf{A}\|_{\mathbf{L}^6} + \|\varphi\|^2_{L^6}\ \|\mathbf{A}-\mathbf{B}\|_{\mathbf{L}^6} \\
&\le (\|\psi\|_{L^6} + \|\varphi\|_{L^6})\ \|\psi-\varphi\|_{L^6}\ \|\mathbf{A}\|_{\mathbf{L}^6} + \|\varphi\|^2_{L^6}\ \|\mathbf{A}-\mathbf{B}\|_{\mathbf{L}^6} \\
&\le C\|U-V\|_{\mathcal{L}^6} \le C_6\|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}, \qquad \text{for all } \gamma \ge \frac{5}{8}.
\end{aligned}
\tag{57}
$$

Next, we estimate I_3 as follows:

$$
\begin{aligned}
I_3 &\leq \||\varphi|^2\varphi - |\varphi|^2\psi\|_{L^2} + \||\varphi|^2\psi - |\psi|^2\psi\|_{L^2} \\
&\leq \|\varphi\|_{L^6}^2 \|\varphi - \psi\|_{L^6} + \|\psi\|_{L^6} \||\varphi| - |\psi|\|_{L^6} \||\varphi| + |\psi|\|_{L^6} \\
&\leq \|\varphi\|_{L^6}^2 \|\varphi - \psi\|_{L^6} + \|\psi\|_{L^6} \|\varphi - \psi\|_{L^6} (\|\varphi\|_{L^6} + \|\psi\|_{L^6}) \\
&\leq C\|U - V\|_{\mathcal{L}^6} \leq C_6\|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}, \qquad \text{for all } \gamma \geq \frac{5}{8}. \qquad (58)
\end{aligned}
$$

Finally, we estimate I_4. From Poincaré's inequality, we have

$$\|\Phi_1 - \Phi_2\|_{L^2} \leq C'\|\nabla(\Phi_1 - \Phi_2)\|_{\mathbf{L}^2}, \tag{59}$$

where C' is a constant depending on Ω. Suppose that

$$f = \frac{i}{2}\left\{\overline{\psi}(\mathbf{D}_A\psi) - \psi(\overline{\mathbf{D}_A\psi})\right\} \tag{60}$$

$$g = \frac{i}{2}\left\{\overline{\varphi}(\mathbf{D}_B\varphi) - \varphi(\overline{\mathbf{D}_B\varphi})\right\} \tag{61}$$

From (14), we have

$$\Delta(\Phi_1 - \Phi_2) = \operatorname{div}(f - g). \tag{62}$$

Now,

$$
\begin{aligned}
\|\nabla(\Phi_1 - \Phi_2)\|_{\mathbf{L}^2}^2 &= -\int_\Omega (\Phi_1 - \Phi_2) \operatorname{div}(f - g)\, dx \\
&= \int_\Omega (\nabla(\Phi_1 - \Phi_2))\, (f - q)\, dx \\
&\leq \|\nabla(\Phi_1 - \Phi_2)\|_{\mathbf{L}^2} \|f - g\|_{\mathbf{L}^2} \qquad (63)
\end{aligned}
$$

From (59), (63) and estimates of I_5 and I_6, we have

$$\|\nabla(\Phi_1 - \Phi_2)\|_{\mathbf{L}^2} \leq \|f - g\|_{\mathbf{L}^2} \leq C_5 I_5 + C_6 I_6 \leq K\|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}, \tag{64}$$

$$\|\Phi_1 - \Phi_2\|_{\mathbf{L}^2} \leq C'\|f - g\|_{\mathbf{L}^2} \leq C'C_5 I_5 + C'C_6 I_6 \leq K'\|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}. \tag{65}$$

for all $\gamma \geq 5/8$, where K and K' are constants. Therefore, from (46), (47), (64), (65) and Gagliardo-Nirenberg's inequality, we have

$$
\begin{aligned}
I_4 &\leq \|\Phi_1\psi - \Phi_2\psi\|_{L^2} + \|\Phi_2\psi - \Phi_2\varphi\|_{L^2} \\
&\leq \|\Phi_1 - \Phi_2\|_{L^{12/5}} \|\psi\|_{L^{12}} + \|\Phi_2\|_{L^{12/5}} \|\psi - \varphi\|_{L^{12}} \\
&\leq \|\nabla(\Phi_1 - \Phi_2)\|_{\mathbf{L}^2}^{1/4} \|\Phi_1 - \Phi_2\|_{L^2}^{3/4} \|\psi\|_{L^{12}} + \|\nabla\Phi_2\|_{\mathbf{L}^2}^{1/4} \|\Phi_2\|_{L^2}^{3/4} \|\psi - \varphi\|_{L^{12}} \\
&\leq C\|\mathcal{A}^\gamma U - \mathcal{A}^\gamma V\|_{\mathcal{L}^2}, \qquad \text{for all } \gamma \geq \frac{5}{8}. \qquad (66)
\end{aligned}
$$

Combining (52), (54)-(58) and (66), the proof of Proposition 3 is completed. Q.E.D.

Using (42) and (43), we have

Proposition 3 *If $d = 2$, $\alpha \geq 1/2$ and $U \in D(\mathcal{A})$, then $\mathcal{B}(U)$ is well-defined and*

$$\|\mathcal{B}(U)\|_{\mathcal{L}^2} \leq C\left\{\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2} + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^2 + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^3 + \|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}^4\right\}, \tag{67}$$

where C is a constant depending on β, κ, and Ω. Moreover, if $U, V \in D(\mathcal{A})$ then

$$\|\mathcal{B}(U) - \mathcal{B}(V)\|_{\mathcal{L}^2} \leq C\|\mathcal{A}^\alpha U - \mathcal{A}^\alpha V\|_{\mathcal{L}^2}, \tag{68}$$

where C is a constant depending on β, κ, $\|\mathcal{A}^\alpha U\|_{\mathcal{L}^2}$, $\|\mathcal{A}^\alpha V\|_{\mathcal{L}^2}$ and Ω.

The proof is similar to those of the case of $d = 3$ and omit it.

From (44) and (67), it follows that the domain of $\mathcal{B}$ can be extended by continuity to $D(\mathcal{A}^\alpha)$ ($\alpha \geq \frac{5}{8}$ ($d = 3$), $\alpha \geq 1/2$ ($d = 2$)). Therefore, (44), (45), (67) and (68) hold for every $U, V \in D(\mathcal{A}^\alpha)$. Therefore, the condition of Theorem 1 is satisfied. We complete the proof of Theorem 2. Q.E.D.

3 Proof of Theorem 3.

For $d = 2$ and $U_0 \in D(\mathcal{A}^{1/2})$, from Theorem 2, we have a local strong solution $U \in C([0, T_0) : \mathcal{L}^2(\Omega)) \cap C((0, T_0) : D(\mathcal{A})) \cap C^1((0, T_0) : \mathcal{L}^2(\Omega))$ $T_0 = T_0(\|U_0\|_{1/2})$. In order to establish global existence of solution, it is sufficient to obtain a propri estimate

$$\sup_{0 \leq t < T} \|\mathcal{A}^{1/2} U\|_{\mathcal{L}^2} \leq C, \tag{69}$$

for any $T > 0$.

From (1)-(2) with (3)-(10), we have

$$\begin{aligned}
&\|\mathbf{D}_A\psi(t)\|_{\mathbf{L}^2}^2 + \|\mathrm{rot}\mathbf{A}(t)\|_{\mathbf{L}^2}^2 + \frac{\kappa}{2}\|1 - |\psi(t)|^2\|_{L^2}^2 \\
&\quad + \int_0^t \|\psi_t - i\Phi\psi\|_{L^2}^2 dt + 2\int_0^t \|\mathbf{A}_t - \nabla\Phi\|_{\mathbf{L}^2}^2 dt \\
&= \|\mathbf{D}_{A_0}\psi_0\|_{\mathbf{L}^2}^2 + \|\mathrm{rot}\mathbf{A}\|_{\mathbf{L}^2}^2 + \frac{\kappa}{2}\|1 - |\psi_0|^2\|_{L^2} \\
&\leq 2\|\nabla\psi_0\|_{\mathbf{L}^2}^2 + 2\|\mathbf{A}_0\|_{\mathbf{L}^4}^2 \, \|\psi_0\|_{L^4}^2 + 2\|\nabla\mathbf{A}_0\|_{\mathbf{L}^2}^2 + \kappa|\Omega| + \kappa\|\psi_0\|_{L^4}^4.
\end{aligned} \tag{70}$$

Thus, we have

$$\sup_{0 \leq t < \infty} \|\mathbf{D}_A\psi\|_{\mathbf{L}^2} \leq C, \tag{71}$$

$$\sup_{0 \leq t < \infty} \|\mathrm{rot}\mathbf{A}\|_{\mathbf{L}^2} \leq C, \tag{72}$$

$$\sup_{0 \leq t < \infty} \|1 - |\psi|^2\|_{L^2} \leq C, \tag{73}$$

$$\int_0^T \|\psi_t - i\Phi\psi\|_{L^2}^2 dt \leq C, \tag{74}$$
$$\int_0^T \|\mathbf{A}_t - \nabla\Phi\|_{\mathbf{L}^2}^2 dt \leq C, \tag{75}$$

for any $T > 0$, here C is a constant which depends only on $\|U_0\|_{\mathcal{H}^1}$ and Ω. Since $\|\nabla\mathbf{A}\|_{\mathbf{L}^2}$ and $\|\mathrm{rot}\mathbf{A}\|_{\mathbf{L}^2}$ are equivalent norms in $\mathbf{H}^1(\Omega) \cap \mathbf{H}_\sigma(\Omega)$ and $\|\mathbf{A}\|_{\mathbf{L}^2} \leq C\|\mathrm{rot}\mathbf{A}\|_{\mathbf{L}^2}$ for any $\mathbf{A} \in \mathbf{H}^1(\Omega) \cap \mathbf{H}_\sigma(\Omega)$, (72) yields that

$$\sup_{0\leq t<\infty} \|\mathbf{A}(t)\|_{\mathbf{H}^1} \leq C. \tag{76}$$

Using Sobolev's inequality, we have

$$\sup_{0\leq t<\infty} \|\mathbf{A}(t)\|_{\mathbf{L}^p} \leq C, \qquad \text{for any } 1 \leq p \leq 6 \ (d = 3), \tag{77}$$
$$\sup_{0\leq t<\infty} \|\mathbf{A}(t)\|_{\mathbf{L}^p} \leq C, \qquad \text{for any } 1 \leq p \leq \infty \ (d = 2). \tag{78}$$

Hence, we have

$$\|\psi(t)\|_{L^4}^4 \leq 2\|1 - |\psi|^2\|_{L^2}^2 + 2|\Omega| \tag{79}$$

From (73), we get

$$\sup_{0\leq t<\infty} \|\psi(t)\|_{L^4} \leq C. \tag{80}$$

Now,

$$\begin{aligned} \|\nabla\psi\|_{\mathbf{L}^2} &\leq \|\nabla\psi - i\mathbf{A}\psi\|_{\mathbf{L}^2} + \|\mathbf{A}\psi\|_{\mathbf{L}^2} \\ &\leq \|\mathbf{D}_A\psi\|_{\mathbf{L}^2} + 2\|\mathbf{A}\|_{\mathbf{L}^4}^2 + 2\|\psi\|_{L^4}^2 \end{aligned} \tag{81}$$

Therefore, from (71), (77) and (80), we have

$$\sup_{0\leq t<\infty} \|\nabla\psi(t)\|_{\mathbf{L}^2} \leq C. \tag{82}$$

Since, $\|\psi\|_{L^2} \leq |\Omega|^{1/4}\|\psi\|_{L^4}$, we have

$$\sup_{0\leq t<\infty} \|\psi(t)\|_{L^2} \leq C. \tag{83}$$

Therefore, from (82) and (83), it follows that

$$\sup_{0\leq t<\infty} \|\psi(t)\|_{H^1} \leq C. \tag{84}$$

Hence, we obtain

$$\sup_{0\leq t<\infty} \|U(t)\|_{\mathcal{H}^1}^2 \leq \sup_{0\leq t<\infty} \|\psi(t)\|_{H^1}^2 + \sup_{0\leq t<\infty} \|\mathbf{A}(t)\|_{\mathbf{H}^1}^2 \leq M_0, \tag{85}$$

where M_0 is a constant which depends only on $\|U_0\|_{\mathcal{H}^1}$ and Ω. but not on t. This implies

$$\sup_{0\leq t<\infty} \|\mathcal{A}^{1/2}U\|_{\mathcal{L}^2} < C. \tag{86}$$

This completes the proof of Theorem 3.

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