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Stability of solitary waves for coupled Klein-Gordon-Schrödinger equations in one space dimension

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1 Introduction and Main Result

In this note we consider the stability of solitary wave solutions for the Yukawa coupled Klein-Gordon-Schrödinger equations in one space dimension:

$$i\partial_t u+\partial_x^2 u=uv, \qquad (t,x)\in \mathbf{R}\times\mathbf{R}, \tag{1.1}$$

$$\partial_t^2 v-\partial_x^2 v+v=-|u|^2, \qquad (t,x)\in \mathbf{R}\times\mathbf{R}. \tag{1.2}$$

Here, $u=u(t,x)$ and $v=v(t,x)$ describe a complex scalar neucleon field and a real scalar meson field, respectively (see Fukuda and M. Tsutsumi [3] and Yukawa [11]). In Section 5 of [3], Fukuda and M. Tsutsumi showed that (1.1)–(1.2) admits the following two types of exact solitary wave solutions (1.3)–(1.4) and (1.5)–(1.6):

(I) when $\lambda^2<1$ and $\mu=\lambda^2/4+1/(1-\lambda^2)$,

$$u(t,x)=\frac{3}{2\sqrt{1-\lambda^2}}\operatorname{sech}^2\left(\frac{x-\lambda t}{2\sqrt{1-\lambda^2}}\right)\exp[i\mu t+i(\lambda/2)(x-\lambda t)], \tag{1.3}$$

$$v(t,x)=-\frac{3}{2(1-\lambda^2)}\operatorname{sech}^2\left(\frac{x-\lambda t}{2\sqrt{1-\lambda^2}}\right), \tag{1.4}$$

(II) when $\lambda^2 = 1$ and $\mu > 1/4$,

$$\begin{aligned} u(t,x) &= \sqrt{2(\mu - 1/4)}\,\mathrm{sech}\left(\sqrt{\mu - 1/4}\,(x - \lambda t)\right) \\ &\quad \times \exp[i\mu t + i(\lambda/2)(x - \lambda t)], \qquad (1.5) \\ v(t,x) &= -2(\mu - 1/4)\,\mathrm{sech}^2\left(\sqrt{\mu - 1/4}\,(x - \lambda t)\right), \qquad (1.6) \end{aligned}$$

and they proposed a problem of whether the solitary wave solutions are stable or not. The purpose of this note is to give a partial answer to the problem.

To explain our results precisely, we prepare some function spaces and functionals. Let $X = H^1(\mathbf{R};\mathbf{C}) \times H^1(\mathbf{R};\mathbf{R}) \times L^2(\mathbf{R};\mathbf{R})$ be a real Hilbert space with the inner product

$$\begin{aligned} ((u,v,w),(\psi,\phi,\varphi))_X &= 2\,\mathrm{Re}\int_{\mathbf{R}} \left(\partial_x u(x)\overline{\partial_x \psi(x)} + u(x)\overline{\psi(x)}\right) dx \\ &\quad + \int_{\mathbf{R}} (\partial_x v(x)\partial_x \phi(x) + v(x)\phi(x) + w(x)\varphi(x))\, dx \end{aligned}$$

for (u,v,w), $(\psi,\phi,\varphi) \in X$. Then, (1.1)–(1.2) is written as abstract Hamiltonian system in the form

$$\frac{d}{dt}\vec{u}(t) = JE'(\vec{u}(t)), \qquad (1.7)$$

where

$$\vec{u}(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \in X, \quad J = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and E is the energy functional on X defined by

$$E(u,v,w) = \int_{\mathbf{R}} \left\{ |\partial_x u|^2 + |u|^2 v + \frac{1}{2}\left(w^2 + (\partial_x v)^2 + v^2\right)\right\} dx. \qquad (1.8)$$

The energy functional $E(\vec{u})$ is invariant under the action $T(\alpha,\beta)$ of the group defined by

$$T(\alpha,\beta)(u,v,w)(x) = (e^{i\beta}u(x+\alpha), v(x+\alpha), w(x+\alpha))$$

for α, $\beta \in \mathbf{R}$ and $(u,v,w) \in X$. Associated with the group action $T(\alpha,\beta)$, we define two conserved functionals on X, the momentum P and the charge Q, by

$$P(u,v,w) = \int_{\mathbf{R}} (i\bar{u}\partial_x u + w\partial_x v)\, dx, \qquad (1.9)$$

$$Q(u,v,w) = \int_{\mathbf{R}} |u|^2 dx. \qquad (1.10)$$

The following global existence of solutions of the Cauchy problem for (1.7) is known.

Proposition 1.1 *For any $\vec{u}_0 \in X$ there exists a unique solution $\vec{u} \in C(\mathbf{R}; X)$ of* (1.7) *with $\vec{u}(0) = \vec{u}_0$ satisfying*

$$E(\vec{u}(t)) = E(\vec{u}_0), \quad P(\vec{u}(t)) = P(\vec{u}_0), \quad Q(\vec{u}(t)) = Q(\vec{u}_0), \quad t \in \mathbf{R}. \tag{1.11}$$

For λ, $\mu \in \mathbf{R}$, we put

$$S_{\lambda,\mu}(\vec{u}) = E(\vec{u}) + \lambda P(\vec{u}) + \mu Q(\vec{u}), \quad \vec{u} \in X.$$

Then, we have

$$\begin{aligned} S_{\lambda,\mu}(u,v,w) &= \int_{\mathbf{R}} \Big\{ \left| \partial_x (e^{-i\lambda x/2} u) \right|^2 + (\mu - \lambda^2/4)|u|^2 + |u|^2 v \\ &\quad + \frac{1}{2}\left((w + \lambda \partial_x v)^2 + (1-\lambda^2)(\partial_x v)^2 + v^2 \right) \Big\} \, dx, \end{aligned} \tag{1.12}$$

and $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$ is a solution of (1.7) if $\vec{\psi}_{\lambda,\mu}$ is a solution of $S'_{\lambda,\mu}(\vec{\psi}) = 0$. Note that $S'_{\lambda,\mu}(\psi, \phi, \varphi) = E'(\psi,\phi,\varphi) + \lambda P'(\psi,\phi,\varphi) + \mu Q'(\psi,\phi,\varphi) = 0$ is equivalent to

$$-\partial_x^2 \psi + i\lambda \partial_x \psi + \mu \psi + \psi \phi = 0, \tag{1.13}$$

$$-(1-\lambda^2)\partial_x^2 \phi + \phi + |\psi|^2 = 0, \tag{1.14}$$

$$\varphi + \lambda \partial_x \phi = 0, \tag{1.15}$$

and also that by $\psi(x) = e^{i(\lambda/2)x}\tilde{\psi}(x)$, (1.13) is transformed into

$$-\partial_x^2 \tilde{\psi} + (\mu - \lambda^2/4)\tilde{\psi} + \tilde{\psi}\phi = 0. \tag{1.16}$$

Thus, when $\lambda^2 < 1$ and $\mu = \lambda^2/4 + 1/(1-\lambda^2)$, if we put

$$\psi_{\lambda,\mu}(x) = \frac{3}{2\sqrt{1-\lambda^2}} \operatorname{sech}^2 \left(\frac{x}{2\sqrt{1-\lambda^2}} \right) \exp[i(\lambda/2)x], \tag{1.17}$$

$$\phi_{\lambda,\mu}(x) = -\frac{3}{2(1-\lambda^2)} \operatorname{sech}^2 \left(\frac{x}{2\sqrt{1-\lambda^2}} \right), \tag{1.18}$$

$$\varphi_{\lambda,\mu}(x) = -\lambda \partial_x \phi_{\lambda,\mu}(x), \tag{1.19}$$

then $\vec{\psi}_{\lambda,\mu} = (\psi_{\lambda,\mu}, \phi_{\lambda,\mu}, \varphi_{\lambda,\mu})$ is a solution of $S'_{\lambda,\mu}(\vec{\psi}) = 0$, and when $\lambda^2 = 1$ and $\mu > 1/4$, if we put

$$\psi_{\lambda,\mu}(x) = \sqrt{2(\mu - 1/4)} \operatorname{sech}\left(\sqrt{\mu - 1/4}\, x \right) \exp[i(\lambda/2)x], \tag{1.20}$$

$$\phi_{\lambda,\mu}(x) = -2(\mu - 1/4) \operatorname{sech}^2 \left(\sqrt{\mu - 1/4}\, x \right), \tag{1.21}$$

$$\varphi_{\lambda,\mu}(x) = -\lambda \partial_x \phi_{\lambda,\mu}(x), \tag{1.22}$$

then $\vec{\psi}_{\lambda,\mu} = (\psi_{\lambda,\mu}, \phi_{\lambda,\mu}, \varphi_{\lambda,\mu})$ is a solution of $S'_{\lambda,\mu}(\vec{\psi}) = 0$.

Definition. We say that a subset Σ of X is *stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property. If $\vec{u}_0 \in X$ satisfies $\inf\{\|\vec{u}_0 - \vec{\psi}\|_X : \vec{\psi} \in \Sigma\} < \delta$, then the solution $\vec{u}(t)$ of (1.7) with $\vec{u}(0) = \vec{u}_0$ exists for all $t \in \mathbf{R}$ and satisfies

$$\sup_{t\in\mathbf{R}} \inf\{\|\vec{u}(t) - \vec{\psi}\|_X : \vec{\psi} \in \Sigma\} < \varepsilon.$$

Moreover, let $\vec{\psi}_{\lambda,\mu}$ be a solution of $S'_{\lambda,\mu}(\vec{\psi}) = 0$. We say that the solitary wave solution $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$ is *stable* if $\{T(\alpha,\beta)\vec{\psi}_{\lambda,\mu} : \alpha,\ \beta \in \mathbf{R}\}$ is stable.

We are now in a position to state our main result in this note.

Theorem II. *Let $\lambda^2 = 1$ and $\mu > 1/4$. Then, the solitary wave solution $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$ given by* (1.20)–(1.22) *is stable for any $\mu > 1/4$.*

Remark. In my lecture at the conference, I announced that when $\lambda^2 < 1$ and $\mu = \lambda^2/4 + 1/(1-\lambda^2)$, the solitary wave solution $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$ given by (1.17)–(1.19) is stable if λ^2 is sufficiently close to 1. However, after the conference, I found a mistake in the proof. So, the stability of $T(\lambda t, \mu t)\vec{\psi}_{\lambda,\mu}$ given by (1.17)–(1.19) seems to be still an open problem.

2 Proof of Theorem II

In this section, we give the proof of Theorem II, basically along the argument in [7].

When $\lambda^2 = 1$ and $\mu > 1/4$, we obtain the following basic identity from (1.12).

$$\begin{aligned} S_{\lambda,\mu}(u,v,w) &= \int_{\mathbf{R}} \left\{ \left| \partial_x (e^{-i\lambda x/2} u) \right|^2 + (\mu - 1/4)|u|^2 - \frac{1}{2}|u|^4 \right\} dx \\ &\quad + \frac{1}{2} \int_{\mathbf{R}} \left\{ (w + \lambda \partial_x v)^2 + \left(|u|^2 + v \right)^2 \right\} dx. \end{aligned} \tag{2.1}$$

Associated with the identity (2.1), we define for $\rho > 0$

$$S^0_{\lambda,\mu}(u) = \int_{\mathbf{R}} \left\{ \left| \partial_x (e^{-i\lambda x/2} u) \right|^2 + (\mu - 1/4)|u|^2 - \frac{1}{2}|u|^4 \right\} dx, \tag{2.2}$$

$$Q^0(u) = \int_{\mathbf{R}} |u|^2 dx, \tag{2.3}$$

$$I^0(\rho) = \inf\{S^0_{\lambda,\mu}(u) : u \in H^1(\mathbf{R}),\ Q^0(u) = \rho\}, \tag{2.4}$$

$$\Sigma^0(\rho) = \{u \in H^1(\mathbf{R}) : S^0_{\lambda,\mu}(u) = I^0(\rho),\ Q^0(u) = \rho\}. \tag{2.5}$$

For α, $\beta \in \mathbf{R}$ and $u \in L^2(\mathbf{R})$, we define

$$T_1(\alpha)u(x) = u(x+\alpha), \quad T_2(\beta)u(x) = e^{i\beta}u(x).$$

Lemma 2. 1 *Assume that* $\lambda^2 = 1$ *and* $\mu > 1/4$. *Let* $\psi_{\lambda,\mu}$ *be the function defined by* (1.20). *Then, we have*

$$\Sigma^0(\rho(\mu)) = \{T_1(\alpha)T_2(\beta)\psi_{\lambda,\mu} : \alpha, \beta \in \mathbf{R}\},$$

where $\rho(\mu) = Q^0(\psi_{\lambda,\mu}) = 4\sqrt{\mu - 1/4}$.

Lemma 2. 2 *Let* $\rho > 0$. *If* $\{u_j\} \subset H^1(\mathbf{R})$ *satisfies* $S^0_{\lambda,\mu}(u_j) \to I^0(\rho)$ *and* $Q^0(u_j) \to \rho$, *then there exist* $\{\alpha_j\} \subset \mathbf{R}$, *a subsequence of* $\{T_1(\alpha_j)u_j\}$ *(we still denote it by the same letter) and* $\psi \in \Sigma^0(\rho)$ *such that*

$$T_1(\alpha_j)u_j \to \psi \quad \textit{strongly in} \quad H^1(\mathbf{R}).$$

Lemma 2. 2 is proved by using the concentration compactness method introduced by Lions [6]. For the proofs of Lemmas 2. 1 and 2. 2 , see Cazenave and Lions [1]. From Lemmas 2. 1 and 2. 2 and the conservation laws, one can show the stability of solitary wave solutions for the single nonlinear Schrödinger equation (for details, see [1]).

Following the idea by Cazenave and Lions [1], we consider the following minimization problem:

$$I(\rho) = \inf\{S_{\lambda,\mu}(\vec{u}) : \vec{u} \in X,\ Q(\vec{u}) = \rho\}, \tag{2.6}$$

$$\Sigma(\rho) = \{\vec{u} \in X : S_{\lambda,\mu}(\vec{u}) = I(\rho),\ Q(\vec{u}) = \rho\}. \tag{2.7}$$

From Lemma 2. 1 and (2.1), we have

Lemma 2. 3 *Assume that* $\lambda^2 = 1$ *and* $\mu > 1/4$. *For any* $\rho > 0$, *we have* $I(\rho) = I^0(\rho)$ *and*

$$\Sigma(\rho) = \left\{\vec{\psi} = (\psi, \phi, \varphi) : \psi \in \Sigma^0(\rho),\ \phi = -|\psi|^2,\ \varphi = -\lambda\partial_x\phi\right\}. \tag{2.8}$$

Moreover, let $\vec{\psi}_{\lambda,\mu} = (\psi_{\lambda,\mu}, \phi_{\lambda,\mu}, \varphi_{\lambda,\mu})$ *be the vector in* X *given by* (1.20)–(1.22). *Then, we have*

$$\Sigma(\rho(\mu)) = \left\{T(\alpha,\beta)\vec{\psi}_{\lambda,\mu} : \alpha, \beta \in \mathbf{R}\right\}, \tag{2.9}$$

where $\rho(\mu) = Q(\vec{\psi}_{\lambda,\mu}) = 4\sqrt{\mu - 1/4}$.

Proof. First, we note that $S^0_{\lambda,\mu}(u) \le S_{\lambda,\mu}(\vec{u})$ holds for all $\vec{u} = (u, v, w) \in X$, so that we have $I^0(\rho) \le I(\rho)$. We put

$$\Sigma_1(\rho) = \left\{\vec{\psi} = (\psi, \phi, \varphi) : \psi \in \Sigma^0(\rho),\ \phi = -|\psi|^2,\ \varphi = -\lambda\partial_x\phi\right\}.$$

If $\vec{\psi} = (\psi, \phi, \varphi) \in \Sigma_1(\rho)$, then we have $Q(\vec{\psi}) = Q^0(\psi) = \rho$ and

$$I(\rho) \le S_{\lambda,\mu}(\vec{\psi}) = S^0_{\lambda,\mu}(\psi) = I^0(\rho) \le I(\rho).$$

Thus, we have $I(\rho) = I^0(\rho)$ and $\vec{\psi} \in \Sigma(\rho)$. Conversely, if $\vec{\psi} = (\psi, \phi, \varphi) \in \Sigma(\rho)$, then we have $Q(\vec{\psi}) = Q^0(\psi) = \rho$ and

$$I^0(\rho) \le S^0_{\lambda,\mu}(\psi) \le S_{\lambda,\mu}(\vec{\psi}) = I(\rho) = I^0(\rho).$$

Thus, we have $\vec{\psi} \in \Sigma_1(\rho)$. Hence, we obtain (2.8). (2.9) follows from Lemma 2. 1 and (2.8). This completes the proof. □

Lemma 2. 4 *Let $\rho > 0$ and $\vec{\psi}_0 \in \Sigma(\rho)$. If $\{\vec{u}_j\} = \{(u_j, v_j, w_j)\} \subset X$ satisfies*

$$E(\vec{u}_j) \to E(\vec{\psi}_0), \quad P(\vec{u}_j) \to P(\vec{\psi}_0), \quad Q(\vec{u}_j) \to Q(\vec{\psi}_0). \tag{2.10}$$

then there exist $\{\alpha_j\} \subset \mathbf{R}$, a subsequence of $\{T(\alpha_j, 0)\vec{u}_j\}$ (we still denote it by the same letter) and $\vec{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in \Sigma(\rho)$ such that

$$T(\alpha_j, 0)\vec{u}_j \to \vec{\psi}_1 \quad \textit{strongly in} \quad X.$$

Proof. First, we note that by the Gagliardo-Nirenberg-Sobolev inequality and (2.10), we see that $\{\vec{u}_j\}$ is a bounded sequence in X, and

$$S_{\lambda,\mu}(\vec{u}_j) = E(\vec{u}_j) + \lambda P(\vec{u}_j) + \mu Q(\vec{u}_j) \to S_{\lambda,\mu}(\vec{\psi}_0) = I(\rho). \tag{2.11}$$

Since we have

$$I(\rho) = I^0(\rho) \le S^0_{\lambda,\mu}(u_j) \le S_{\lambda,\mu}(\vec{u}_j),$$

it follows from (2.11) and (2.10) that

$$S^0_{\lambda,\mu}(u_j) \to I^0(\rho), \quad Q^0(u_j) = Q(\vec{u}_j) \to \rho.$$

Thus, by Lemma 2. 2 , there exist $\{\alpha_j\} \subset \mathbf{R}$ and a subsequence of $\{T_1(\alpha_j)u_j\}$ (we still denote it by the same letter) and $\tilde{\psi} \in \Sigma^0(\rho)$ such that

$$T_1(\alpha_j)u_j \to \tilde{\psi} \quad \text{strongly in} \quad H^1(\mathbf{R}). \tag{2.12}$$

Since $\{\vec{u}_j\}$ is bounded in X, so is $\{T(\alpha_j, 0)\vec{u}_j\}$. Thus, there exists a subsequence $\{\vec{u}_j^1\} = \{(u_j^1, v_j^1, w_j^1)\}$ of $\{T(\alpha_j, 0)\vec{u}_j\}$ and $\vec{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in X$ such that

$$\vec{u}_j^1 \rightharpoonup \vec{\psi}_1 \quad \text{weakly in} \quad X. \tag{2.13}$$

By (2.12) and (2.13), we have $\psi_1 = \tilde{\psi} \in \Sigma^0(\rho)$ and

$$u_j^1 \to \psi_1 \quad \text{strongly in} \quad H^1(\mathbf{R}). \tag{2.14}$$

Moreover, from (2.1) and (2.14), we have

$$|u_j^1|^2 + v_j^1 \to 0 \qquad \text{strongly in} \qquad L^2(\mathbf{R}), \tag{2.15}$$

$$w_j^1 + \lambda\partial_x v_j^1 \to 0 \qquad \text{strongly in} \qquad L^2(\mathbf{R}). \tag{2.16}$$

From (2.13)–(2.16), we have

$$v_j^1 \to \phi_1 = -|\psi_1|^2 \qquad \text{strongly in} \qquad L^2(\mathbf{R}), \tag{2.17}$$

$$\partial_x v_j^1 \rightharpoonup \partial_x \phi_1 \qquad \text{weakly in} \qquad L^2(\mathbf{R}), \tag{2.18}$$

$$w_j^1 \rightharpoonup \varphi_1 = -\lambda\partial_x\phi_1 \qquad \text{weakly in} \qquad L^2(\mathbf{R}). \tag{2.19}$$

Since $\psi_1 \in \Sigma^0(\rho)$, $\phi_1 = -|\psi_1|^2$ and $\varphi_1 = -\lambda\partial_x\phi_1$, it follows from Lemma 2. 3 that $\vec{\psi}_1 = (\psi_1, \phi_1, \varphi_1) \in \Sigma(\rho)$. Finally, we have to show the strong convergence of $\{\partial_x v_j^1\}$ and $\{w_j^1\}$ in $L^2(\mathbf{R})$. By the definition (1.9) and the convergences in (2.10) and (2.14), we have

$$\begin{aligned}\int_{\mathbf{R}} w_j^1 \partial_x v_j^1 dx &= P(\vec{u}_j^1) - \int_{\mathbf{R}} i\overline{u_j^1}\partial_x u_j^1 dx \\ &= P(\vec{u}_j) - \int_{\mathbf{R}} i\overline{u_j^1}\partial_x u_j^1 dx \to P(\vec{\psi}_0) - \int_{\mathbf{R}} i\overline{\psi_1}\partial_x\psi_1 dx.\end{aligned}$$

Since $\vec{\psi}_0, \vec{\psi}_1 \in \Sigma(\rho)$, from Lemma 2. 3 , we have $P(\vec{\psi}_0) = P(\vec{\psi}_1)$. Thus, we have

$$\int_{\mathbf{R}} w_j^1 \partial_x v_j^1 dx \to \int_{\mathbf{R}} \varphi_1 \partial_x \phi_1 dx. \tag{2.20}$$

Therefore, by (2.16) and (2.18)–(2.20), we have

$$\begin{aligned}
\lambda^2\|\partial_x\phi_1\|_{L^2}^2 + \|\varphi_1\|_{L^2}^2 &\le \liminf_{j\to\infty}\left(\lambda^2\|\partial_x v_j^1\|_{L^2}^2 + \|w_j^1\|_{L^2}^2\right) \\
&= \liminf_{j\to\infty}\left(\|w_j^1 + \lambda\partial_x v_j^1\|_{L^2}^2 - 2\lambda\int_{\mathbf{R}} w_j^1\partial_x v_j^1\,dx\right) \\
&= -2\lambda\int_{\mathbf{R}}\varphi_1\partial_x\phi_1\,dx = \lambda^2\|\partial_x\phi_1\|_{L^2}^2 + \|\varphi_1\|_{L^2}^2.
\end{aligned}$$

Here we have used the fact that $\varphi_1 + \lambda\partial_x\phi_1 = 0$ in the last identity. Hence, we obtain

$$\partial_x v_j^1 \to \partial_x\phi_1 \quad \text{strongly in} \quad L^2(\mathbf{R}), \tag{2.21}$$

$$w_j^1 \to \varphi_1 \quad \text{strongly in} \quad L^2(\mathbf{R}). \tag{2.22}$$

This completes the proof. □

Proof of Theorem II. By Lemma 2. 3 , it is enough to show that $\Sigma(\rho)$ is stable for any $\rho > 0$. We prove it by contradiction. Suppose that $\Sigma(\rho)$ is not stable. Then, by the definition, there exist a constant $\varepsilon_0 > 0$ and sequences $\{\vec{u}_{0j}\} \subset X$ and $\{t_j\} \subset \mathbf{R}$ such that

$$\lim_{j\to\infty} \inf\{\|\vec{u}_{0j} - \vec{\psi}\|_X : \vec{\psi} \in \Sigma(\rho)\} = 0 \tag{2.23}$$

and

$$\inf\{\|\vec{u}_j(t_j) - \vec{\psi}\|_X : \vec{\psi} \in \Sigma(\rho)\} \ge \varepsilon_0, \tag{2.24}$$

where $\vec{u}_j(t)$ is the solution of (1.7) with $\vec{u}_j(0) = \vec{u}_{0j}$. By (2.23) and the conservation laws (1.11), we have

$$\begin{aligned}
E(\vec{u}_j(t_j)) &= E(\vec{u}_{0j}) \to E(\vec{\psi}_0), \\
P(\vec{u}_j(t_j)) &= P(\vec{u}_{0j}) \to P(\vec{\psi}_0), \\
Q(\vec{u}_j(t_j)) &= Q(\vec{u}_{0j}) \to Q(\vec{\psi}_0)
\end{aligned}$$

for some $\vec{\psi}_0 \in \Sigma(\rho)$. Thus, by Lemma 2. 4 , there exist $\{\alpha_j\} \subset \mathbf{R}$, a subsequence of $\{T(\alpha_j, 0)\vec{u}_j(t_j)\}$ (we still denote it by the same letter) and $\vec{\psi}_1 \in \Sigma(\rho)$ such that

$$T(\alpha_j, 0)\vec{u}_j(t_j) \to \vec{\psi}_1 \quad \text{strongly in} \quad X.$$

However, this contradicts (2.24). Hence, $\Sigma(\rho)$ is stable. □

3 Concluding Remarks

When $\lambda^2 > 1$ and $\mu = \lambda^2/4 + 1/(2(\lambda^2 - 1))$, (1.1)–(1.2) also admit the following exact solitary wave solutions (see [10]):

$$\begin{aligned} u(t,x) &= \frac{3}{\sqrt{\lambda^2-1}} \operatorname{sech}\left(\frac{x-\lambda t}{\sqrt{2(\lambda^2-1)}}\right) \tanh\left(\frac{x-\lambda t}{\sqrt{2(\lambda^2-1)}}\right) \\ &\quad \times \exp[i\mu t + i(\lambda/2)(x-\lambda t)], \qquad (3.1) \\ v(t,x) &= -\frac{3}{\lambda^2-1} \operatorname{sech}^2\left(\frac{x-\lambda t}{\sqrt{2(\lambda^2-1)}}\right). \qquad (3.2) \end{aligned}$$

The variational characterizations and the stability problem for (3.1)–(3.2) seem to be open problems.

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