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ON THE EVOLUTION OF A HIGH ENERGY VORTICITY IN AN IDEAL FLUID

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1.Introduction.

The motion of a two dimensional incompressible homogeneous ideal fluid is governed by the system of the partial differential equations called the Euler equations. On the other hand, the vortex model is also used by many researchers to study the motion of the fluid. In the vortex model, the (scalar) vorticity is assumed to be concentrated in some points evolving according to the system of the ordinary differential equations called the Kirchhoff-Routh equations. (3.2) in section 3 is an example of a system of the Kirchhoff-Routh equations.

However, if some parts of the vorticity concentrate in some points, the fluid never has finite kinetic energy. Moreover, the solution of the Kirchhoff-Routh equations does not constitute a solution of the Euler equations even in such a weak sense as Definition 2.1 in this note.

Therefore, we want to understand the solutions of the Kirchhoff-Routh equations in terms of the solutions of the Euler equations with high but finite kinetic energies.

To this purpose, we define the high energy vorticities and establish the structure theorems of them (Theorem A and Theorem B). Then, we

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consider the limit of the energy diverging sequence of the weak solutions of the two-dimensional incompressible Euler equation (Theorem C).

Remark. The details of this note are in [O].

2.On two demensional incompressible ideal fluids.

Let $\Omega \subset \boldsymbol{R}^2$ be a simply connected bounded domain with smooth boundary $\partial\Omega$. We consider the motion of the incompressible homogeneous ideal fluid with unit density in Ω.

The Euler equations are as follows:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p \quad \text{in} \quad \Omega \times (0,T), \tag{2.1}$$

$$\operatorname{div} u = \frac{\partial}{\partial x^1} u^1 + \frac{\partial}{\partial x^2} u^2 = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{2.2}$$

$$u \cdot n = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{2.3}$$

where $u(x,t) = (u^1(x,t), u^2(x,t))$ is the velocity field and $p(x,t)$ is the (scalar) pressure.

Applying the curl operator to (2.1), we obtain the evolution equation of the vorticity $\omega(x,t) = \operatorname{curl} u(x,t) = \frac{\partial}{\partial x^1} u^2(x,t) - \frac{\partial}{\partial x^2} u^1(x,t)$:

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\, \omega = 0. \tag{2.4}$$

Because Ω is simply connected, it is well known that u satisfying (2.2) and (2.3) admits the representation

$$u = \nabla^{\perp} G\omega = \left(\frac{\partial}{\partial x^2} G\omega, -\frac{\partial}{\partial x^1} G\omega\right), \tag{2.5}$$

where $G\omega(x,t) = \int_\Omega g(x,y)\omega(y,t)dy$ and $g(x,y)$ is the Green function of $-\Delta$ with 0-Dirichlet boundary condition.

Using (2.5), we can eliminate u from (2.4) as follows:

$$\frac{\partial \omega}{\partial t} + (\nabla^{\perp} G\omega \cdot \nabla)\, \omega = 0. \tag{2.6}$$

Therefore, the vorticity evolves by itself according to (2.6), which we call the Euler equation for the vorticity, or simply, the Euler equation in this note. We consider solutions of (2.6).

It is well-known that a sufficiently smooth solution of the vorticity equation conserves several quantities during the evolution. Especially, the kinetic energy $(1/2)\int_\Omega |u(x,t)|^2 dx$, which is equal to $E(\omega(\cdot,t)) = \frac{1}{2}\int_\Omega g(x,y)\omega(x,t)\omega(y,t)dxdy$, and $\int_\Omega f(\omega(x,t))dx$ for any smooth function $f(\cdot)$ are conserved. Indeed, multiply each side of (2.6) by $G\omega$ and $f'(\omega)$ respectively and integrate them. Moreover, $\|\omega(\cdot,t)\|_{L^q(\Omega)}$ for $1 \le q \le \infty$ is conserved. Furthermore, if ω is positive initially, then ω is also positive at every following time.

In our analysis, a vorticity should be a weak solution in the sense of Definition 2.1 below staying in the high energy class $E(q,s,K)$ defined in section 3. In particular, we consider the weak solutions in $L^\infty(0,T;L^q(\Omega))$ for $q > 1$.

Using $(\nabla^{\perp} G\omega \cdot \nabla)\, \omega = \mathrm{div}(\omega \nabla^{\perp} G\omega)$, we have the following weak expression of (2.6):

$$\int_0^T \int_\Omega [\omega \frac{\partial \varphi}{\partial t} + \omega(\nabla^{\perp} G\omega \cdot \nabla)\varphi] dxdt = 0 \tag{2.7}$$

for every scalar function $\varphi \in \mathcal{D}(\Omega \times (0,T))$. From the elliptic regularity theory and the Sobolev embedding, (2.7) has the meaning if $\omega \in L^\infty(0,T;L^{4/3}(\Omega))$. However, we can extend (2.7) as follows:

Definition 2.1. $\omega(x,t) \in L^\infty((0,T);L^1(\Omega))$ is a weak solution of the Euler equation if $\omega(x,t)$ satisfies

$$\int_0^T [\int_\Omega \omega \frac{\partial \varphi}{\partial t} dx + \int_\Omega \int_\Omega \hat{H}_\varphi(x,y,t)\omega(x,t)\omega(y,t)dxdy]dt = 0 \qquad (2.8)$$

for every scalar function $\varphi \in \mathcal{D}(\Omega \times (0,T))$, where

$$\begin{aligned}\hat{H}_\varphi(x,y,t) &= H_\varphi(x,y,t) + \nabla_x^\perp h(x,y)\cdot\nabla\varphi(x,t),\\ H_\varphi(x,y,t) &= -\frac{1}{4\pi}\frac{(x-y)^\perp\cdot[\nabla\varphi(x,t)-\nabla\varphi(y,t)]}{|x-y|^2},\\ h(x,y) &= g(x,y) - \frac{1}{2\pi}\log|x-y|^{-1}.\end{aligned}$$

(2.8) is equivalent to (2.7) if $\omega \in L^\infty(0,T;L^{4/3}(\Omega))$. For the precise derivation of (2.8) from (2.7), see [S] (or [O]).

We know the following facts on the existence of a weak solution of the Euler equation:

Facts 2.2. *(Lions[L] and Yudovich[Y]) For every $\omega_0 \in L^q(\Omega)$ for $1 < q \le \infty$, there exists a weak solution $\omega \in L^\infty(0,\infty;L^q(\Omega))$ such that $\omega \in C([0,\infty);L^p(\Omega))$ for all $1 \le p < q$ and $\omega(\cdot,0) = \omega_0(\cdot)$. Moreover, this ω conserves $E(\omega)$, $\|\omega\|_{L^r(\Omega)}$ for every $1 \le r \le q$, and the positivity of ω.*

Remark. For $q = \infty$, Yudovich proves the above facts [Y]. For $1 < q < \infty$, we obtain Facts 2.2 from the results by Lions [L], though his notion of the weak solution of the Euler equations is different from us.

3. The results.

We consider the following classes of the vorticities:

$$P(\Omega) = \{\omega \in L^1(\Omega) : \omega \geq 0 \text{ and } \|\omega\|_{L^1(\Omega)} = 1\}.$$
$$P_q(\Omega) = P(\Omega) \cap L^q(\Omega) \quad \text{for some } 1 < q \leq \infty.$$
$$P_q(\Omega, s) = \{\omega \in P_q(\Omega) : 0 \leq \|\omega\|_{L^q(\Omega)} \leq s\} \quad \text{for some } s > 0.$$

Let I_ε be the characteristic function of $B_\varepsilon(0)$. Then $\lambda I_\varepsilon(x - x_0)$ for some $x_0 \in \Omega$ is a typical element of $P_q(\Omega, s)$ for sufficiently large s if λ, ε and s that satisfy

$$\pi\varepsilon^2\lambda = 1 \quad \text{and} \quad \pi\varepsilon^2\lambda^q = s^q, \quad \text{i.e.,} \quad \varepsilon = (\pi s^{q'})^{-1/2}, \tag{3.1}$$

where $q' = q/(q-1)$ for $1 < q < \infty$ and $q' = 1$ for $q = \infty$. In the rest of this note, λ, ε and s always satisfy these relations (3.1), Moreover, if ε_i is given for example, λ_i and s_i are determined by $\pi\varepsilon_i^2\lambda_i = 1$ and $\pi\varepsilon_i^2\lambda_i^q = s_i^q$.

The following facts are standard:

Fact 3.1.

$$E(q, s) := \sup_{\omega \in P_q(\Omega, s)} E(\omega) < \infty \quad \textit{and} \quad E(q, s) \longrightarrow \infty \quad \textit{as} \quad s \longrightarrow \infty.$$

Now, we define the class of the high energy vorticities of $P_q(\Omega, s)$ as

$$E(q, s, K) := \{\omega \in P_q(\Omega, s) : E(q, s) - K \leq E(\omega) \leq E(q, s)\},$$

for some $K > 0$. Then, we obtain the following results:

Theorem A. *There exists a constant* $\tilde{s}_0 = \tilde{s}_0(\Omega)$ *such that for every* $s \geq \tilde{s}_0$*, every* $\omega \in E(q,s,K)$*, and every* $\gamma > 0$*,*

$$r_\omega(\gamma) := \inf\{r > 0 : \sup_{x\in\Omega} \int_{\Omega\cap B_r(x)} \omega(y)\,dy \geq 1-\gamma\} \leq \varepsilon \exp(C_0/\gamma),$$

where C_0 *is a positive constant independent of* s, ω*, and* γ*.*

Theorem A implies that every $\omega \in E(q,s,K)$ concentrates near its center $\bar{x}_\omega := \int_\Omega x\omega(x)dx$, for example.

Theorem B. *Fix* $1 < q \leq \infty$ *and* $K > 0$*. There exists a constant* $\tilde{K} \geq K$ *satisfying the following properties. If we choose any sequence* $s_n \longrightarrow \infty$ *as* $n \longrightarrow \infty$ *and* $\omega_n \in E(q,s_n,K)$ *such that there exists a limiting center* $\bar{x}_n (:= \bar{x}_{\omega_n}) \longrightarrow \bar{x}_\infty \in \bar{\Omega}$ *as* $n \longrightarrow \infty$*, then*

$$\bar{x}_\infty \in \Omega_{\tilde{K}} := \{x \in \Omega : \max_{x\in\Omega} H(x) - \tilde{K} \leq H(x) \leq \max_{x\in\Omega} H(x)\},$$

where $H(x) = (1/2)h(x,x)$ *and* $h(x,y) = g(x,y) - (1/2\pi)\log|x-y|^{-1}$*.*

Theorem C. *If we choose any sequence* $s_n \longrightarrow \infty$ *as* $n \longrightarrow \infty$ *and* $\omega_n(x,t) \in L^\infty(0,T;L^1(\Omega))$ *that is a weak solution the Euler equation such that* $\omega_n(\cdot,t) \in E(q,s_n,K)$ *for a.e.* $t \in (0,T)$*. Furthermore, if there is a limiting path of center* $\bar{x}_n(t) (:= \bar{x}_{\omega_n(\cdot,t)}) \longrightarrow \bar{x}_\infty(t)$ *in* $L^\infty(0,T)^2$ *weakly* $*$ *as* $n \longrightarrow \infty$*, then, for almost every* t*, this* $\bar{x}_\infty(t)$ *is equal to a solution of*

$$\frac{dz}{dt} = \nabla^\perp H(z) \tag{3.2}$$

staying in $\Omega_{\tilde{K}}$*.*

Remark 3.1. The equations (3.2) are called the Kirchhoff-Routh equations of one vortex with unit intensity (i.e., a vorticity consists of a Dirac measure). See, e.g.,[T2].

Remark 3.2. It is easy to see that there exists a sequence that satisfies the hypothesis of Theorem B and Theorem C, because Ω is bounded and we know Facts 2.2.

Remark 3.3. Our results relate to Turkington's results on $E(\infty, s, 0)$ in [T1], closely. See also [T2].

4.Sketch of the proofs.

4.1. Theorem A.

Instead of $E(q,s,K)$, we consider

$$F(q,s,K) = \{\omega \in P_q(\Omega,s) : F(q,s) - K \leq F(\omega) \leq F(q,s)\},$$

where

$$\begin{aligned} F(q,s) &= \sup_{\omega \in P_q(\Omega,s)} F(\omega)(< \infty), \\ F(\omega) &= \frac{1}{2}\int_\Omega N\omega(x)\omega(x)dx, \\ N\omega(x) &= \frac{1}{2\pi}\int_\Omega \log\frac{1}{|x-y|}\omega(y)dy. \end{aligned}$$

Then, we prove the following theorem:

Theorem 4.1. *There exists $s_1 > 0$ depending only on Ω such that for every $s \geq s_1$, every $K > 0$, every $\omega \in F(q,s,K)$, and every $\gamma > 0$,*

$$r_\omega(\gamma) \leq \varepsilon \exp(C_1/\gamma),$$

where C_1 is a constant depending on K but independent of ω, s, and γ.

Now, Theorem A follows the fact that $E(q,s,K) \subset F(q,s,K')$ for some $K' > 0$, which we can see from the following energy estimate:

Facts 4.2.

$$(1)\ E(\omega) \le F(\omega) + C_2, \quad \textit{especially} \quad E(q,s) \le F(q,s) + C_2,$$
$$(2)\ F(q,s) + \sup_{x\in\Omega} H(x) - C_3 \le E(q,s) \quad \textit{for sufficiently large } s, \tag{4.1}$$

where C_2 and C_3 are constants independent of s.

The estimate (1) is easily obtained because $h(x,y)$ is bounded from the above. On the other hand, the estimate of (2) is obtained by calculating the energy of the typical element $\lambda I_\varepsilon(x - x_0) \in P_q(\Omega, s)$, where $x_0 \in \Omega$ satisfies $H(x_0) = \sup_{x\in\Omega} H(x)$.

The following estimate of the Newton potential $N\omega(x)$ of $\omega(x)$ is the key in the proof of Theorem 4.1:

Lemma 4.3. *For every $\varepsilon > 0$, every $R \ge 1$ and every $\omega \in P_q(\Omega)$ for $1 < q \le \infty$,*

$$N\omega(x) \le \frac{1}{2\pi}\log\frac{1}{\varepsilon} + \frac{C_4}{2\pi}\varepsilon^{2/q'}\|\omega\|_{L^q(\Omega)} - \frac{1}{2\pi}\log R \int_{\Omega\setminus B_{R\varepsilon}(x)} \omega(y)dy,$$

where C_4 is a constant depending only on q.

Proof. We have the following decomposition of $N\omega(x)$:

$$N\omega(x) = \frac{1}{2\pi}[\log\frac{1}{\varepsilon} + \int_{\Omega\cap B_\varepsilon(x)} \log\frac{\varepsilon}{|x-y|}\omega(y)dy + \int_{\Omega\setminus B_\varepsilon(x)} \log\frac{\varepsilon}{|x-y|}\omega(y)dy].$$

It is easy to see that

$$\int_{\Omega\cap B_\varepsilon(x)} \log\frac{\varepsilon}{|x-y|}\omega(y)dy \le \varepsilon^{2/q'} C_4\|\omega\|_{L^q(\Omega)},$$

where $C_4 = \|\log|x|\|_{L^{q'}(B_1)} < \infty$. On the other hand, as $R \geq 1$ and $\omega \geq 0$, we obtain

$$\begin{aligned}\int_{\Omega\setminus B_\varepsilon(x)} \log\frac{\varepsilon}{|x-y|}\omega(y)dy &\leq \int_{\Omega\setminus B_{R\varepsilon}(x)} \log\frac{\varepsilon}{|x-y|}\omega(y)dy \\ &\leq -\log R\int_{\Omega\setminus B_{R\varepsilon}(x)} \omega(y)dy. \quad \square\end{aligned}$$

Corollary 4.4. *For every sufficiently large s and every $\omega \in P_q(\Omega, s)$,*

$$F(\omega) \leq \frac{1}{4\pi}\log\frac{1}{\varepsilon} + \frac{C_5}{4\pi} - \frac{1}{4\pi}\log R \inf_{x\in\Omega}\int_{\Omega\setminus B_{R\varepsilon}(x)} \omega(y)dy. \tag{4.2}$$

Proof. $F(\omega) \leq \sup_{x\in\Omega} N\omega(x)$ provided $\omega \in P_q(\Omega)$. $\square$

Proof of Theorem 4.1. Fix $x_0 \in \Omega$. Then $\lambda I_\varepsilon(x - x_0) \in P_q(\Omega, s)$ for sufficiently large s. Therefore, we have

$$F(q,s) \geq F(\lambda I_\varepsilon(x-x_0)) = \frac{1}{4\pi}\log\frac{1}{\varepsilon} + \frac{C_6}{4\pi}. \tag{4.3}$$

Using (4.2) and (4.3), for every $R \geq 1$ and every $\omega \in F(q, s, K)$, we have

$$\begin{aligned}\log R \inf_{x\in\Omega}\int_{\Omega\setminus B_{R\varepsilon}(x)} \omega(y)dy &\leq 4\pi[\frac{1}{4\pi}\log\frac{1}{\varepsilon} - F(\omega)] + C_5 \\ &\leq 4\pi[F(q,s) - F(\omega)] - C_6 + C_5 \\ &\leq 4\pi K + C_5 - C_6. \end{aligned}\tag{4.4}$$

Now, we take $C_1 > \max\{4\pi K + C_5 - C_6, 0\}$. Then, if $R > 1$, we can rewrite (4.4) as

$$\sup_{x\in\Omega}\int_{\Omega\cap B_{R\varepsilon}(x)} \omega(y)dy \geq 1 - C_1/\log R. \tag{4.5}$$

For every $\gamma > 0$, let R satisfy $\gamma = C_1/\log R$, that is, $R = \exp(C_1/\gamma)$. Then, $R > 1$, since $C_1 > 0$. Therefore, using (4.5), we obtain

$$\inf\{r > 0 : \sup_{x\in\Omega}\int_{\Omega\cap B_r(x)} \omega(x)dx \geq 1 - \gamma\} \leq R\varepsilon = \varepsilon\exp(C_1/\gamma). \quad \square$$

4.2. Theorem B.

Let $\gamma = \gamma(s) = -C_0'(\log \varepsilon)^{-1}$ for some fixed $C_0' > C_0$. Then

$$\gamma(s) \longrightarrow 0 \quad \text{and} \quad r(s) := \varepsilon \exp(C_0/\gamma(s)) \longrightarrow 0 \quad \text{as} \quad s \longrightarrow \infty.$$

Therefore, $\omega_n \longrightarrow \delta(x - \bar{x}_\infty)$ weakly in the sense of the measures.

On the other hand, using the energy estimate (4.1), we have, for sufficiently large n,

$$\begin{aligned} F(q, s_n) + &\sup_{x\in\Omega} H(x) - C_3 - K \\ &\leq E(q, s_n) - K \\ &\leq E(\omega_n) = F(\omega_n) + \frac{1}{2}\int_\Omega\int_\Omega h(x,y)\omega_n(x)\omega_n(y)dxdy \\ &\leq F(q, s_n) + \frac{1}{2}\int_\Omega\int_\Omega h(x,y)\omega_n(x)\omega_n(y)dxdy, \end{aligned}$$

that is,

$$\sup_{x\in\Omega} H(x) - K - C_3 \leq \frac{1}{2}\int_\Omega\int_\Omega h(x,y)\omega_n(x)\omega_n(y)dxdy. \tag{4.6}$$

Since $H(x) = (1/2)h(x,x) \longrightarrow -\infty$ as $x \longrightarrow \partial\Omega$, we can see that $\bar{x}_\infty \in \Omega$. Then, the righthand side of (4.6) converges to $H(\bar{x}_\infty)$ because $\omega_n \longrightarrow \delta(x - \bar{x}_\infty)$. Therefore, we obtain Theorem B with $\tilde{K} \geq K + C_3$.

4.3. Theorem C.

The following proof is essentially equal to that of Theorem 3.2 in [T2].

Instead of considering the motion of the center of the vorticity $\bar{x}_n(t)$, we consider the motion of more regular function

$$\tilde{x}_n(t) = \int_\Omega x\xi(x)\omega_n(x,t)dx,$$

where $\xi(x) \in C_0^\infty(\Omega)$ is a fixed function satisfying

$$\begin{cases} 0 \le \xi(x) \le 1 & \text{for} \quad \forall x \in \Omega, \\ \xi(x) \equiv 1 & \text{for} \quad \forall x \in \Omega_{L_1}. \end{cases}$$

Here $L_1 > \tilde{K}$ is a fixed constant.

It is easy to see that

$$\|\bar{x}_n - \tilde{x}_n\|_{L^\infty(0,T)^2} = o(1) \quad \text{as} \quad n \longrightarrow \infty. \tag{4.7}$$

Moreover, we have the following fact:

Lemma 4.5. *Let $\omega \in L^\infty(0,T;L^1(\Omega))$ be a weak solution of the Euler equation. Then $\tilde{x}(t) = \int_\Omega x\xi(x)\omega(x,t)dt \in W^{1,\infty}(0,T)^2$. Especially,*

$$\frac{d}{dt}\tilde{x}^i = \int_\Omega \int_\Omega \hat{H}_{x^i\xi}(x,y)\omega(x,t)\omega(y,t)dxdy \quad \textit{in} \quad \mathcal{D}'(0,T).$$

Proof. Insert a test function $\eta(t)x^i\xi(x)$ for $\eta(t) \in \mathcal{D}(0,T)$ into (2.8). □

The following theorem is the main part of the proof of Theorem C:

Theorem 4.6. *For every $0 < T < \infty$ and every $\sigma > 0$, there exists a constant s_2 depending on T and σ that satisfies following properties. Let a weak solution of the vorticity equation $\omega(x,t) \in L^\infty(0,T;L^1(\Omega))$ satisfy $\omega(\cdot,t) \in E(q,s,K)$ for a.e. $t \in (0,T)$ for some $s \ge s_2$. Then there exists $z(t)$ that is a solution of (3.2) staying in Ω_{L_1} such that*

$$\|\tilde{x}(t) - z(t)\|_{W^{1,\infty}(0,T)^2} \le \sigma.$$

Sketch of the proof. At every $t \in (0,T)$ such that $\omega(\cdot,t) \in E(q,s,K)$, we may assume that $B_{r(s)}(\tilde{x}(t)) \subset \Omega_{L_1}$ for a.e. $t \in (0,T)$ if s is sufficiently large. Now, we take $T_1 \in (0,T)$ such that $\omega(x,T_1) \in E(q,s,K)$, and $z(t)$ that is a solution of (3.2) satisfying $z(T_1) = \tilde{x}(T_1)$. Then, $z(t)$ stays in Ω_{L_1} because $z(t)$ conserves $H(z(t))$. Furthermore, for a. e. t, we have

$$|\frac{d}{dt}z^i(t) - \frac{d}{dt}\tilde{x}^i(t)| \leq J_1 + J_2 + J_3,$$

where

$$\begin{aligned}
J_1 &= |(\nabla^{\perp}H)^i(z(t)) - (\nabla^{\perp}H)^i(\tilde{x}(t))|, \\
J_2 &= |(\nabla^{\perp}H)^i(\tilde{x}(t)) \\
&\qquad - \int_{B_{r(s)}(\tilde{x}(t))}\int_{B_{r(s)}(\tilde{x}(t))} \hat{H}_{x^i\xi}(x,y)\omega(x,t)\omega(y,t)dxdy|, \\
J_3 &= |\int_{B_{r(s)}(\tilde{x}(t))}\int_{B_{r(s)}(\tilde{x}(t))} \hat{H}_{x^i\xi}(x,y)\omega(x,t)\omega(y,t)dxdy \\
&\qquad - \int_{\Omega}\int_{\Omega} \hat{H}_{x^i\xi}(x,y)\omega(x,t)\omega(y,t)dxdy|.
\end{aligned}$$

It is easy to see that

$$J_1 \leq C_7|z(t) - \tilde{x}(t)|$$

where C_7 is a constant depending only on L_1, because $\nabla^{\perp}H(x)$ is uniformly continuous over Ω_{L_1}. It is also easy to see that

$$J_3 \leq C_8(\|\omega\|_{L^1(\Omega)} + \|\omega\|_{L^1(B_{r(s)}(\tilde{x}(t)))})\|\omega\|_{L^1(\Omega\backslash B_{r(s)}(\tilde{x}(t)))} \leq 2C_8\gamma = o(1),$$

where a constant C_8 and $o(1)$ is also independent of t and ω.

By the way, we have

$$\begin{aligned}
&\int_{B_{r(s)}(\tilde{x}(t))}\int_{B_{r(s)}(\tilde{x}(t))} \hat{H}_{x^i\xi}(x,y)\omega(x,t)\omega(y,t)dxdy \\
&\qquad = \int_{B_{r(s)}(\tilde{x}(t))}\int_{B_{r(s)}(\tilde{x}(t))} (\nabla_x^{\perp}h)^i(x,y)\omega(x,t)\omega(y,t)dxdy,
\end{aligned}$$

because $H_{x^i\xi}(x,y) \equiv 0$ if $x, y \in B_{r(s)}(\tilde{x}(t)) \subset \Omega_{L_1}$. Therefore

$$J_2 = |(\nabla^\perp H)^i(\tilde{x}(t)) - \int_{B_{r(s)}(\tilde{x}(t))}\int_{B_{r(s)}(\tilde{x}(t))} (\nabla_x^\perp h)^i(x,y)\omega(x,t)\omega(y,t)dxdy| = o(1) \quad \text{as} \quad s \longrightarrow \infty,$$

where $o(1)$ is also independent of t and ω.

We can summarize the above calculations as follows:

$$|\frac{dz}{dt}(t) - \frac{d\tilde{x}}{dt}(t)| \le C_9|z(t) - \tilde{x}(t)| + o(1) \quad \text{as} \quad s \longrightarrow \infty \tag{4.8}$$

for a.e. $t \in (0,T)$, where $o(1)$ is independent of t and ω.

Then, using the Gronwall inequality, we obtain Theorem 4.6. □

Now, we sketch the proof of Theorem C. We may assume $T < \infty$. Using (4.7) and Theorem 4.6, we can construct $\{z_n(t)\}$, which are the solutions of (3.2) and $z_n \longrightarrow \bar{x}_\infty$ in $L^\infty(0,T)^2$. As $\{z_n(t)\}$ are the solutions of (3.2), it is easy to see that the $\bar{x}_\infty$ is equal to a solution of (3.2) for a. e. $t \in (0,T)$. Therefore, we obtain Theorem C.

References

[L] P. L. Lions, "Mathematical topics in fluid mechanics, Volume 1. Incompressible Models", Oxford University Press, Oxford, 1996.

[O] H. Ohtsuka, On the evolution of a high energy vorticity in an ideal fluid, Kyushu J. Math. (to appear).

[S] S. Schochet, The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation, *Comm. Partial Differential Equations* **20** (1995), 1077–1104.

[T1] B. Turkington, On steady vortex flow in two dimensions, I, *Comm. Partial Differential Equations* **8** (1983), 999–1030.

[T2] ______, On the evolution of a concentrated vortex in an ideal fluid, *Arch. Rational Mech. Anal.* **97** (1987), 75–87.

[Y] V. I. Yudovich, Non-stationary flow of an ideal incompressible liquid, *Zh. Vychisl Mat. i Mat. Fiz.* **3** (1963), 1032–1066 (Russian); *Comput. Math. Math. Phys.* **3** (1963), 1407–1456. (English)

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