数理解析研究所講究録
1077 巻 1999 年 105-139

An introduction to Ocneanu's theory of double triangle algebras for subfactors and classification of irreducible connections on the Dynkin diagrams

SATOSHI GOTO (SOPHIA UNIVERSITY)
後藤 聡史 (上智大学)

January 19, 1999

1 Introduction

The classification of subfactors of the hyperfinite II_1 factor is one of the most important and stimulating problems in the theory of operator algebras since V. F. R. Jones initiated his celebrated index theory for subfactors in [16]. The strongest form of the classification has been obtained by S. Popa based on his notion of *strong amenability* in [26]. In the early stage it was known that hyperfinite II_1 subfactors with index less than four has one of the Dynkin diagrams A_n, D_n, E_6, E_7 and E_8 as their principal graphs. For the complete classification of subfactors of the hyperfinite II_1 factor A. Ocneanu introduced the notion of *paragroup* in [20]. And by the paragroup theory the classification of subfactors of the hyperfinite II_1 factor with finite index and finite depth is reduced to the classification of *flat bi-unitary connections* on the (dual) principal graphs.

The paragroup theory as well as its importance are now widely spread and more and more people have become to work on the theory. The importance of paragroup theory is not only because it is complete invariant for subfactors of the hyperfinite II_1 factor with finite index and finite depth but also because it has deep relations to many other theories in mathematics and mathematical physics. Actually it has been revealed that there are striking relations between the paragroup theory and other theories such as exactly solvable integrable lattice models, quantum groups, topological quantum field theories (TQFT) both in the sense of Turaev-Viro ([32]) based on triangulation and in the sense of Reshetikhin-Turaev ([27]) based on surgery, and rational conformal field theories (RCFT) in the sense of Moore-Seiberg ([19]) and so on. (See for example [6], [11], [9], [10], [22], [23], [28], [33], [34]. All of these relations are explained in [13].)

In 1995 A. Ocneanu gave a series of lectures on subfactor theory at The Fields Institute from April 19 to 25. In his lectures ([24]) he introduced a new algebra called *double triangle algebra* by using the notion of essential paths and extension of Kauffman-Lins' Temperley-Lieb recoupling theory. He also gave many applications of his result. Among their applications he raised particularly five problems in his talks at Aarhus in June 1995, which consists of one problem concerned with TQFT, one concerned with RCFT and three concerned with subfactor theory. There he showed that his method gives essentially one solution of them.

Among the solutions of the five problems the most fundamental result is the complete classification of irreducible bi-unitary connections on the Dynkin diagrams $A_n, D_n, E_{6,7,8}$ and other solutions will follow from it. More precisely the irreducible connections on the Dynkin diagrams here means the irreducible connections on the four graphs which have the

Dynkin diagram K and L as the two horizontal graphs. (We call such a connection a K-L *bi-unitary connection.*) And the classification here means the classification of irreducible bi-unitary connections up to gauge choice, which is finer than the classification up to isomorphisms.

The main purpose of this paper is to give a detailed proof of the classification of irreducible connections on the Dynkin diagrams. As we mentioned in the beginning the classification in more restricted case when the four graphs are all the same Dynkin diagrams has been done in order to classify subfactors with index less than four. So the classification of connections itself is very important for this purpose.

Another example in which bi-unitary connections on the Dynkin diagrams naturally appear is the construction of a series of subfactors given by Goodman-de la Harpe-Jones ([14]). These subfactors are called *Goodman-de la Harpe-Jones subfactors.* (We call them GHJ subfactors in short. A. Ocneanu calls the same subfactors *Jones-Okamoto subfactors* because S. Okamoto computed their principal graphs [25].) They are constructed from A-K bi-unitary connections, where A represents the Dynkin diagrams A_n and K is one of the A-D-E Dynkin diagrams. The principal graphs of these subfactors are easily obtained by a simple method but the dual principal graphs as well as their fusion rules are much more difficult to compute. The most important example is the subfactor with index $3+\sqrt{3}$ which is constructed from the embedding of the string algebra of A_{11} to that of E_6, i.e., it is obtained from an A_{11}-E_6 bi-unitary connection. In this particular case it happens that it is not very difficult to compute the dual principal graph (see [18], [13, Section 11.6]). But it is more difficult to determine its fusion rule. Actually D. Bisch has tried to compute the fusion rule just from the graph but there were five possibilities and it turned out that the fusion rule can not be determined from the graph only. Some more information is needed and Y. Kawahigashi obtained the fusion rule as an application of paragroup actions in [18]. In his lectures at The Fields Institute A. Ocneanu gave a solution to this problem of determining the dual principal graphs and their fusion rules as one of some applications of his theory of double triangle algebra ([24]). In particular, the fusion rule algebra of all K-K bi-unitary connection is used to determine the fusion rule of GHJ subfactors which correspond to the Dynkin diagram K. Here we would like to mention that some of recent works has revealed a surprising relation between GHJ subfactors and conformal inclusions ([37], [2], [3], [4]). Furthermore some generalization of the construction of GHJ subfactors has been obtained by F. Xu ([35], [36]) and J. Böckenhauer-D. E. Evans ([2], [3], [4]).

Another striking and unexpected observation which A. Ocneanu has found ([24]) is the relation between fusion rule algebras of all K-K connections for a Dynkin diagram K he obtained and affine SU(2) modular invariants corresponding to the graph K. The A-D-E classification of affine SU(2) modular invariants has been obtained in [7], (see also [31]). A. Ocneanu showed some interpretations of off-diagonal terms of these modular invariant matrices corresponding to D_n and $E_{6,7,8}$ in his lectures [24] by using a notion of essential paths. In December 1997, he introduced the notion of *quantum Kleinian invariants* which is the quantum version of Kleinian invariant and he showed another new explanation of the off-diagonal terms. After A. Ocneanu's work on the fusion rule algebras of K-K bi-unitary connections essentially the same fusion rule algebras are constructed from conformal inclusions of SU(2) Wess-Zumino-Witten models by F. Xu ([37]) and J. Böckenhauer-D. E. Evans ([3], [4]) in flat cases. Note that the non-flat case, i.e., the case of D_{odd} and E_7 can not be obtained from their approach using conformal inclusions. Moreover the theory of double triangle algebra is recently used to generalize the result to the case of conformal inclusions of SU(n) WZW models by J. Böckenhauer-D. E. Evans-Y.

Kawahigashi ([5]). They showed that A. Ocneanu's observation of the relation between fusion rule algebras and modular invariant matrices holds true for some more general cases including the case of SU(n) WZW models.

Now we give a brief outline of the contents in this paper. In the next section we will give some definitions and terminologies concerning A. Ocneanu's double triangle algebras and we also fix some notations. Though all of the definitions of important notions such as essential paths, gaps of finite graphs and chiral projectors are given in [24], we did not omit them for reader's convenience because they are indispensable for the classification of connections. We refer readers to [24] for more details.

In section 3 we define some operations on the set of connections such as direct sum, conjugation, irreducible decomposition and composition (product). These operations are first defined by A. Ocneanu in [24] and later M. Asaeda-U. Haagerup clarified the correspondence between these operations on connections and those on bimodules ([1]). In order to deal with the system of connections closed under these operations we define a notion of *horizontally conjugate pair* of connections and give a natural identification between connections. We also give some equivalence relations on connections. We will emphasize the difference of *vertical gauge choice* and *total gauge choice* and will make it clear that the *vertical gauge choice* is the right equivalence relation to deal with a system of connections. This point is also clarified by M. Asaeda-U. Haagerup ([1]). In order to make the most of M. Asaeda-U. Haagerup's notion of *generalized open string bimodule* we will show that Frobenius reciprocity holds for the system of connections.

Section 4 is devoted to show the correspondence between irreducible K-L bi-unitary connections and irreducible $*$-representations of the double triangle algebras on the graphs K and L. This is the most important tool to classify the irreducible connections. Though a detailed proof of the correspondence is given in his original paper [24], some more details are necessary for our purpose. So we will supply it here and as a corollary we show an important correspondence between some special minimal central projections of the double triangle algebra and some irreducible bi-unitary connections on the Dynkin diagrams. We also show the relation between the fusion rule algebra of K-K bi-unitary connections and the center of the double triangle algebra. Actually it turns out that the fusion rule algebra is isomorphic to the center of double triangle algebra with different product from original one. One will notice that the notion of *horizontally conjugate pair* and the equivalence relation *vertical gauge choice* are both natural to deal with this correspondence.

Finally in section 5, the classification result is explained in each case of the Dynkin diagrams $A_n, D_n, E_{6,7,8}$. In the procedure to get this result we also obtain the new fusion rule algebras which consists of all K-K bi-unitary connections. It also provides a simple proof of the flatness of D_{2n}, E_6 and E_8 connections. Hence we get another proof of the complete classification of subfactors of the hyperfinite II_1 factor with index less than 4 by this method. The flat part of non-flat connections D_{2n+1} and E_7 are also obtained easily. By putting together all the cases of A-D-E we will obtain some important structural result on the fusion rule algebras including a partial commutativity of the fusion rule algebra.

2 Preliminaries and Notations

In this section we give definitions of essential paths on finite graphs and the double triangle algebras for the sake of completeness. We also fix some notations. We refer readers to [24] for the details.

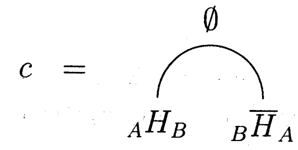


Figure 1: Creation operator

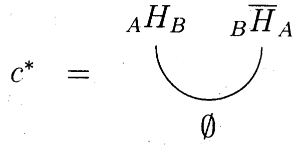


Figure 2: Annihilation operator

2.1. Wenzl projectors and Essential paths. Consider a finite oriented bipartite graph $\mathcal{G}$. We denote even vertices of the graph $\mathcal{G}$ by $\mathrm{Vert}^0\mathcal{G}$ and its odd vertices by $\mathrm{Vert}^1\mathcal{G}$. We denote an abelian C^*-algebra with basis $v \in \mathrm{Vert}^0\mathcal{G}$ [resp. $v \in \mathrm{Vert}^1\mathcal{G}$] by A [resp. B]. We use a notation H for a Hilbert space with basis $e \in \mathrm{Edge}^{(0,1)}\mathcal{G}$. Here $\mathrm{Edge}^{(0,1)}\mathcal{G}$ represents the set of oriented edges of $\mathcal{G}$ with orientation from $\mathrm{Vert}^0\mathcal{G}$ to $\mathrm{Vert}^1\mathcal{G}$. The Hilbert space H defined as above becomes an A-B bimodule with the action defined as follows, $x \cdot \xi \cdot y = \delta_{x,s(\xi)}\delta_{r(\xi),y}\xi$ for $x \in \mathrm{Vert}^0\mathcal{G}$, $y \in \mathrm{Vert}^1\mathcal{G}$ and $\xi \in \mathrm{Edge}^{(0,1)}\mathcal{G}$. Here $s(\xi)$ [resp. $r(\xi)$] means the source (starting point) [resp. range (end point)] of the edge ξ. The Hilbert space H has the conjugate Hilbert space $\bar{H}$ with the basis consists of the orientation reversed edges $\bar{\xi} \in \mathrm{Edge}^{(1,0)}\mathcal{G}$. Then the conjugate Hilbert space $\bar{H}$ becomes a B-A bimodule. We denote the adjacency matrix of $\mathcal{G}$ by $\Delta_{\mathcal{G}}$ and the Perron-Frobenius eigenvector for $\Delta_{\mathcal{G}}$ by μ which satisfy the condition $\Delta_{\mathcal{G}}\mu = \beta\mu$. Here β is the Perron-Frobenius eigenvalue of $\Delta_{\mathcal{G}}$.

Now we define the *annihilation operator* $c \in \mathrm{Hom}({}_AH_B \otimes {}_B\bar{H}_A, A)$ by the following.

$$c(\xi \otimes \overline{\eta}) = \delta_{\xi,\eta}\mu(r(\xi))^{1/2}\mu(s(\xi))^{-1/2} \cdot s(\xi)$$

for $\xi, \eta \in \mathrm{Edge}\,\mathcal{G}$.

Its adjoint operator is called the *creation operator* which is given by the following.

$$c^*(x) = \sum_{\xi \in \mathrm{Edge}\,\mathcal{G},\ s(\xi)=x} \xi \otimes \bar{\xi}\ \mu(r(\xi))^{1/2}\mu(s(\xi))^{-1/2}.$$

We draw diagrams for the operators c and c^* as in Figures 1 and 2.

We remark that the composition of these operators $c \cdot c^*$ becomes scalar multiplication operator β. We define the *Jones projection* on the path spaces on the graph $\mathcal{G}$ by $e = \beta^{-1}c^* \cdot c$. For these operators we draw pictures as in Figures 3 and 4.

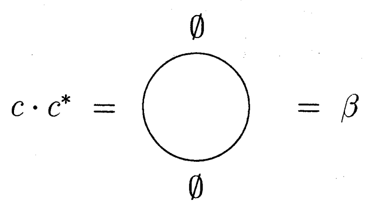


Figure 3:

$$e = \beta^{-1}c^* \cdot c = \beta^{-1}$$

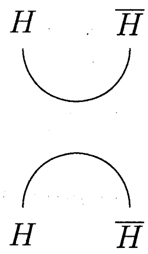


Figure 4: The Jones projection

$$p_n = 1 - e_1 \vee e_2 \vee \cdots \vee e_{n-1} \quad =$$

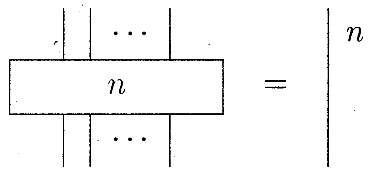


Figure 5: Wenzl projector p_n

We denote the set of paths of a graph $\mathcal{G}$ with length n by $\mathrm{Path}^{(n)}\mathcal{G}$, i.e. $\mathrm{Path}^{(n)}\mathcal{G} = \{\xi = (\xi_1, \xi_2, \cdots, \xi_n) \mid \xi_k \in \mathrm{Edge}\,\mathcal{G},\ s(\xi_{k+1}) = r(\xi_k)\}$, and denote the Hilbert space with orthonormal basis $\xi \in \mathrm{Path}^{(n)}\mathcal{G}$ by $\mathrm{HPath}^{(n)}\mathcal{G}$. Note that n times (relative) tensor products ${}_AH_B \otimes {}_B\bar{H}_A \otimes \cdots \otimes {}_AH_B(\mathrm{or}\, {}_B\bar{H}_A)$ produce the path Hilbert space $\mathrm{HPath}^{(n)}\mathcal{G}$. We can define the sequence of creation/annihilation operators $c_1, c_2, \ldots c_{n-1}$ and that of the Jones projections $e_1, e_2, \ldots e_{n-1}$ on the Hilbert space $\mathrm{HPath}^{(n)}\mathcal{G}$ depending on the position where they act.

Definition 2.1 The *Wenzl projectors* p_n on $\mathrm{HPath}^{(n)}\mathcal{G}$ is defined by $p_n = 1 - e_1 \vee e_2 \vee \cdots \vee e_{n-1}$. We draw the picture in Figure 5 for the Wenzl projector p_n. The space of *essential paths* with length n on a graph $\mathcal{G}$ is defined by $\mathrm{EssPath}^{(n)}\mathcal{G} = p_n \cdot \mathrm{HPath}^{(n)}\mathcal{G}$. We denote the space of essential paths of a graph $\mathcal{G}$ with length n, with starting point x and end point y by $\mathrm{EssPath}^{(n)}_{x,y}\mathcal{G}$.

We remark that the space of essential paths can be defined as follows.

$$\begin{aligned}\mathrm{EssPath}^{(n)}\mathcal{G} &= \{\xi \in \mathrm{HPath}^{(n)}\mathcal{G} \mid e_k\xi = 0 \text{ for } k = 1, 2, , \cdots, n-1\} \\ &= \{\xi \in \mathrm{HPath}^{(n)}\mathcal{G} \mid c_k\xi = 0 \text{ for } k = 1, 2, , \cdots, n-1\}.\end{aligned}$$

The following *Moderated Pascal rule* is quite useful to count a dimension of essential paths.

$$\dim \mathrm{EssPath}^{(n+1)}_{a,x}\mathcal{G} = \sum_{\xi \in \mathrm{Edge}\,\mathcal{G}, r(\xi) = x} \dim \mathrm{EssPath}^{(n)}_{a,s(\xi)}\mathcal{G} - \dim \mathrm{EssPath}^{(n-1)}_{a,x}\mathcal{G}$$

For the proof of this rule, see [24, Section 5].

2.2. Extension of recoupling model and the double triangle algebra. Next we define an extended model of Kauffman-Lins' recoupling theory ([17]) from a viewpoint of subfactor theory by using the notion of essential paths. First we remark that the recoupling model for $q = e^{i\pi/N}$ a root of unity can be realized by using the fusion rule algebra of sector (or bimodule) and quantum 6j-symbols arising from the Jones' subfactor with principal

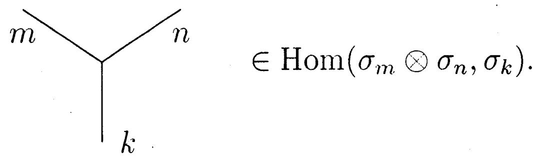


Figure 6: An intertwiner

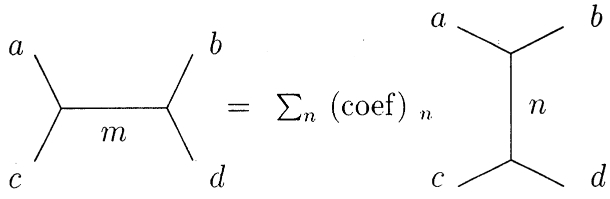


Figure 7: Recoupling

graph A_{N-1}. For example the trivalent vertex as in Figure 6 represent an intertwiner in $\mathrm{Hom}(\sigma_m \otimes \sigma_n, \sigma_k)$. Here σ_j is an irreducible sector (or bimodule) corresponding to j-th vertex from the distinguished vertex $*$ of the principal graph of type A_{N-1} Jones' subfactors. The other notions in the recoupling theory such as θ-evaluations, tetrahedral nets and (quantum) 6j-symbols will be interpreted in terms of sectors and intertwiners arising from the Jones' subfactor. Especially we have the recoupling as in Figure 7. Here the coefficient is given as follows.

$$(\mathrm{coef})_n = \left\{ \begin{matrix} a & b & n \\ c & d & m \end{matrix} \right\} = \frac{\mathrm{Tet}\begin{bmatrix} a & b & n \\ c & d & m \end{bmatrix} \Delta_n}{\theta(a,b,n)\ \theta(c,d,n)}$$

Here $\theta(a,b,c)$ means the θ-evaluation and

$$\mathrm{Tet}\begin{bmatrix} a & b & n \\ c & d & m \end{bmatrix}$$

represents a value of the tetrahedral net. (See [17], [24, Section 12].) The special case when $m = 0$ is given in Figure 8 and we use this to define the convolution product of the double triangle algebras.

Now fix a recoupling model A which corresponds to a Perron-Frobenius eigenvalue β and let K be one of the Dynkin diagrams $A_n, D_n, E_{6,7,8}$ with the same Perron-Frobenius eigenvalue. We draw a picture for an essential path $\xi \in \mathrm{EssPath}^{(n)}_{x,y} K$ as in the left hand

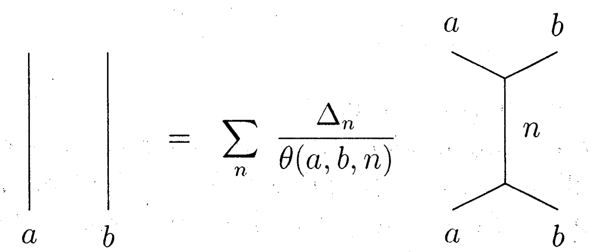


Figure 8:

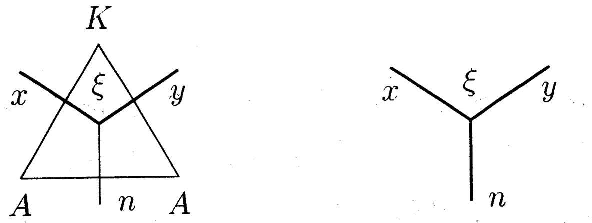


Figure 9:

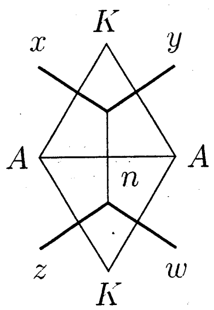


Figure 10:

side of Figure 9 or we simply draw the picture in the right hand side. (See [24, Section 10].) Then a *double triangle algebra* $\mathcal{A}$ is defined as an algebra which elements are linear combinations of pairs of essential paths as in Figure 10. We call a double triangle algebra defined on the graphs K and L which correspond to upper and lower horizontal graphs respectively a *K-L double triangle algebra*. Two products are defined on this algebra. One is $\cdot$ product defined as in Figure 11. The other product called *convolution product* is defined as in Figure 12 and denoted by $*$. We decompose the element of the right hand side in this figure as in Figure 13 by using recoupling and the equality in Figure 14. The $*$-operation for the convolution product on the double triangle algebra is given by Figure 15.

2.3 Ocneanu's chiral projectors. We recall that special elements in $(\mathcal{A}, *)$ is defined by

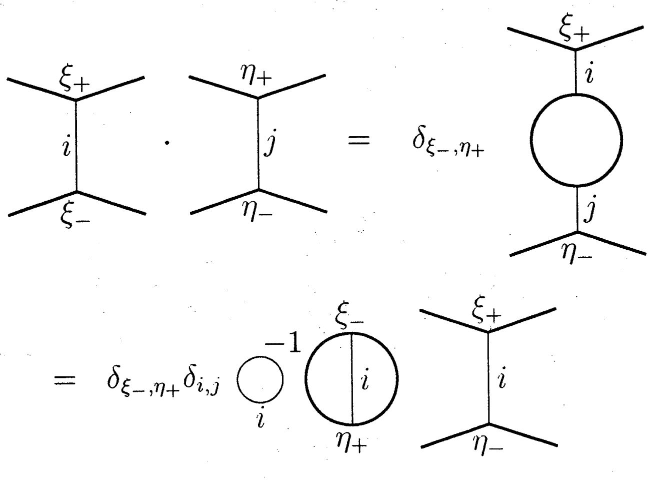


Figure 11: $\cdot$ product on the double triangle algebra

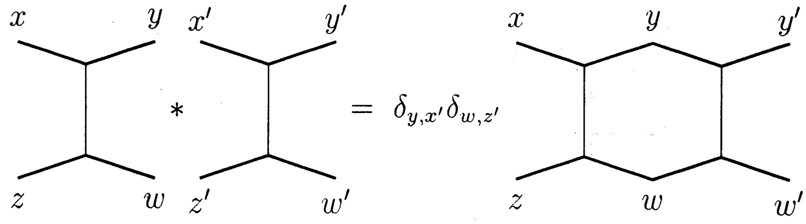


Figure 12: The convolution product on the double triangle algebra

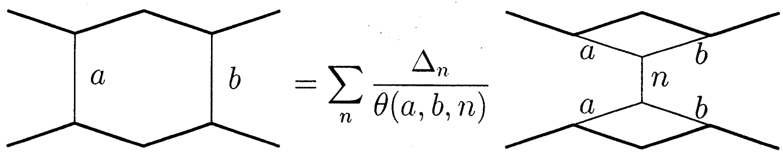


Figure 13:

Figure 16 and Figure 17 and give the definition of Ocneanu's *chiral projectors.*

Definition 2.2 The *chiral projectors* Ψ_+ and Ψ_- which are central projections in the double triangle algebra $(\mathcal{A}, *)$ are defined as in Figure 18 and 19. The product of the two chiral projectors $\Psi_+ * \Psi_-$ is called the *ambichiral projector* and is denoted by $\Psi_\pm$.

2.4 Gaps on the Dynkin diagrams and minimal central projections. The *gap* and *0-gap* of a finite graph G are numbers (positive integer or ∞) defined by the following. (See [24, Section 17])

$$\begin{aligned}\mathrm{gap}(G) &\equiv \min\{n > 0 \mid \mathrm{EssPath}^{(n)}_{a,a} G \neq 0 \ \text{ for all } \ a \in \mathrm{Vert}G\},\\ 0 - \mathrm{gap}(G) &\equiv \min\{n > 0 \mid \mathrm{EssPath}^{(n)}_{0,0} G \neq 0\}.\end{aligned}$$

Here 0 represents the distinguished vertex of the Coxeter graph G, i.e., the vertex of G which has the smallest Perron-Frobenius weight. The gaps and the 0-gaps of the Dynkin diagrams are given in the following table.

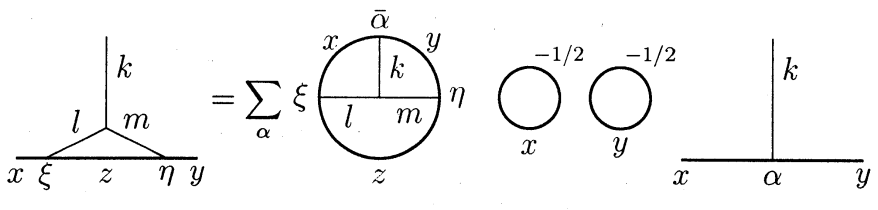


Figure 14:

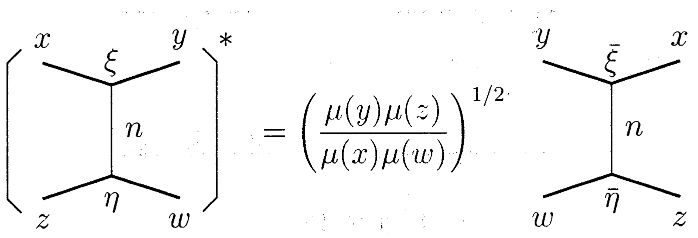


Figure 15: The $*$-operation for the convolution product

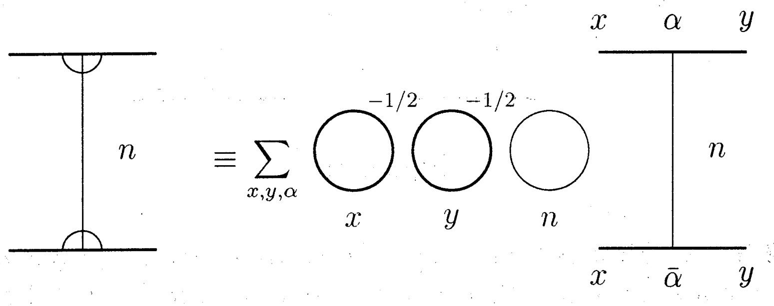


Figure 16:

Graph G	gap(G)	0-gap(G)
A_n	∞	∞
D_{2n+1}	∞	$4n-2$
D_{2n}	$4n-4$	$4n-4$
E_6	6	6
E_7	16	8
E_8	10	10

Let K be a connected finite bipartite graph and $(\mathcal{A}, *)$ a double triangle algebra on K endowed with the convolution product. The following proposition shows that there are two finite family of minimal central projections $\{p_k^+\}_k$ and $\{p_k^-\}_k$ on the double triangle algebra $(\mathcal{A}, *)$.

Proposition 2.3 ([24, Proposition 17.3, Corollary 17.4]) *Two elements p_k^+ and p_k^- defined by Figure 20 and 21 are minimal central projections if $k < \mathrm{gap}(K)/2$.*

For orthogonality of these projections we have the following propositions.

Proposition 2.4 ([24, Proposition 18.1]) *The minimal central projections p_k^+ (resp. p_k^-) are mutually orthogonal if $k < \mathrm{gap}(K)/2$.*

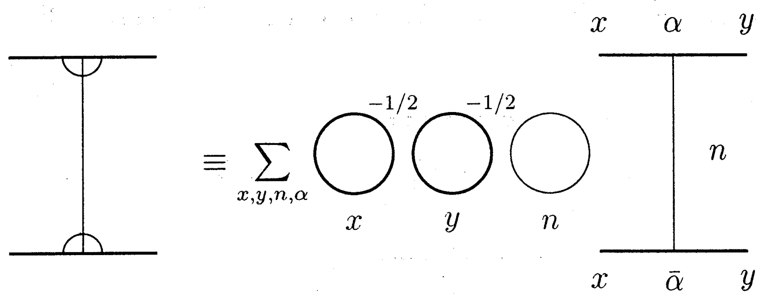


Figure 17:

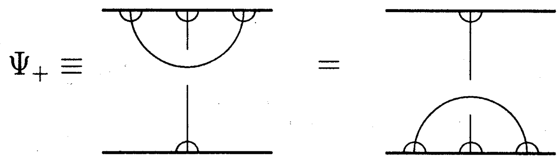


Figure 18:

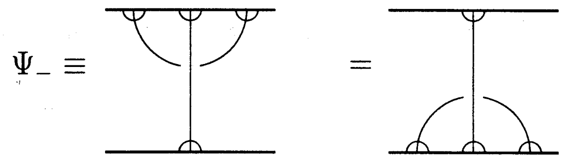


Figure 19:

Proposition 2.5 ([24, Proposition 18.2]) *The minimal central projections p_k^+ and p_l^- as in the previous section are mutually orthogonal if $k \neq l$ and $k + l <$ 0-gap(K).*

3 Operations and equivalence relations of connections, conjugate pairs and Asaeda-Haagerup's generalized open string bimodules

In this section we define some operations on the set of connections such as direct sum, composition, irreducible decomposition and conjugation etc. (See [24, Section 20].) We also define some equivalence relations on it. We shall give a natural identification of connections and define the above operations on the set of equivalence classes of connections with the identification. These operations are originally defined by Ocneanu ([24, Section 20]). Later Asaeda and Haagerup ([1]) introduced the notion of generalized open string bimodules which is a generalization of open string bimodule of Ocneanu ([20]) and Sato ([30]) and they clarified the relation between connections and bimodules. We remark that the identification of connections given in this section is different Asaeda-Haagerup's setting. Here we will also give their original definitions for reader's convenience.

Remark 3.1 In this paper connections on four graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ as in Figure 22 are always assumed to have connected horizontal graphs $\mathcal{G}_0$ and $\mathcal{G}_2$. We do not assume that the vertical graphs $\mathcal{G}_1$ and $\mathcal{G}_3$ are connected. The word *connection* always means *bi-unitary connection* in this paper. So we will often use the word *connection* instead of '*bi-unitary connection*' for simplicity.

First we define the notion of direct sum, composition, irreducibility and conjugation on the set of bi-unitary connections. (See [24, Section 20], [1, Section 3].)

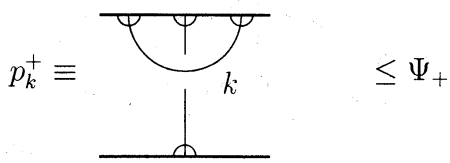


Figure 20:

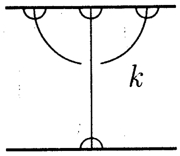
$p_k^- \equiv$  $\leq \Psi_-$

Figure 21:

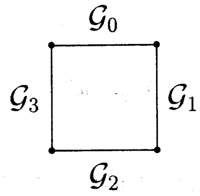


Figure 22:

Definition 3.2 ([24, Definition 20.2]) Let W_1 and W_2 be two bi-unitary connections on four graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and $\mathcal{G}_0, \mathcal{G}'_1, \mathcal{G}_2, \mathcal{G}'_3$ respectively, then a *direct sum* of these bi-unitary connections is a bi-unitary connection W on four graphs $\mathcal{G}_0, \mathcal{G}_1 \sqcup \mathcal{G}'_1, \mathcal{G}_2, \mathcal{G}_3 \sqcup \mathcal{G}'_3$ defined by the following. See Figure 23.

$$W\left(\begin{array}{ccc} & \xi_0 & \\ \xi_3 & & \xi_1 \\ & \xi_2 & \end{array}\right) = \begin{cases} W_1\left(\begin{array}{ccc} & \xi_0 & \\ \xi_3 & & \xi_1 \\ & \xi_2 & \end{array}\right), & \text{if} \quad \xi_0 \in \mathcal{G}_0, \xi_1 \in \mathcal{G}_1, \xi_2 \in \mathcal{G}_2, \xi_3 \in \mathcal{G}_3, \\ W_2\left(\begin{array}{ccc} & \xi_0 & \\ \xi_3 & & \xi_1 \\ & \xi_2 & \end{array}\right), & \text{if} \quad \xi_0 \in \mathcal{G}_0, \xi_1 \in \mathcal{G}'_1, \xi_2 \in \mathcal{G}_2, \xi_3 \in \mathcal{G}'_3, \\ 0, & \text{otherwise.} \end{cases}$$

We denote a direct sum bi-unitary connection W of two bi-unitary connections W_1 and W_2 by $W_1 \oplus W_2$.

Definition 3.3 ([24, Section 20]) Let W be a connection on four graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and W' a connection on other four graphs $\mathcal{H}_0 = \mathcal{G}_2, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ which has the common

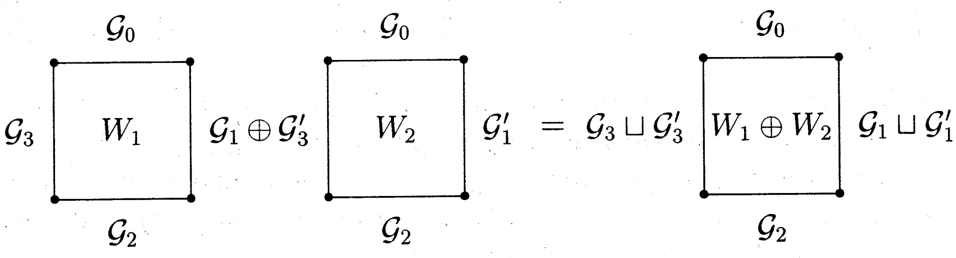


Figure 23: Direct sum of two bi-unitary connections

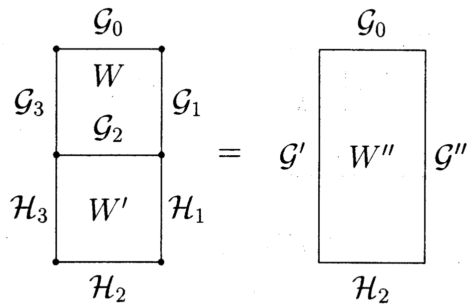


Figure 24: Composition (or product) of two connections

graph $\mathcal{H}_0 = \mathcal{G}_2$, then we define a composition of them by a bi-unitary connection W'' obtained by connecting these graphs, making products of both connections and summing them over all the common horizontal edges as in Figure 24. Here the vertical graphs $\mathcal{G}'$ and $\mathcal{G}''$ of the composed connection W'' will change by this construction. We denote the composite connection W'' by $W \cdot W'$ or simply WW'.

Definition 3.4 ([24, Definition 20.3]) A bi-unitary connection W on four graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are called *reducible* if there exist two bi-unitary connections W_1 and W_2 on four graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ and $\mathcal{G}_0, \mathcal{G}_1', \mathcal{G}_2, \mathcal{G}_3'$ respectively such that the direct sum of them produces W as in Figure 25 up to *vertical* gauge choice. We call a bi-unitary connection W *irreducible* if it is not reducible.

Remark 3.5 We remark that this definition of reducibility is the same as Asaeda-Haagerup's ([1, Section 3]). In general the reducibility up to *vertical* gauge choices is different from that up to *total* gauge choices. But the next lemma shows that both definition coincides if two horizontal graphs $\mathcal{G}_0$ and $\mathcal{G}_2$ as in Figure 22 are trees, i.e., if both graphs only have single edges and have no cycle. (cf. [1, Remark, Section 3].) Especially in the case when the horizontal graphs are both among the Dynkin diagrams $A_n, D_n, E_{6,7,8}$ the two equivalence relations coincide.

Lemma 3.6 *If one of the two horizontal graphs $\mathcal{G}_0$ is a tree, then any gauge choices on $\mathcal{G}_0$ can be forced to put on vertical gauge choice. In particular if the two horizontal graphs $\mathcal{G}_0$ and $\mathcal{G}_2$ are both trees, then two equivalence relation 'total gauge choice' and 'vertical gauge choice' on the set of connections on the four graphs coincide.*

Proof Take a gauge choice $\alpha \in \mathbf{C}$ ($|\alpha| = 1$) on an edge of $\mathcal{G}_0$. We can easily see that the same gauge choice can be given by taking a gauge choice on vertical graphs $\mathcal{G}_1$ and $\mathcal{G}_3$ which consists only gauge α on some edges of $\mathcal{G}_1$ and α^{-1} on some edges on $\mathcal{G}_3$. (This can be shown by taking a vertical gauge step by step.) The existence of such a gauge choice is assured by the fact that there is no multiple edge and no cycle on the graph $\mathcal{G}_0$. $\square$

Remark 3.7 *The condition in the above lemma that the graph $\mathcal{G}_0$ does not have any cycle is necessary. Consider the case when all the four graphs are $A_{2n+1}^{(1)}$ ($n \geq 1$). It is easy to see that you can not force to put horizontal gauge choice to vertical ones in these examples. This shows that even if the two horizontal graphs consist of single edges, it may happen that the above two equivalence relations do not coincide.*

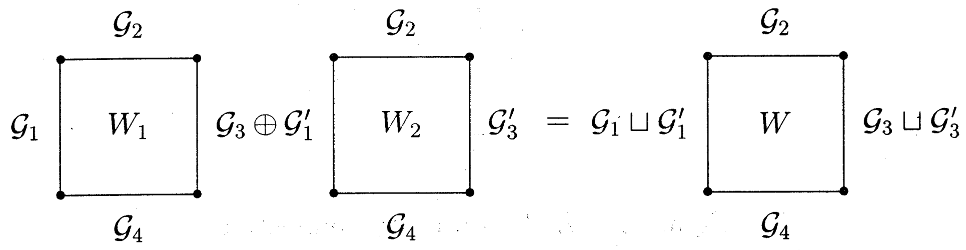


Figure 25: Reducibility of a bi-unitary connection W

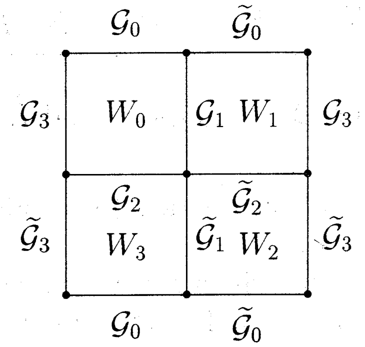


Figure 26: Four connections associated to a connection W_0

In this paper an *equivalence relation* on connections will mean *vertical gauge choice* unless otherwise stated. Before we define a natural identification of connections we remark first that four different connections which are transferred each other by renormalization rule are associated to one bi-unitary connection W_0 as in Figure 26, where $\tilde{\mathcal{G}}$ represents the graph $\mathcal{G}$ with reversed orientation.

Definition 3.8 We say two connections W_0 and W_1 (resp. W_0 and W_3) as in Figure 26 are *horizontally* (resp. *vertically*) *conjugate*. We denote them by W_0^h and W_0^v respectively. We will call vertically conjugate connection W_0^v simply a *conjugate* connection of W_0. We define a *conjugation* operation on a set of connections by a vertical conjugation in this sense. The conjugation operation is often denoted by $\bar{\cdot}$. So we will also adopt the notation $\bar{W}$ for a (vertically) conjugate connection of W.

In this paper, we regard two horizontally conjugate connections as the same one. In other words, we always consider a connection W_0 as a pair of two connections (W_0, W_1) which are horizontally conjugate each other. We call such a pair horizontally conjugate pair. By this identification the relation '*vertically conjugate*' still make sense. Moreover, we consider the two equivalence relation '*total gauge choice*' and '*vertical gauge choice*' on this identified set of connections. Again they are still equivalence relation on it.

In the following we consider the set of connections which have fixed common horizontal graphs. We will use the following terminology. (See [24, Section 20].)

Definition 3.9 Let K and L be two connected finite bipartite graphs. A bi-unitary connection on four graphs is called a *K-L bi-unitary connection* if it has the graph K as an upper horizontal graph and the graph L as a lower horizontal graph as in Figure 27.

Note that we can naturally define the operations such as direct sum, conjugation and irreducible decomposition on the set of equivalence classes of connections with the above

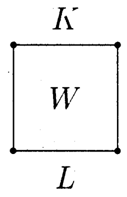


Figure 27: K-L bi-unitary connection

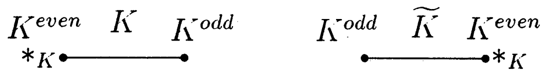


Figure 28: A graph K and $\widetilde{K}$

identification. Before we define the composition (or product) operation on it we need some more notations and terminology.

We often denote a graph K with a distinguished vertex $*_K$ by a pair $(K, *_K)$. Let $(K, *_K)$ be a connected finite bipartite graph with a distinguished vertex. A vertex of K with the same (resp. different) colour as $*_K$ is called *even* (resp. *odd*) vertex. The set of even and odd vertices are denoted by K^{even} and K^{odd} respectively. In the following whenever we consider a horizontal graph $(K, *_K)$, the notation K will represent the graph K with their even vertices on the left hand and odd vertices on the right hand. The graph K with reversed orientation, that is, one with their odd vertices on the left and even vertices on the right will be denoted by $\widetilde{K}$. (See Figure 28.)

Note that when two horizontal graphs K and L are connected finite bipartite with distinguished vertices, four kinds of K-L bi-unitary connections will be distinguished by the above notation which respects the orientations of the graphs. That is, there are four kinds of K-L bi-unitary connections depending on which graph K or $\widetilde{K}$ and L or $\widetilde{L}$ they actually have as horizontal graphs.

For a given connection we can naturally associate the index of its generalized open string bimodule [1]. We call it an *index* of a connection. This value is the same as the square root of the index of subfactor constructed from the connection.

Now we define a composition (or product) of two (pairs of) connections.

Definition 3.10 Let $(K, *_K)$, $(L, *_L)$ and $(M, *_M)$ be three connected finite bipartite graphs with distinguished vertices. Let α be a K-L bi-unitary connection and β a L-M bi-unitary connection. We regard these as two pairs of connections (α, α^h) and (β, β^h) as in the above setting. Define a *composition* (or *product*) of two (pairs of) connections α and β by a (pair of) connection obtained by the following procedure. When α and β have the common graph L as bottom and top graphs respectively, then we just make a product $\alpha \cdot \beta$ and regard this as a pair of connections $(\alpha \cdot \beta, \alpha^h \cdot \beta^h)$. When one of α and β have the graph L as a top or bottom graph and the other have the graph $\widetilde{L}$, then we make a product $\alpha \cdot \beta^h$ or a product $\alpha^h \cdot \beta$. Again we regard it as a pair of connections $(\alpha \cdot \beta^h, \alpha^h \cdot \beta)$. A product of two (pairs of) connections α and β in the above sense will be simply denoted by $\alpha \cdot \beta$ (or $\alpha\beta$) when it does not cause any confusion.

¿From the above definition we are now ready to deal with a system of K-K bi-unitary connections which is closed under direct sum, product, conjugation and irreducible decomposition. The precise definition is given in the following.

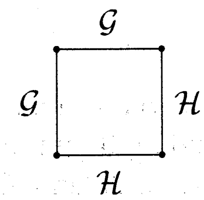


Figure 29:

Definition 3.11 Let K be a connected finite bipartite graph and ${}_K\mathcal{W}_K$ be a set of equivalence classes of horizontally conjugate pairs of K-K bi-unitary connections with respect to the equivalence relation 'vertical gauge choice'. Then a set ${}_K\mathcal{W}_K$ is called a *system of K-K bi-unitary connections* or simply a *K-K bi-unitary connection system* if it is closed under direct sum, product, conjugation and irreducible decomposition. Another word '*fusion rule algebra of K-K bi-unitary connections*' will often be used in the same meaning. A K-K bi-unitary connection system is said to be *finite* if it contains only finitely many irreducible connections. The rule of irreducible decomposition of products of two irreducible connections in a system is called the *fusion rule* of the system.

Remark 3.12 *Note that all the above operations are well-defined on a set of equivalence classes of horizontally conjugate pairs of K-K bi-unitary connections with respect to the equivalence relation 'vertical gauge choice'. It should be remarked that if we adopt an equivalence relation 'total gauge choice' instead of 'vertical gauge choice', composition of two connection will not necessarily be well-defined.*

Example 3.13 Obviously there are two trivial examples of K-K bi-unitary connection systems. One is a system which consists of only trivial (identity) connection as its irreducible object. The other is a system consisting of all K-K bi-unitary connections. For any given family of K-K bi-unitary connections there exists a system generated by them. So the word *generator* of a system makes sense in this setting.

Example 3.14 Let $N \subset M$ be a subfactor with finite index and finite depth. Then we obtain a (flat) bi-unitary connection W on the four graphs as in Figure 29 by Ocneanu's Galois functor. Where $\mathcal{G}$ is the principal graph and $\mathcal{H}$ is the dual principal graph. As usual we denote the (vertical) conjugate connection of W by $\bar{W}$. The composition of two connections W and $\bar{W} = W^v$ produce a $\mathcal{G}$-$\mathcal{G}$ bi-unitary connection $W\bar{W}$ and a $\mathcal{H}$-$\mathcal{H}$ bi-unitary connection $\bar{W}W$. A system of $\mathcal{G}$-$\mathcal{G}$ bi-unitary connections generated by $W\bar{W}$ is finite by the finite depth assumption. This can be regarded the same system as N-N bimodules arising from the subfactor $N \subset M$. Another system of $\mathcal{H}$-$\mathcal{H}$ connections generated by a connection $\bar{W}W$ also corresponds to a system of M-M bimodule arising from the subfactor. This shows that for every subfactor with finite index and finite depth we can associate two systems of bi-unitary connections. We will obtain many non-trivial finite systems of connections in this way.

Example 3.15 Another fundamental example is a system generated by one bi-unitary connection, i.e., a *singly generated* system. Let $(K, *_K)$ and $(L, *_L)$ be two connected finite bipartite graphs with the same Perron-Frobenius eigenvalue. Any K-L bi-unitary connection W yields a hyperfinite II_1 subfactor $N \subset M$ by a string algebra construction. This connection W generates a system of four kinds of generalized open string bimodules

([1]) which has the same fusion rule as the system of bimodule arising from the subfactor $N \subset M$. By looking at the corresponding bi-unitary connections, we get a system of four kinds of connections, i.e., K-K, K-L, L-K and L-L connections. Especially we obtain K-K and L-L connection system in this way. If the subfactor $N \subset M$ has finite depth, then both systems will be finite. And if it has infinite depth, they become infinite systems.

Remark 3.16 *The procedure in Example 3.15 looks similar to that of Example 3.14. Actually the former is the special case of the latter. Here we remark that the latter is much more general because the connections appear in the former example is always flat and infinite system can not be obtained by the former procedure. Moreover a flat connection obtained by the Galois functor is very special because they have (dual) principal graphs as the four graphs. The following example shows this speciality of flat connections obtained by the Galois functor. Flat connections obtained by Sato's procedure ([29, Theorem 2.1]) which is a generalization of the example of [8] show that any finite depth subfactor generated by a (not necessarily flat) bi-unitary connection W can be reconstructed by a different flat connection $W_f \cdot W$ which is a horizontally composed connection by its flat part connection W_f (See [29], [30]). Hence there are many examples of finite depth subfactors which are constructed by a flat connection that does not come from the Galois functor.*

Example 3.17 Let K be one of the Dynkin diagrams $A_n, D_n, E_{6,7,8}$. It is known that there are at most two non-equivalent bi-unitary connections on the four graphs which are all the same graph K and trivially connected. (There are only one non-equivalent connections in the case of A_n and exactly two mutually complex conjugate non-equivalent connections in the case of $D_n, E_{6,7,8}$.) We call them *fundamental connections* of the A-D-E Dynkin diagrams. We denote one of them by W and the other by $\widetilde{W}$. Let $N \subset M$ be a subfactor constructed by W in the horizontal direction.

First we consider the system generated by a single connection W. In this case the system is finite and has the same fusion rule as that of the system of bimodules arising from the subfactor $N \subset M$.

Next consider the system generated by the two bi-unitary connections W and $\widetilde{W}$. In this paper we will mainly deal with this system. It turns out that this system is finite and all the irreducible K-K bi-unitary connections appear in this system. (See section 5.)

Remark 3.18 *It is a remarkable fact that there are only finitely many K-K irreducible bi-unitary connections on the Dynkin diagrams K. This is one of the very special properties of the Dynkin diagrams. We should compare it to the following example. Consider the case when all the four graphs are the same graph as the principal graph of Goodman-de la Harpe-Jones subfactor with index $3+\sqrt{3}$ (see Figure 30) which arise from an embedding of the A_{11} string algebra to that of E_6 ([14], [25]). In this case there exists one parameter family of (hence uncountably many) non-equivalent bi-unitary connections on the four graphs. These connections are automatically irreducible by the criterion of Asaeda-Haagerup [1, Corollary 2, Section 3] This means that even if we fix not only the two horizontal graphs but all the four graphs it can happen that uncountably many irreducible non-equivalent connections exist on the graphs. The author would like to express his thanks to Y. Kawahigashi for pointing out this example.*

Remark 3.19 *In this paper we mainly deal with the case when the graph K is one of the A-D-E Dynkin diagrams. It is natural to take the usual distinguished vertex $*_K$ (i.e.,*

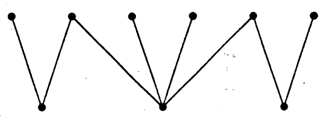


Figure 30: The principal graph of the Goodman-de la Harpe-Jones subfactor

the vertex with smallest Perron-Frobenius eigenvalue) in these cases and we do so in the following. We denote a hyperfinite II_1 *factor generated by the string algebra on the graph* K *with the starting point* $*_K$ *by the same notation* K. *For a reversed graph* $\widetilde{K}$ *we will take the vertex next to* $*_K$ *as a starting point of string algebra and again we denote its generating factor by the same* $\widetilde{K}$. *Then by generalized open string algebra construction ([1]) we can associate two different bimodules for a given horizontally conjugate pair of* K-K *bi-unitary connections. Again we deal with the pair of generalized open string bimodules as a corresponding bimodule to the original pair of connections. By working on this correspondence between pairs of connections and pairs of bimodule, we can show that Frobenius reciprocity holds for the system of connections as in the next proposition.*

Proposition 3.20 *Let* K, L *and* M *be three connected finite bipartite graphs with the same Perron-Frobenius eigenvalue. Let* ${}_K\alpha_L$, ${}_L\beta_M$ *and* ${}_K\gamma_M$ *be three (pairs of) irreducible bi-unitary connections which are* K-L, L-M *and* K-M *respectively. If* γ *appears* n *times in the composite connection* $\alpha\beta$, *then* α *appears* n *times in* $\gamma\bar{\beta}$ *and* β *appears* n *times in* $\bar{\alpha}\gamma$.

Proof Choose and fix distinguished vertices from even and odd vertices of each graphs K, L and M. Then apply the correspondence between (pairs of) bi-unitary connections and (pairs of) generalized open string bimodules. We get the result from the Frobenius reciprocity for bimodules ([22], [13, Section 9.8]). □

Let α be a K-L bi-unitary connection. Then from the rule of irreducible decomposition of finite product connections ${}_Kid_K \cdot {}_K\alpha_L \cdot {}_L\bar{\alpha}_K \cdots {}_K\alpha_L$ (or ${}_L\alpha_K$) we get a graph which is similar to the principal graph of a subfactor. Here ${}_Kid_K$ denotes an identity K-K connection. We call this graph the *principal fusion graph* of a connection α. Then we can show the following by Asaeda-Haagerup's criterion of irreducible decomposition of connections [1, Claim 1, Section 3]. This is what we observed in Example 3.14 and 3.15.

Proposition 3.21 *Let* α *be a* K-L *bi-unitary connection. Then the principal fusion graph of* α *is the same as the principal graph of the subfactor constructed from the connection* α.

4 Correspondence between connections and $*$-representations of double triangle algebras

Let a graph K be one of the Dynkin diagrams $A_n, D_n, E_{6,7,8}$ and $(\mathcal{A}, *)$ be the double triangle algebra corresponding to the graph K with the convolution product. This is a finite dimensional C^*-algebra and is isomorphic to a finite direct sum of matrix algebras as follows.

$$(\mathcal{A}, *) \cong \oplus_{i \in I} H_i \otimes \overline{H_i} \cong \oplus_{i \in I} \mathrm{End}(H_i).$$

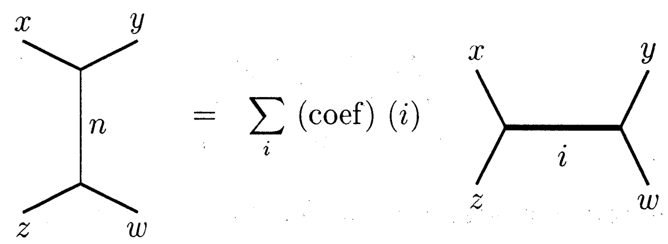


Figure 31:

Here the index i's are labelled by minimal central projections in $(\mathcal{A}, *)$ and H_i's are corresponding finite dimensional Hilbert spaces. This identification of elements in $(\mathcal{A}, *)$ is written as an extension of recoupling as in Figure 31.

Now we give the following theorem which means that we have only to find all the minimal central projections in $(\mathcal{A}, *)$ in order to classify all irreducible K-K bi-unitary connections.

Theorem 4.1 *There is one-to-one correspondence between unitary equivalence classes of irreducible matricial $*$-representations of the K-K double triangle algebra $(\mathcal{A}, *)$ and equivalence classes of irreducible K-K bi-unitary connections.*

Remark 4.2 *Equivalence relation on bi-unitary connections considered in this theorem is 'vertical gauge choice'. But it coincides with 'total gauge choice' in this case as we stated in Remark 3.5. (See Lemma 3.6.)*

Proof A proof for the construction of a bi-unitary connection for a given $*$-representation and vice versa is shown in [24, Section 15]. So we only have to show that two unitarily equivalent $*$-representation give two equivalent connections up to vertical gauge choice and vice versa. It is easy to see that two equivalent connections up to vertical gauge choice give two unitarily equivalent $*$-representation because the unitary matrix which transfer one representation to the other is given by the unitary matrix of gauge choice. For the other direction, first we have to find the vertical edges connecting two horizontal graphs K. Let Φ be a matricial $*$-representation of the K-K double triangle algebra $(\mathcal{A}, *)$. Because elements $a_{x,y} \in \mathcal{A}$ $(x, y \in \mathrm{Vert}K)$ as in Figure 32 are mutually orthogonal projections, the matrices $\Phi(a_{x,y})$ are diagonalized with only 0 and 1 in the diagonal entries by a certain unitary. We draw edges connecting the vertices x and y with the same numbers as that of 1 in the diagonalized matrix $\Phi(a_{x,y})$. In this way we get the vertical edges connecting the two horizontal graphs. We label these vertical edges by some index set Λ. Then we define a connection value of a rectangle as in Figure 33 by the number $\Phi_{\lambda,\mu}(\xi \otimes \eta)$ for $\xi, \eta \in \mathrm{EssPath}^{(n)}(K)$ and $\lambda, \mu \in \Lambda$. Here $\xi \otimes \eta$ is an element in $\mathcal{A}$ as in Figure 34 and $\Phi_{\lambda,\mu}(\xi \otimes \eta)$ represents a (λ, μ)-th entry of the matrix $\Phi(\xi \otimes \eta)$. If we restrict the map from $\mathcal{A}$ to the complex numbers $\mathbf{C}$ defined as above to $\mathrm{EssPath}^{(1)}(K) \otimes \mathrm{EssPath}^{(1)}(K)$, we get the connection map. Now the bi-unitarity condition of this connection follows easily from the fact that Φ is a $*$-representation. (See [24, Section 15].) Note that in this procedure to get a connection W^{Φ} from a given $*$-representation Φ, it is easy to see that if the representation Φ is reducible, then the connecting vertical edges as well as the connections W^{Φ} defined on the four graphs decompose into some irreducible components which corresponds to the irreducible components of the $*$-representation Φ. So the irreducibility is preserved by this correspondence. Now it is easy to see that the

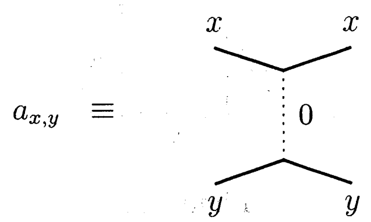


Figure 32:

$$u_{\lambda,\mu}(\xi,\eta) = \begin{tikzcd} a \arrow[r, "\xi"] \arrow[d, "\lambda"'] & b \arrow[d, "\mu"] \\ c \arrow[r, "\eta"'] & d \end{tikzcd}$$

Figure 33:

two unitarily equivalent $*$-representation gives the two equivalent bi-unitary connections up to vertical gauge choice. Moreover one will notice that a given $*$-representation gives rise to two horizontally conjugate connections at the same time. $\square$

Remark 4.3 *The above correspondence in Theorem 4.1 hold true for the case of K-L double triangle algebras and K-L bi-unitary connections. The proof is exactly the same as the proof of Theorem 4.1. But we remark that we do not have minimal central projections $p_k^{\pm}$ as in Section 2 in the case of K-L double triangle algebras when $K \neq L$.*

Applying the above theorem to the concrete $*$-representation corresponding to the minimal central projection $p_1^{\pm}$ as in Section 2, we get the following important result.

Corollary 4.4 *Let K be one of the Dynkin diagrams $A_n, D_n, E_{6,7,8}$. The minimal central projections $p_1^{\pm}$ of the K-K double triangle algebra correspond to the two mutually complex conjugate (flat) bi-unitary connections on the four graphs which are all the same Dynkin diagram K as in 35. In particular $p_1^+ = p_1^-$ in the case of the Dynkin diagram A_n.*

Proof By looking at the shape of the minimal central projections $p_1^{\pm}$, the vertical graphs of the corresponding irreducible K-K connections are the graph K itself. Because we know that there are two mutually complex conjugate non-equivalent connections on the four graphs as in Figure 35 when the graph K is one of $D_n, E_{6,7,8}$. And there is only one bi-unitary connection when the graph K is one of A_n. (See [13, Theorem 11.22]. We remark that Theorem 11.22 in [13] states the isomorphic classes of connections whereas it is also true for the equivalence classes.) The difference of two connections corresponding

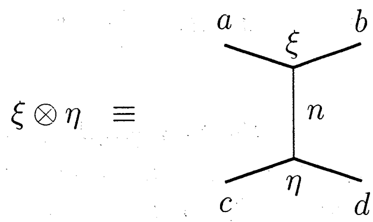


Figure 34:

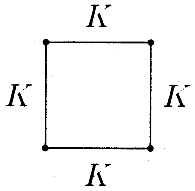


Figure 35:

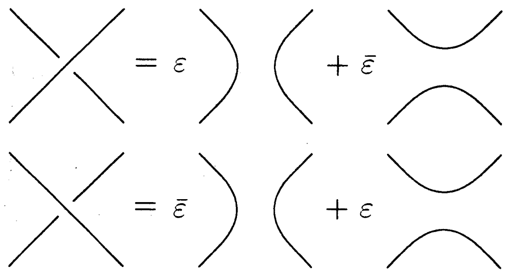


Figure 36:

to p_1^+ and p_1^- is shown by the relation in Temperley-Lieb recoupling theory as in Figure 36 which represents the difference of positive and negative crossing. Where ε is a complex number given by $ie^{i\pi/2h}$ with the Coxeter number h. In particular it shows that these two connections are mutually complex conjugate. $\square$

¿From Theorem 4.1 every minimal central projection p in the K-K double triangle algebra corresponds to an irreducible K-K connection W_p. By the definition of product of two connections and the above correspondence, the $\cdot$ product of two minimal central projections p and q corresponds to the product of two irreducible bi-unitary connections W_p and W_q. So by decomposing the product connection $W_p \cdot W_q$ into irreducible ones and using the correspondence between irreducible connections and minimal central projections, we get a linear combination of minimal central projections with positive integer coefficient. This means that the center of the K-K double triangle algebra $\mathcal{Z} = \mathcal{Z}(\mathcal{A}, *)$ is closed under the $\cdot$ product operation. And this shows the fact that the fusion rule of K-K bi-unitary connections is given by the $\cdot$ product of corresponding minimal central projections. So we get the following.

Corollary 4.5 *Let K be one of the Dynkin diagrams $A_n, D_n, E_{6,7,8}$. Then the fusion rule algebra of K-K bi-unitary connections is isomorphic to the center $\mathcal{Z}$ of the K-K double triangle algebra $(\mathcal{A}, *)$ with $\cdot$ product, i.e. $(\mathcal{Z}, \cdot)$.*

5 Classification of irreducible bi-unitary connections on the Dynkin diagrams

5.1 Classification of irreducible A-A bi-unitary connections.

Let A be one of the Dynkin diagram A_n. We first classify all (irreducible) A-A bi-unitary connections.

Proposition 5.1 *Let $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be the four graphs connected as in Figure 22. Suppose that both the upper graph $\mathcal{G}_0$ and the lower graph $\mathcal{G}_2$ are A and suppose there is a bi-unitary connection on the four graphs. Then the connecting vertical graphs $\mathcal{G}_1$ and $\mathcal{G}_3$ are* *uniquely determined by the initial condition, i.e., the condition of edges connected to the distinguished vertex $*$ of the upper graph A. Moreover such a connection is unique up to vertical gauge choice.*

Proof Because the string algebra on the graph A is generated only by Jones projections, the vertical graphs $\mathcal{G}_1$ and $\mathcal{G}_3$ are uniquely determined by looking at the dimension of essential paths with starting point corresponding to the initial condition, i.e., the vertices of the lower graph A connected to the distinguished vertex of the upper graph A. (See [13, Section 11.6].) The connection on the four graphs can be decomposed into irreducible ones. And an irreducible A-A connection have only one initial edge by the criterion in [1, Section 3, Claim 1] and the fact that it is automatically flat ([13, page 593]). The choice of the initial edge is one-to-one correspondent to the vertex of the (lower) graph A. Hence the uniqueness (up to vertical gauge choice) of the connections on the four graphs is proved by the uniqueness of irreducible connections corresponding to each vertex of A. We know that there is at least one bi-unitary connection on the four graphs with an initial edge corresponding to each vertex of the lower graph A. (Note that a corresponding commuting square is given in [25].) So the proof will end to show that these are the only irreducible bi-unitary connections up to vertical gauge choice. This is done by using the correspondence between $*$-representations of the A-A double triangle algebras and A-A bi-unitary connections. More precisely we have to show the following equality.

$$\sum_{a\in \mathrm{Vert}A}\left(\sum_{k=0}^{n-1}\dim \mathrm{EssPath}_a^{(k)}A\right)^2=\sum_{k=0}^{n-1}\left(\sum_{a\in\mathrm{Vert}A}\dim\mathrm{EssPath}_a^{(k)}A\right)^2$$

This equality can be easily obtained by a direct computation. □

Remark 5.2 *The above method also works for the classification of irreducible A-K bi-unitary connections for arbitrary Dynkin diagrams K with the same Coxeter number as A. The only different point is the last equality concerning the dimensions of the double triangle algebras. In the case of general Dynkin diagram K, we have to show the following equality instead of the above one.*

$$\sum_{x\in \mathrm{Vert}K}\left(\sum_{k=0}^{m}\dim \mathrm{EssPath}_x^{(k)}K\right)^2=\sum_{k=0}^{m}\left(\sum_{a\in\mathrm{Vert}A}\dim\mathrm{EssPath}_a^{(k)}A\right)\left(\sum_{x\in\mathrm{Vert}K}\dim\mathrm{EssPath}_x^{(k)}K\right)$$

Here m is the maximal length of essential paths on A and K which is the same as (the Coxeter number)-2. This also can be shown by a direct computation in each case. But here we give another proof based on estimates of the global index in the following.

Remark 5.3 *The uniqueness of irreducible A-K bi-unitary connection corresponding to each vertex of K does not seem to be obvious though we know the uniqueness of corresponding commuting square up to isomorphism. Here we remark that an isomorphic class of commuting square corresponds an isomorphic class of connections (See [13, Definition 10.11] for the definition of isomorphic connections) and it does not imply the uniqueness of equivalent connections up to (vertical) gauge choice.*

5.2 Classification of irreducible A-K bi-unitary connections.

Let A be one of the Dynkin diagrams A_n and K one of the Dynkin diagrams $A_n, D_n, E_{6,7,8}$ with the same Coxeter number as A. Before going into the details of the classification, we will show that a simple consideration on the fusion rule algebra leads an important consequence, that is, the system of bi-unitary connections which consist of all irreducible A-A, A-K, K-A and K-K connections are finite. It means that the numbers of all equivalence classes of irreducible A-K and K-K bi-unitary connections are finite. Moreover we can measure its size explicitly. To show this first we need the next lemma.

Lemma 5.4 *Let A and K be as above. Then all irreducible K-K (resp. A-A) bi-unitary connections are obtained from the product ${}_K\bar{\alpha}_A \cdot {}_A\beta_K$ (resp. ${}_A\alpha_K \cdot {}_K\bar{\beta}_A$) for some two irreducible A-K bi-unitary connections α and β.*

Proof We give a proof for the case of K-K bi-unitary connections because the same proof also works for the case of A-A bi-unitary connections. This is an easy consequence of Frobenius reciprocity. Let ${}_Kw_K$ be any irreducible K-K bi-unitary connection. Then take (any) irreducible A-K bi-unitary connection ${}_A\alpha_K$ and make a product of them. Take an irreducible A-K connection ${}_A\beta_K$ in the irreducible decomposition of the product ${}_A\alpha_K \cdot {}_Kw_K$, i.e., we have ${}_A\alpha w_K \succ {}_A\beta_K$. So we get ${}_K\bar{\alpha}\beta_K \succ {}_Kw_K$ by Frobenius reciprocity. □

Theorem 5.5 *Let K be one of the Dynkin diagrams $A_n, D_n, E_{6,7,8}$. Then the numbers of all equivalence classes of irreducible A-K and K-K bi-unitary connections are finite. Moreover, they have the same global index as that of the system of all irreducible A-A bi-unitary connections.*

Proof The case when the graph K is A_n is shown in the previous section. So we consider the other cases. The system of all A-A bi-unitary connections are obtained from a finite set of irreducible A-K connections by the previous lemma. We choose and fix such a finite set and consider the system generated by one A-K bi-unitary connection ${}_Aw_K$ which is the (finite) direct sum of all the A-K connections we have chosen. It was shown that the set of all irreducible A-A connections are finite. So this system contains only finitely many different irreducible A-K connections because of the local finiteness of the principal fusion graph of the generator ${}_Aw_K$.

We claim that all irreducible A-K connections appear in this system. Otherwise we have an irreducible A-K connection ${}_Az_K$ which do not appear in this system. If we take an A-K connection $w \oplus z$ as an generator, we get a different system having the same set of irreducible A-A connections and strictly larger set of irreducible A-K connections than before. Thus we get two systems of four kinds of connections consisting of irreducible A-A, A-K, K-A and K-K connections which are both generated by one A-K connection. Now it is easy to see that proof of the equality $\sum_{{}_NX_N}[{}_NX_N] = \sum_{{}_NY_M}[{}_NY_M] = \sum_{{}_MZ_M}[{}_MZ_M]$ for the estimates of global index for subfactor $N \subset M$ still works in the case of singly generated connection system. Here the summations run over all irreducible bimodule appear in the system generated by ${}_NM_M$. (See [13, Proposition 12.25] for the proof of subfactor case.) So we get a contradiction from the estimates of the global index of the two systems. Hence the system must contain all the irreducible A-K connections.

Now the same argument shows that we have finitely many irreducible K-K connections and these are the all irreducible K-K connections by the previous lemma. So the system of four kinds of connections consisting of *all* irreducible A-A, A-K, K-A and K-K connections is generated by one A-K connection w. Applying the estimates of the global indices of this system we get the result. □

By this estimates of global index we can easily classify all irreducible A-K bi-unitary connections as in the following proposition.

Proposition 5.6 *Let $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be the four graphs connected as in Figure 22. Suppose that the upper graph $\mathcal{G}_0$ is A and the lower graph $\mathcal{G}_2$ is K and suppose there is a bi-unitary connection on the four graphs. Then the connecting vertical graphs $\mathcal{G}_1$ and $\mathcal{G}_3$ are uniquely determined by the initial condition, i.e., the condition of edges connected to the distinguished vertex of the graph A. Moreover such a connection is unique up to vertical gauge choice.*

Proof The proof of the first assertion is exactly the same as Proposition 5.1. So we have only to show the uniqueness of irreducible connections corresponding to each vertex of K. Again we know that there is at least one bi-unitary connection on the four graphs with an initial edge corresponding to each vertex of K ([25]). So we show that these are the only irreducible bi-unitary connections up to vertical gauge choice.

In the case of $K = D_n$, we know that there is an irreducible A-K connections with index $\sqrt{2}$ which correspond to the vertex $*$ of D_n. We denote it by ${}_A\alpha_K$. Take one of the two non-equivalent fundamental connections on D_n and denote it by ${}_K w_K$. Take a finite product $\alpha w\bar{w}\cdots w$ (or $\bar{w}$) and decompose them. In this way we get some irreducible A-K connections. We remark that the fusion graph with initial vertex α and generator w has the same Perron-Frobenius eigenvalue as the index of the connection w. So it must be one of A-D-E Dynkin diagrams. It is easy to see that irreducible connections corresponding to any choice of the initial vertex of K appears in this procedure. So the graph has vertices at least as many as that of D_n. ¿From the estimates of global index, these are all the irreducible A-K connections because of the equality $|A_{2n+1}| = 2\cdot|D_n|$. Here $|K|$ represents the global index corresponding to all the vertices of the graph K.

In the case of $K = E_{6,7,8}$, the same proof as in the case of D_n works. So we have only to show the following estimates of global indices, i.e. $|E_6| = |\alpha_{E_6}|^2\cdot|A_{11}|$, $|E_7| = |\alpha_{E_7}|^2\cdot|A_{17}|$, and $|E_8| = |\alpha_{E_8}|^2\cdot|A_{29}|$. Here α_K represents a connection corresponding to an initial edge connected to the distinguished vertex $*$ of the graph K and $|\alpha_K|$ denotes its index. These are shown by Wenzl's index formula [14, Theorem 4.3.3]. Actually the square of the indices of connections α_K as above are the same as the indices of the corresponding GHJ subfactors, which are exactly the quotient of two global indices of K and A, i.e. $|\alpha_K|^2 = |K|/|A|$ for $K = E_{6,7,8}$ from Wenzl's index formula. So the above equalities hold. □

¿From Proposition 5.1 and Proposition 5.6 we get the following theorem.

Theorem 5.7 *Let K be one of the Dynkin diagrams $A_n, D_n, E_{6,7,8}$. There is a one-to-one correspondence between vertices of the graph K and equivalence classes of irreducible A-K bi-unitary connections.*

5.3 Classification of irreducible K-K bi-unitary connections.

¿From the one-to-one correspondence between minimal central projections of the K-K double triangle algebra and K-K bi-unitary connections, for every minimal central projection p of the K-K double triangle algebra we can associate an index of the subfactor generated by the corresponding connections. We call the square root of the index of the subfactor corresponding to a minimal central projection p an *index* of the projection p and we denote it by $\mathrm{d}(p)$. Because two equivalent bi-unitary connections give rise to an isomorphic subfactor, this definition is well-defined. Moreover if two minimal central projections p and q coincide, they must have the same index, i.e. $d(p) = d(q)$.

For the case of the Dynkin diagrams K we have special central projections Ψ_+ and Ψ_- which is called *chiral projectors* (see Section 2.3). We call the subset of $(\mathcal{Z}, \cdot)$ which consists of minimal central projections contained in Ψ_+ (resp. Ψ_-) *chiral left part* (resp. *chiral right part*). Because the chiral left part (resp. chiral right part) coincide with the set of minimal central projections appears in the system generated by p_1^+ (resp. p_1^-) they form two fusion rule subalgebras of $(\mathcal{Z}, \cdot)$. The intersection of chiral left part and chiral right part is called *ambichiral part* of the fusion rule algebra $(\mathcal{Z}, \cdot)$ and it corresponds to the ambichiral projector $\Psi_\pm$. In the following we use the notations $\mathcal{Z}_l$, $\mathcal{Z}_r$ and $\mathcal{Z}_a$ to represent the fusion rule subalgebras of chiral left part, chiral right part and ambichiral part respectively.

Proposition 5.8 *Let K be one of Dynkin diagrams $A_n, D_n, E_{6,7,8}$. Suppose we have a fusion rule subalgebra $\mathcal{B}$ of the fusion rule algebras of all irreducible K-K connections $\mathcal{Z}$. Then $\mathcal{Z}$ decomposes into left cosets of $\mathcal{B}$ and right cosets of $\mathcal{B}$, i.e. we have subsets $X, Y \subset \mathcal{Z}$ of irreducible connections (representatives of left and right cosets) such that $\mathcal{Z} = \bigcup_{x\in X} x \cdot \mathcal{B} = \bigcup_{y \in Y} \mathcal{B} \cdot y$, $x \cdot \mathcal{B} \cap x' \cdot \mathcal{B} = \emptyset$ if $x \neq x' \in X$ and $\mathcal{B} \cdot y \cap \mathcal{B} \cdot y' = \emptyset$ if $y \neq y' \in Y$.*

Proof We will give a proof for the left cosets. We have only to show the following; $x \cdot \mathcal{B} \cap x' \cdot \mathcal{B} = \emptyset$ for irreducible $x, x' \in \mathcal{Z}$, then $x \cdot \mathcal{B} = x' \cdot \mathcal{B}$. Suppose we have $x \cdot \mathcal{B} \cap x' \cdot \mathcal{B} = \emptyset$ for $x, x' \in \mathcal{Z}$. Then there are irreducible K-K connections $b, b' \in \mathcal{B}$ and $z \in \mathcal{Z}$, such that $x \cdot b \succ z$ and $x' \cdot b' \succ z$. Hence $x \cdot \mathcal{B} \supset x \cdot b \cdot \mathcal{B} \supset z \cdot \mathcal{B}$ and $x' \cdot \mathcal{B} \supset x' \cdot b' \cdot \mathcal{B} \supset z \cdot \mathcal{B}$ holds. From the Frobenius reciprocity, we have $z \cdot \bar{b} \succ x$ and $z \cdot \bar{b}' \succ x'$. So the converse inclusions $z \cdot \mathcal{B} \supset z \cdot \bar{b} \cdot \mathcal{B} \supset x \cdot \mathcal{B}$ and $z \cdot \mathcal{B} \supset z \cdot \bar{b}' \cdot \mathcal{B} \supset x' \cdot \mathcal{B}$ holds. Thus we have $x \cdot \mathcal{B} = x' \cdot \mathcal{B} = z \cdot \mathcal{B}$. □

Remark 5.9 *It is easy to see that this proposition holds true for more general fusion rule algebras such as those treated in Hiai-Izumi [15]. We only need the property of Frobenius reciprocity. For example any fusion rule algebras of bimodule (or sectors) arising from subfactors have this coset decomposition property.*

The following can be easily shown from Proposition 5.8 and the estimates of global indices (Theorem 5.5).

Corollary 5.10 *If the chiral left part $\mathcal{Z}_l$ does not coincide with the chiral right part $\mathcal{Z}_r$, then the principal fusion graphs of minimal central projections p_1^+ and p_1^- cannot be the Dynkin diagram of type A. Conversely if one of the principal fusion graphs of p_1^+ and p_1^- is the Dynkin diagram of type A, then we have $\mathcal{Z}_l = \mathcal{Z}_r = \mathcal{Z}_a = \mathcal{Z}$.*

5.3.1 The case of A_n

This is done in the previous subsection 5.1. There is one-to-one correspondence between vertices of the Dynkin diagram A_n and irreducible A_n-A_n bi-unitary connections. In this case the two minimal central projections p_1^+ and p_1^- coincide. The fusion rule graph for the generator $[1] = p_1^+ = p_1^-$ is given in Figure 40.

5.3.2 The case of D_{2n+1}

In this case we have $\mathrm{gap}(D_{2n+1}) = \infty$ and corresponding recoupling system is A_{4n-1}. So there are two series of mutually orthogonal minimal central projections $\{p_k^+\}_{k=0,1,2,\dots,4n-2}$ and $\{p_k^-\}_{k=0,1,2,\dots,4n-2}$. Here p_0^+ and p_0^- coincide and it corresponds to the identity connection. Because all p_k^+ arise from p_1^+ by taking $\cdot$ product, this means that the subfactor arising from the connection corresponding to p_1^+ (which is one of the two non-equivalent bi-unitary connections as in Corollary 4.4) have at least $4n-1$ vertices in its principal graph. From the index value $\mathrm{d}(p_1^+)$ this is possible only when the principal graph is the Dynkin diagram A_{4n-1}. So we have $\mathcal{Z}_l = \mathcal{Z}_r = \mathcal{Z}_a = \mathcal{Z}$ by Corollary 5.10. Hence these are all the minimal central projections. ¿From the facts that $p_1^- \neq p_1^+$, $p_1^- \neq p_{4n-3}^-$ and $\mathrm{d}(p_1^+) = \mathrm{d}(p_1^-) = \mathrm{d}(p_{4n-3}^+) = \mathrm{d}(p_{4n-3}^-)$, the minimal central projection p_1^- must coincide with p_{4n-3}^+. Hence we get $p_k^- = p_{4n-2-k}^+$ from the fusion rule. The fusion graph of the two generator $[1] = p_1^+$ and $[4n-3] = p_1^-$ are given as in Figure 41.

Note that we did not use the fact of non-existence of D_{2n+1} subfactors in the above argument. So it gives another proof of the non-existence of D_{2n+1} subfactors. It also shows that the flat part of D_{2n+1} commuting squares are A_{4n-1}.

5.3.3 The case of D_{2n}

In this case there are two non-equivalent connections with index 1. One is trivial connection and the other comes from the flip of the two tails of the graph D_{2n} (we denote it by ε). We denote the minimal central projections corresponding to the connection ε by p_ε.

Because $p_1^+ \neq p_1^-$, the fusion graph of chiral left part as well as chiral right part can not be A_{4n-3} from Corollary 5.10. So they must be D_{2n} except the case $n = 5, 8$. But in the case of D_{10} and D_{16} we have $\mathrm{gap}(D_{10}) = 16$ and $\mathrm{gap}(D_{16}) = 28$. Hence there is a series of mutually orthogonal minimal central projections $p_0^+, p_1^+, \dots, p_7^+ \in \mathcal{Z}_l$ in the case of D_{10} and $p_0^+, p_1^+, \dots, p_{13}^+ \in \mathcal{Z}_l$ in the case of D_{16}. These shows that the fusion graph of chiral left part (hence chiral right part as well) can not be E_6 or E_8 and they must be D_{10} and D_{16} themselves.

It is easy to see that p_ε does not appear in either $\mathcal{Z}_l$ nor $\mathcal{Z}_r$ by comparing the indices except the case D_4. In the case of D_4, the fusion rule algebra of even vertices of D_4 is the cyclic group $\mathbf{Z}_3$. Hence $p_\varepsilon \notin \mathcal{Z}_l \cup \mathcal{Z}_r$ in this case, either. So we get the coset decomposition $\mathcal{Z} \supset \mathcal{Z}_l \cup \mathcal{Z}_l \cdot p_\varepsilon$. But the estimates of the global indices of the both sets $\mathcal{Z}$ and $\mathcal{Z}_l \cup \mathcal{Z}_l \cdot p_\varepsilon$ shows that these are all the irreducible K-K connections.

¿From the equalities $p_1^- = p_\varepsilon \cdot p_1^+ = p_1^+ \cdot p_\varepsilon$ and $p_\varepsilon^2 = p_\varepsilon \cdot p_\varepsilon = id$ we have $p_2^+ = p_2^-$, which shows that the even vertices of chiral left and right part coincide from fusion rule of D_{2n}. It is easy to see that the odd vertices of $\mathcal{Z}_l$ and $\mathcal{Z}_r$ does not coincide again from the fusion rule. So we obtain the fusion rule graph for two generators $[1] = p_1^+$ and $[1^-] = p_1^-$ as in Figure 42.

5.3.4 The case of E_6

In the case of E_6 we know that $p_1^+ \neq p_1^-$. By looking at the vertical edges of the composite connection corresponding to $p_1^+ \cdot p_1^-$, we can see that this connection is irreducible from the criterion [1, Corollary 2, Section 3] and the Frobenius reciprocity. This means that p_1^- is not in the chiral left part, i.e. $p_1^- \in \mathcal{Z}_l$. So we have coset decomposition $\mathcal{Z} \supset \mathcal{Z}_l \cup \mathcal{Z}_l \cdot p_1^-$ (see Proposition 5.8). The (coset) principal fusion graphs for $\mathcal{Z}_l$ and $\mathcal{Z}_l \cdot p_1^-$ are one of the Dynkin diagrams D_7 or E_6. We cannot have A_{11} from Corollary 5.10. The estimates of global indices show that $|A_{11}| = (3+\sqrt{3})|E_6| = (1 + |p_1^-|^2)|E_6|$. ¿From this together with the following inequality $|E_6| < |D_7| < |A_{11}|$ the both (coset) principal fusion graphs for $\mathcal{Z}_l$ and $\mathcal{Z}_l \cdot p_1^-$ must be E_6. This also shows the flatness of

$\times$	id_N	ρ_2	ρ_4	ρ_1	ρ_3	ρ_5
id_N	id_N	ρ_2	ρ_4	ρ_1	ρ_3	ρ_5
ρ_2	ρ_2	$id_N + 2\rho_2 + \rho_4$	ρ_2	$\rho_1 + \rho_3 + \rho_5$	$\rho_1 + \rho_5$	$\rho_1 + \rho_3 + \rho_5$
ρ_4	ρ_4	ρ_2	id_N	ρ_5	ρ_3	ρ_1
$\bar\rho_1$	$\bar\rho_1$	$\bar\rho_1 + \bar\rho_3 + \bar\rho_5$	$\bar\rho_5$	$id_M + \rho_2'$	ρ_2'	$\rho_2' + \rho_4'$
$\bar\rho_3$	$\bar\rho_3$	$\bar\rho_1 + \bar\rho_5$	$\bar\rho_3$	ρ_2'	$id_M + \rho_4'$	ρ_2'
$\bar\rho_5$	$\bar\rho_5$	$\bar\rho_1 + \bar\rho_3 + \bar\rho_5$	$\bar\rho_1$	$\rho_2' + \rho_4'$	ρ_2'	$id_M + \rho_2'$

Table 1: Multiplication table for $\cdot \otimes_N \cdot$ of the fusion rule algebra of E_6

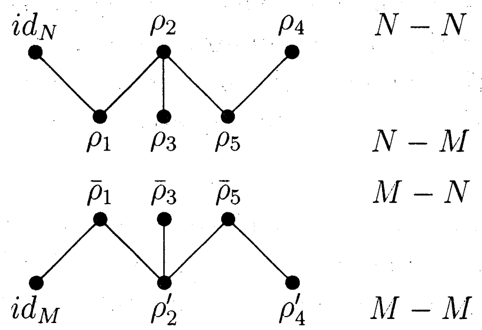


Figure 37: The (dual) principal graph E_6

E_6 connections (see Proposition 3.21) and that the whole system $\mathcal{Z}$ is generated by the two irreducible fundamental connections p_1^+ and p_1^-. Now it is easy to see the fusion rule graph for the two generators $[1] = p_1^+$ and $[1^-] = p_1^-$ is given in Figure 43 from the index values. Here we also give the multiplication table of the E_6 fusion rule algebra in Table 1, where ρ_i's correspond to the vertices of E_6 shown in Figure 37.

5.3.5 The case of E_7

We have $p_1^+ \neq p_1^-$ and the irreducibility of the composite connection $p_1^+ \cdot p_1^-$ can be shown in the same way as E_6 case. So we have coset decomposition $\mathcal{Z} \supset \mathcal{Z}_l \cup \mathcal{Z}_l \cdot p_1^-$ and the (coset) principal fusion graphs for $\mathcal{Z}_l$ and $\mathcal{Z}_l \cdot p_1^-$ are one of the Dynkin diagrams D_{10} or E_7. (Again we cannot have A_{18} from Corollary 5.10.)

In this case the global indices satisfies $|E_7| = 2\beta(3\beta^2 - 15\beta + 18) < |D_{10}| = \beta|E_7| < |A_{17}| = 2\beta|E_7|$, where $\beta = 4\cos^2(\pi/18)$. So we have $|A_{17}| = 2\beta|E_7| > |G_1| + \beta|G_2|$, where G_1 and G_2 denote the (coset) principal fusion graph for $\mathcal{Z}_l$ and $\mathcal{Z}_l \cdot p_1^-$ respectively. Hence the equality only happens when $G_1 = D_{10}$ and $G_2 = E_7$. This shows that the flat part of the E_7 connections is D_{10} and that the whole system $\mathcal{Z}$ is generated by the two fundamental connections p_1^+ and p_1^-. Here we claim $p_2^+ \neq p_2^-$. If $p_2^+ = p_2^-$, from the fusion rule we have $p_1^+ \cdot p_1^+ \cdot p_1^- = 2p_1^- + p_3^-$. But this is impossible because $\dim \mathrm{End}(p_1^+ \cdot p_1^-) = \dim \mathrm{Hom}(p_1^+ \cdot p_1^+ \cdot p_1^-, p_1^-) = 2$ by Frobenius reciprocity, which contradicts the irreducibility of $p_1^+ \cdot p_1^-$. Hence we must have $p_2^+ \neq p_2^-$. Then by looking at the indices and the fusion rules, we get the fusion graph for the two generators $[1] = p_1^+$ and $(0) = p_1^-$ as in Figure 43.

5.3.6 The case of E_8

In this case we also have the coset decomposition $\mathcal{Z} \supset \mathcal{Z}_l \cup \mathcal{Z}_l \cdot p_1^-$ which is shown in the same way as $E_{6,7}$ cases. the (coset) principal fusion graphs for $\mathcal{Z}_l$ and $\mathcal{Z}_l \cdot p_1^-$ are one of the Dynkin diagrams $\breve{D}_{16}$ or E_8. The global indices satisfies $|E_8| < |D_{16}| = (\beta^2 - 2\beta + 2)|E_8| < |A_{17}| = (2\beta^2 - 4\beta + 4)|E_8|$, where $\beta = 4\cos^2(\pi/30)$.

First we show the principal fusion graph of p_1^+ is E_8. ¿From the above coset decomposition, we have the following inequality, $|A_{17}| = (2\beta^2 - 4\beta + 4)|E_8| > |G_1| + \beta|G_2|$. Here G_1 and G_2 denote the (coset) principal fusion graph for $\mathcal{Z}_l$ and $\mathcal{Z}_l \cdot p_1^-$ respectively. If $G_1 = D_{16}$, there are two possibilities, i.e. $p_2^+ = p_2^-$ or $p_2^+ \neq p_2^-$. If $p_2^+ = p_2^-$ holds, the even vertices of the chiral left part $\mathcal{Z}_l$ and those of the chiral right part $\mathcal{Z}_r$ coincide from the fusion rule. And we have the following coset decomposition, $\mathcal{Z} \supset \mathcal{Z}_l \cup \mathcal{Z}_l \cdot p_1^- \cup \mathcal{Z}_l \cdot p_3^-$ because $p_3^- \notin \mathcal{Z}_l \cup \mathcal{Z}_l \cdot p_1^-$. But the smallest possible value of the global indices is $|D_{16}| + \beta|E_8| + \beta(\beta-2)^2|E_8| = (\beta^3 - 3\beta^2 + 3\beta + 2)|E_8| > |A_{29}|$ and this is impossible. If $p_2^+ \neq p_2^-$ holds, then $p_2^- \notin \mathcal{Z}_l \cup \mathcal{Z}_l \cdot p_1^-$ by comparing the indices. So we have the coset decomposition $\mathcal{Z} \supset \mathcal{Z}_l \cup \mathcal{Z}_l \cdot p_1^- \cup \mathcal{Z}_l \cdot p_2^-$. The smallest possible value of the global indices is $|D_{16}| + \beta|E_8| + (\beta-1)^2|E_8| = (2\beta^2 - 3\beta + 3)|E_8| > |A_{29}|$. So again this is impossible. Thus the principal fusion graph of p_1^+ must be E_8. And this shows the flatness of E_8 connections. We label the vertices of the chiral left and right part as in Figure 38.

Next we show that the ambichiral part $\mathcal{Z}_a = \mathcal{Z}_l \cap \mathcal{Z}_r$ is $\{\rho_0, \rho_6 = \rho_6^{\sim}\}$ ¿From gap(E_8) =0-gap$(E_8) = 10$, we have two series of mutually orthogonal minimal central projections $\{p_k^+\}_{k=0,1,2,3,4}$ and $\{p_k^-\}_{k=0,1,2,3,4}$. These are also labelled by $p_k^+ = \rho_k$ and $p_k^+ = \rho_k^{\sim}$. There are two possibilities, i.e. $\rho_2 = \rho_2^{\sim}$ or $\rho_2 \neq \rho_2^{\sim}$ from the index values. If $\rho_2 = \rho_2^{\sim}$, then from the fusion rule (Table 3) we have $\rho_1 \cdot \rho_1 \cdot \rho_1^{\sim} = (\rho_0 + \rho_2) \cdot \rho_1^{\sim} = 2\rho_1^{\sim} + \rho_3^{\sim}$. But this is impossible because $\dim \mathrm{End}(\rho_1 \cdot \rho_1^{\sim}) = \dim \mathrm{Hom}(\rho_1 \cdot (\rho_1 \cdot \rho_1^{\sim}), \rho_1)$ by Frobenius reciprocity and it contradicts the irreducibility of $\rho_1 \cdot \rho_1^{\sim}$. So ρ_2 and $\rho_2^{\sim}$ do not coincide. Then again from the fusion rule we cannot have $\rho_4 = \rho_4^{\sim}$ which contradicts $\rho_2 \neq \rho_2^{\sim}$. Now if $\rho_6 \neq \rho_6^{\sim}$ then we can check the following coset decomposition, $\mathcal{Z} \supset \mathcal{Z}_l \cup \mathcal{Z}_l \cdot \rho_1^{\sim} \cup \mathcal{Z}_l \cdot \rho_2^{\sim} \cup \mathcal{Z}_l \cdot \rho_5^{\sim}$ $\cup \mathcal{Z}_l \cdot \rho_6^{\sim}$ by comparing indices. The smallest possible value of the global index of these cosets is $(2\beta^2 - 3\beta + 3)|E_8| > |A_{29}|$, which contradicts the estimates of global indices (Theorem 5.5). Hence we must have $\rho_6 = \rho_6^{\sim}$. It is easy to see that the odd vertices of $\mathcal{Z}_l$ and $\mathcal{Z}_r$ cannot coincide because of the fusion rule (Table 3) and the fact $\rho_1 = \rho_1^{\sim}$.

Remark 5.11 *Here we remark that the multiplication tables 1, 2 and 3 are for the system of bimodules or sectors. But in the case of K-K connection systems we have different gradings. Actually we have equality $w = \bar{w}$ for the two fundamental connections w of A-D-E Dynkin diagrams. This is impossible in the case of bimodules because w corresponds to an $N-M$ bimodule. In the case of sectors self-conjugate sector $\rho_1 = \bar{\rho}_1$ makes sense, but these have different meaning. ¿From the fact that the fundamental connection w, which corresponds to ρ_1 in the multiplication tables, is self-conjugate in our sense as a pair of connections, we can easily get $\rho_k = \bar{\rho}_k$ for odd k and $\rho_k' = \rho_k$ for even k in the case of E_6 and E_8. Again one will notice that $\rho_k' = \rho_k$ for even k does not make sense even in the case of sectors because the left hand side is in* Sect(M) *while the right hand side is in* Sect(N). *But in our setting of K-K connection systems such things can happen. Hence we can read the multiplication $\bar{\rho}_1 \cdot \rho_2 = \bar{\rho}_1 + \bar{\rho}_3$ and $\bar{\rho}_1 \cdot \rho_3 = \rho_2' + \rho_4'$ as $\rho_1 \cdot \rho_2 = \rho_1 + \rho_3$ and $\rho_1 \cdot \rho_3 = \rho_2 + \rho_4$ respectively for examples in Table 2 and 3.*

Now from the fusion rule we know $\rho_7^{\sim} = \rho_6 \cdot \rho_1^{\sim}$, $\rho_4^{\sim} = \rho_6 \cdot \rho_2^{\sim}$ and $\rho_3^{\sim} = \rho_6 \cdot \rho_5^{\sim}$. So we have the coset decomposition $\mathcal{Z} \supset \mathcal{Z}_l \cup \mathcal{Z}_l \cdot \rho_1^{\sim} \cup \mathcal{Z}_l \cdot \rho_2^{\sim} \cup \mathcal{Z}_l \cdot \rho_5^{\sim}$. And the estimates of the global indices of both hand side is $|A_{29}| = (2\beta^2 - 4\beta + 4)|E_8|$ and $|E_8| + \beta|E_8|$ $+(\beta-1)^2|E_8|$ $+\beta(-\beta^3 + 7\beta^2 - 13\beta + 5)^2|E_8| = (2\beta^2 - 4\beta + 4)|E_8|$, where we used the equality $\beta^4 - 7\beta^3 + 14\beta^2 - 8\beta + 1 = 0$ to compute the right hand side. So we get the equality

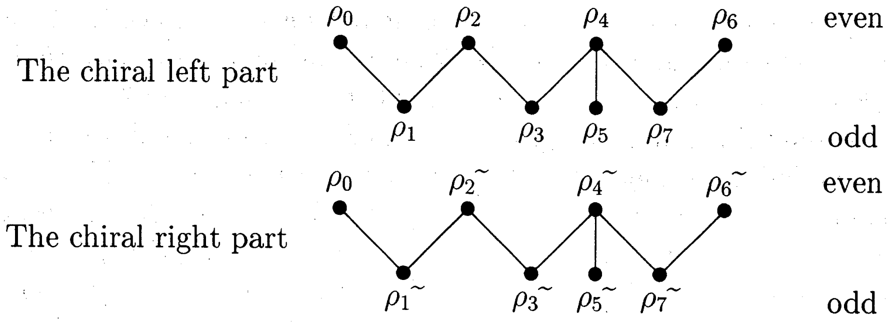


Figure 38: The chiral left and right part of the fusion rule algebra $\mathcal{Z}$ of E_8.

$\times$	id_N	ρ_2	ρ_4	ρ_6
id_N	id_N	ρ_2	ρ_4	ρ_6
ρ_2	ρ_2	$id_N + \rho_2 + \rho_4$	$\rho_2 + 2\rho_4 + \rho_6$	ρ_4
ρ_4	ρ_4	$\rho_2 + 2\rho_4 + \rho_6$	$id_N + 2\rho_2 + \rho_4 + \rho_6$	$\rho_2 + \rho_4$
ρ_6	ρ_6	ρ_4	$\rho_2 + \rho_4$	$id_N + \rho_6$
$\bar\rho_1$	$\bar\rho_1$	$\bar\rho_1 + \bar\rho_3$	$\bar\rho_3 + \bar\rho_5 + \bar\rho_7$	$\bar\rho_7$
$\bar\rho_3$	$\bar\rho_3$	$\bar\rho_1 + \bar\rho_3 + \bar\rho_5 + \bar\rho_7$	$\bar\rho_1 + 2\bar\rho_3 + \bar\rho_5 + 2\bar\rho_7$	$\bar\rho_3 + \bar\rho_5$
$\bar\rho_5$	$\bar\rho_5$	$\bar\rho_3 + \bar\rho_7$	$\bar\rho_1 + \bar\rho_3 + \bar\rho_5 + \bar\rho_7$	$\bar\rho_3$
$\bar\rho_7$	$\bar\rho_7$	$\bar\rho_3 + \bar\rho_5 + \bar\rho_7$	$\bar\rho_1 + 2\bar\rho_3 + \bar\rho_5 + \bar\rho_7$	$\bar\rho_1 + \bar\rho_7$

Table 2: Multiplication table for $\cdot \otimes_N \cdot$ of the fusion rule algebra of E_8 (1)

in the above coset decomposition. This shows that the whole system $\mathcal{Z}$ is generated by the two elements p_1^+ and p_1^-. Finally by looking at the index values and the fusion rule, we obtain the fusion rule graph for the two generators $[1] = p_1^+$ and $[1^\sim] = p_1^-$ as in in Figure 45.

Remark 5.12 *We will explain the meaning of Figures 40 to 45. The white vertices and black vertices represents even and odd vertices respectively. The large double circled vertices denote the ambichiral part. The thick edges and thin edges represent the chiral left graphs and the left coset graphs, which are obtained as Cayley graphs for multiplication by the generator p_1^+ from the left. The thick dotted edges and thin dotted edges represent the chiral right graphs and the right coset graphs, which are obtained as Cayley graphs for*

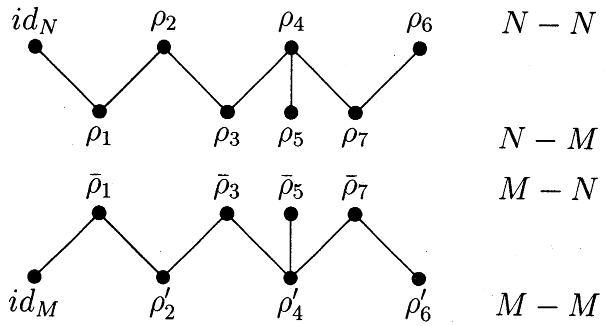


Figure 39: The (dual) principal graph E_8

$\times$	ρ_1	ρ_3	ρ_5	ρ_7
id_N	ρ_1	ρ_3	ρ_5	ρ_7
ρ_2	$\rho_1+\rho_3$	$\rho_1+\rho_3+\rho_5+\rho_7$	$\rho_3+\rho_7$	$\rho_3+\rho_5+\rho_7$
ρ_4	$\rho_3+\rho_5+\rho_7$	$\rho_1+2\rho_3+\rho_5+2\rho_7$	$\rho_1+\rho_3+\rho_5+\rho_7$	$\rho_1+2\rho_3+\rho_5+\rho_7$
ρ_6	ρ_7	$\rho_3+\rho_5$	ρ_3	$\rho_1+\rho_7$
$\bar\rho_1$	$id_M+\rho'_2$	$\rho'_2+\rho'_4$	ρ'_4	$\rho'_4+\rho'_6$
$\bar\rho_3$	$\rho'_2+\rho'_4$	$id_M+\rho'_2+2\rho'_4+\rho'_6$	$\rho'_2+\rho'_4+\rho'_6$	$\rho'_2+2\rho'_4$
$\bar\rho_5$	ρ'_4	$\rho'_2+\rho'_4+\rho'_6$	$id_M+\rho'_4$	$\rho'_2+\rho'_4$
$\bar\rho_7$	$\rho'_4+\rho'_6$	$\rho'_2+2\rho'_4$	$\rho'_2+\rho'_4$	$id_M+\rho'_2+\rho'_4+\rho'_6$

Table 3: Multiplication table for $\cdot \otimes_N \cdot$ of the fusion rule algebra of E_8 (2)

multiplication by the generator p_1^- from the right.

In the procedure to get the complete classification of irreducible K-K bi-unitary connections. We also obtained the complete classification of flat connections and flat part of non-flat connections on the Dynkin diagrams. We state this as the following corollary.

Corollary 5.13 *The (fundamental) bi-unitary connections on the four graphs as in Figure 35 are flat in the case of A_n, D_{2n}, E_6 and E_8. They are not flat in the case of D_{2n+1} and E_7. The flat part of D_{2n+1} and E_7 connections are A_{4n-1} and D_{10} respectively.*

Hence it provides another proof of the complete classification of subfactors of the hyperfinite II_1 factor with index less than 4.

Corollary 5.14 *There is only one subfactor with principal graph A_n for each $n \geq 2$. There is only one subfactor with principal graph D_{2n} for each $n \geq 2$. There are two and only two non-isomorphic subfactors with principal graph E_6 and E_8 respectively. These are all the subfactors of the hyperfinite II_1 factor with index less than 4.*

By examining each case we obtain the following structural result on the fusion rule algebra of all K-K bi-unitary connections.

Theorem 5.15 *Let K be one of the A-D-E Dynkin diagrams. The fusion rule algebras of all K-K bi-unitary connections are generated by the two minimal central projections $p_1^\pm$. Moreover the chiral left part and the chiral right part commutes.*

Remark 5.16 *The first assertion of the above theorem can be shown directly by taking the $\cdot$ product of two chiral projectors $\Psi_+ \cdot \Psi_-$. Minimal central projections appear in this product is contained in the fusion rule subalgebra generated by p_1^+ and p_1^-. And it is not difficult to show that the product contains the identity element of the K-K double triangle algebra by using the non-degenerate braiding on the recoupling system A. This result can be generalized in more abstract setting of double triangle algebra as in [5].*

Remark 5.17 *Though it is not written in detail in [24], Ocneanu showed stronger result than the commutativity of the chiral left and right part. He showed that the chiral left part and chiral right part of the fusion rule algebra of K-K bi-unitary connections has*

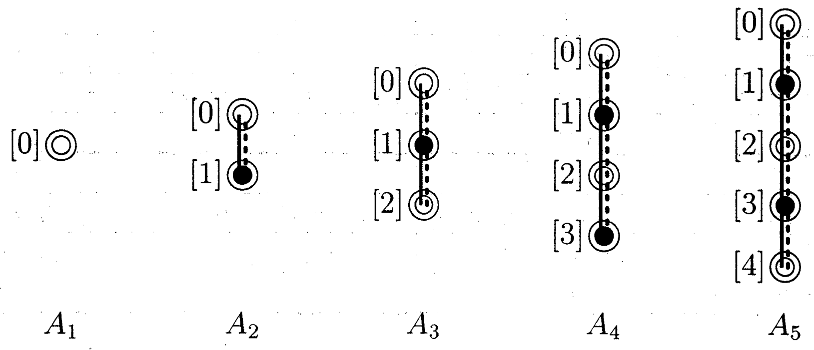


Figure 40: Chiral symmetry for the Coxeter graph A_n

non-degenerated braiding. (See the explanation of the picture "Quantum Symmetry for Coxeter graphs" in [24].) He defined the choice of intertwiner in $\mathrm{Hom}(p_i \cdot p_j, p_j \cdot p_1)$ *graphically and showed the existence of the non-degenerate braiding. This shows that the fusion rule algebra of the ambichiral part has non-degenerate braiding. And it implies the existence of non-degenerate braiding on the system of bimodule corresponding to even vertices of the Dynkin diagram D_{2n} which was shown by D. E. Evans-Y. Kawahigashi in [12].*

References

[1] M. Asaeda & U. Haagerup, *Exotic subfactors of finite depth with Jones indices* $(5+\sqrt{13})/2$ *and* $(5+\sqrt{17})/2$, to appear in Comm. Math. Phys.

[2] J. Böckenhauer & D. E. Evans, *Modular Invariants, Graphs and α-Induction for Nets of Subfactors I*, Comm. Math. Phys **197** (1998), 361–386.

[3] J. Böckenhauer & D. E. Evans, *Modular Invariants, Graphs and α-Induction for Nets of Subfactors II*, to appear in Comm. Math. Phys.

[4] J. Böckenhauer & D. E. Evans, *Modular Invariants, Graphs and α-Induction for Nets of Subfactors III*, in preparation.

[5] J. Böckenhauer, D. E. Evans & Y. Kawahigashi, in preparation.

[6] J. de Boer & J. Goeree, *Markov traces and II_1 factors in conformal field theory*, Comm. Math. Phys. **139** (1991), 267–304.

[7] A. Capelli, C. Itzykson, and J.-B. Zuber, *The A-D-E classification of minimal and $A_1^{(1)}$ conformal invariant theories*, Comm. Math. Phys. **113** (1987), 1–26.

[8] D. E. Evans & Y. Kawahigashi, *The E_7 commuting squares produce D_{10} as principal graph*, Publ. RIMS. Kyoto Univ. **30** (1994), 151–166.

[9] D. E. Evans & Y. Kawahigashi, *Subfactors and conformal field theory*, in "Quantum and non-commutative analysis", 341–369, Kluwer Academic (1993).

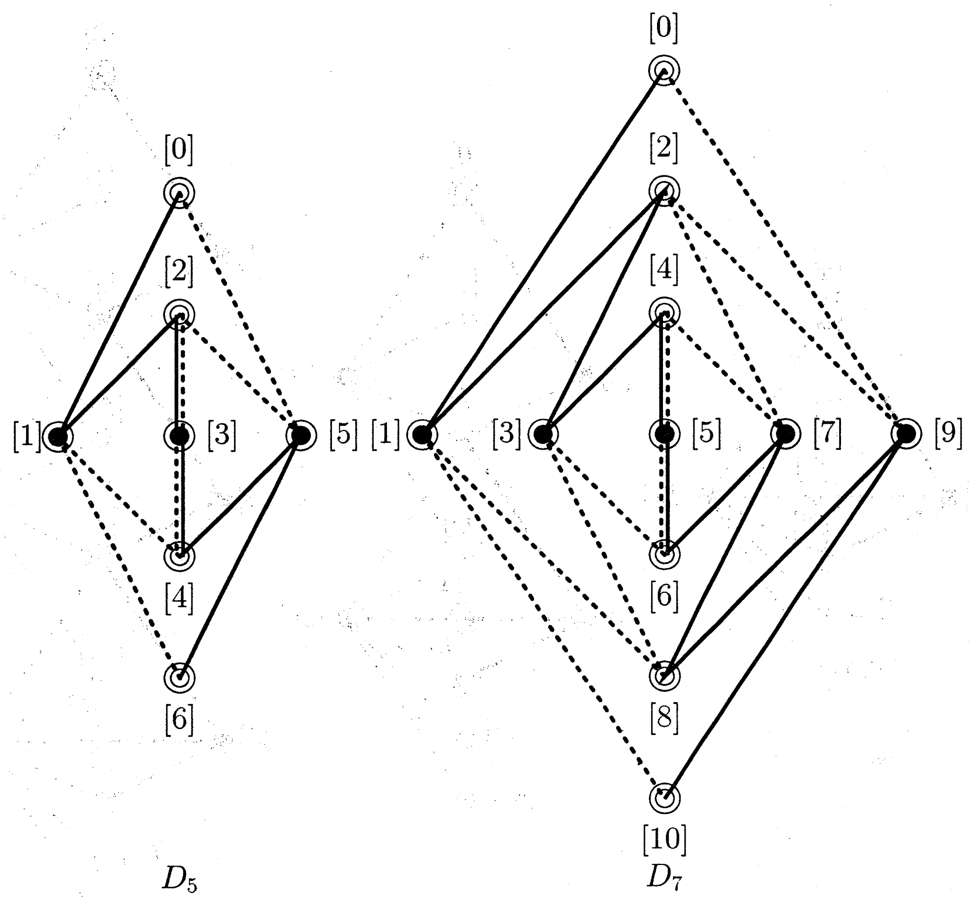


Figure 41: Chiral symmetry for the Coxeter graph D_{odd}

[10] D. E. Evans & Y. Kawahigashi, *On Ocneanu's theory of asymptotic inclusions for subfactors, topological quantum field theories and quantum doubles*, Internat. J. Math. **6** (1995), 205-228.

[11] D. E. Evans & Y. Kawahigashi, *From subfactors to* 3*-dimensional topological quantum field theories and back — a detailed account of Ocneanu's theory —*, Internat. J. Math. **6** (1995), 537-558.

[12] D. E. Evans & Y. Kawahigashi, *Orbifold subfactors from Hecke algebras II - Quantum doubles and braiding*, to appear in Comm. Math. Phys.

[13] D. E. Evans & Y. Kawahigashi, *Quantum symmetries on operator algebras*, Oxford University Press (1998).

[14] F. Goodman, P. de la Harpe & V. F. R. Jones, *Coxeter graphs and towers of algebras*, MSRI Publications, 14, Springer, Berlin (1989).

[15] F. Hiai & M. Izumi, *Amenability and strong amenability for fusion algebras with applications to subfactor theory*, to appear in Internat. J. Math.

[16] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–15.

[17] L. H. Kauffman & S. L. Lins *Temperley-Lieb recoupling theory and invariants of* 3*-manifolds*, Ann. Math. Studies **133**, Princeton University Press, (1995).

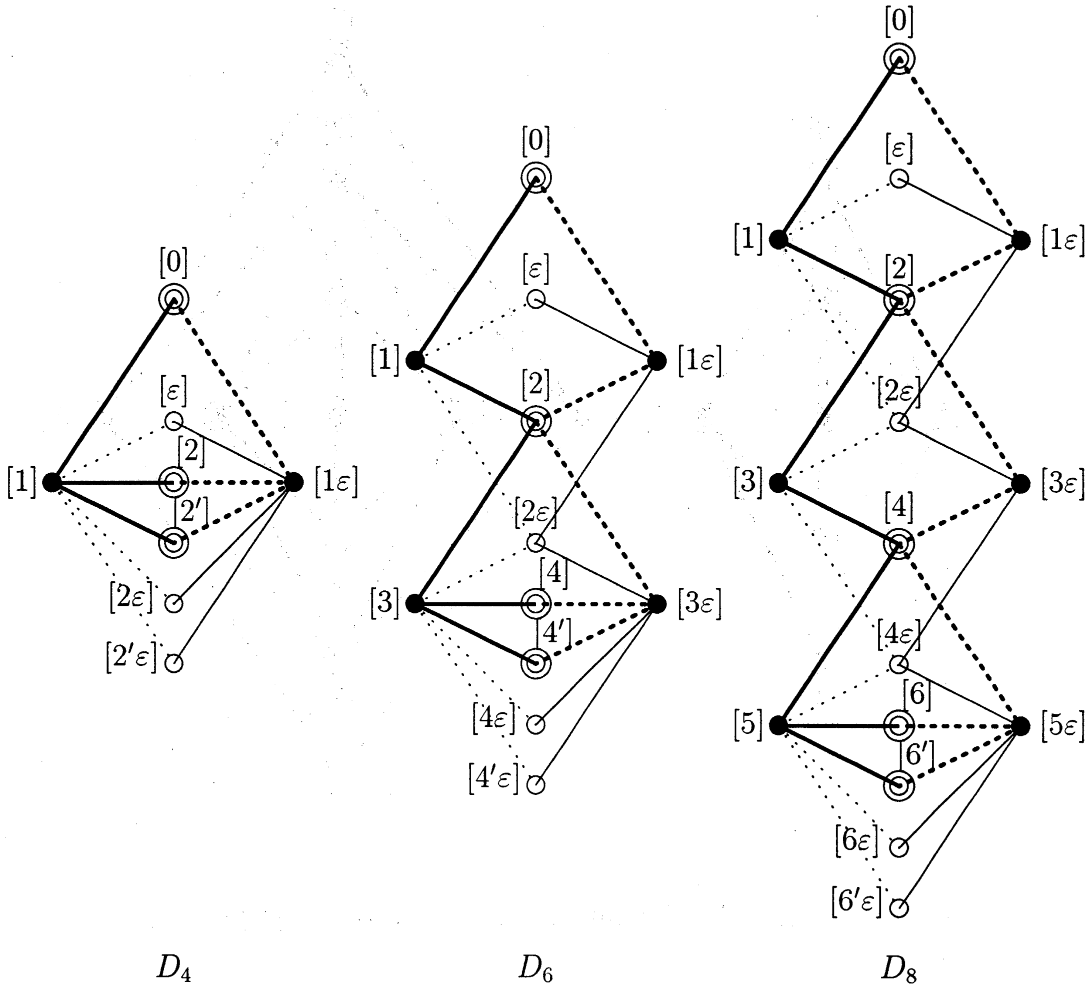


Figure 42: Chiral symmetry for the Coxeter graph D_{even}

[18] Y. Kawahigashi, *Classification of paragroup actions on subfactors*, Publ. RIMS, Kyoto Univ **31** (1995), 481–517.

[19] G. Moore & N. Seiberg, *Classical and quantum conformal field theory*, Comm. Math. Phys. **123** (1989), 177–254.

[20] A. Ocneanu, *Quantized group string algebras and Galois theory for algebras*, in "Operator algebras and applications, Vol. 2 (Warwick, 1987)," London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press, (1988), 119–172.

[21] A. Ocneanu, "Quantum symmetry, differential geometry of finite graphs and classification of subfactors", University of Tokyo Seminary Notes 45, (Notes recorded by Y. Kawahigashi), (1991).

[22] A. Ocneanu, *An invariant coupling between 3-manifolds and subfactors, with connections to topological and conformal quantum field theory*, preprint (1991).

[23] A. Ocneanu, *Chirality for operator algebras*, (recorded by Y. Kawahigashi), "Subfactors — Proceedings of the Taniguchi Symposium, Katata, 1993", World Sci Publishing, 39–63 (1994).

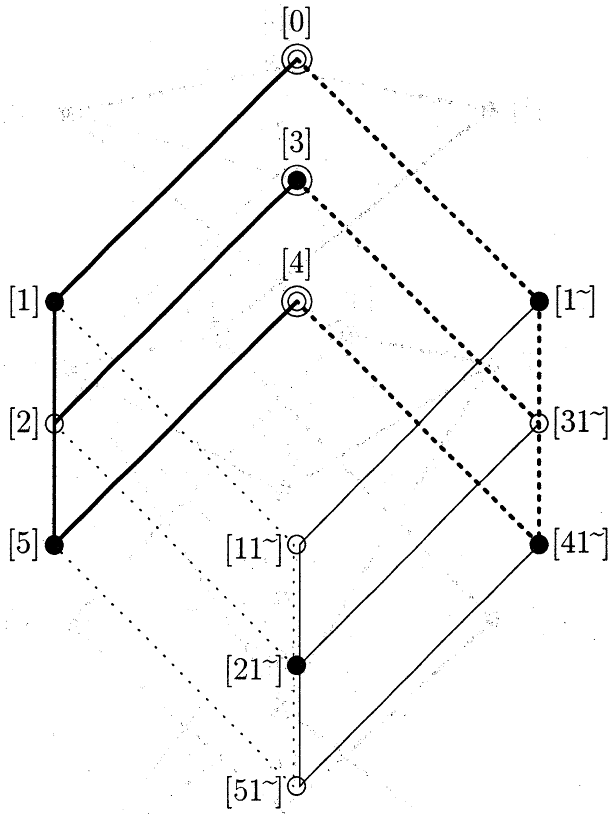


Figure 43: Chiral symmetry for the Coxeter graph E_6

[24] A. Ocneanu, *Paths on Coxeter Diagrams: From Platonic Solids and Singularities to Minimal Models and Subfactors*, (recorded by S. Goto), to appear in The Fields Institute Monographs.

[25] S. Okamoto, *Invariants for subfactors arising from Coxeter graphs*, Current Topics in Operator Algebras, World Scientific Publishing, (1991) 84–103.

[26] S. Popa, *Classification of amenable subfactors of type II*, Acta Math. **172** (1994), 253–345.

[27] N. Yu. Reshetikhin & V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597.

[28] Ph. Roche, *Ocneanu cell calculus and integrable lattice models*, Comm. Math. Phys. **127** (1990), 395–424.

[29] N. Sato, *A relation between two subfactors arising from a non-degenerate commuting square – An answer to a question raised by V.F.R. Jones*, Pacific J. Math **180** (1997), 369–376.

[30] N. Sato, *A relation between two subfactors arising from a non-degenerate commuting square – tensor categories and TQFT's*, Internat. J. Math. **8** (1997), 407–420.

[31] P. Slodowy, *A new A-D-E classification (after A. Capelli, C. Itzykson, J.-B. Zuber)*, Bayreuther Math. Sch. **33** (1990), 197–213.

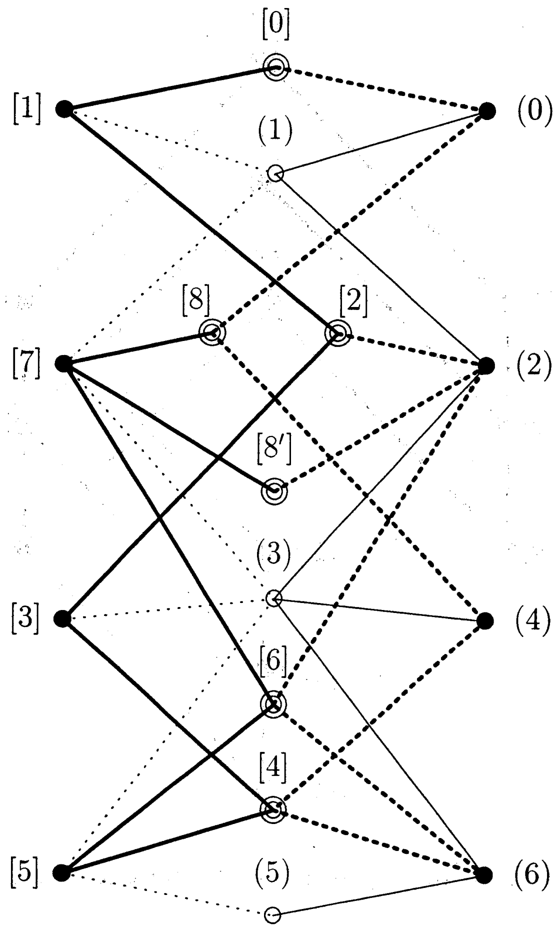


Figure 44: Chiral symmetry for the Coxeter graph E_7

[32] V. G. Turaev & O. Y. Viro, *State sum invariants of* 3*-manifolds and quantum* $6j$*-symbols*, Topology, **31** (1992), 865–902.

[33] H. Wenzl, *Hecke algebras of type A and subfactors*, Invent. Math. **92** (1988), 345–383.

[34] H. Wenzl, *Quantum groups and subfactors of type B, C and D*, Comm. Math. Phys. **133** (1990), 383–432.

[35] F. Xu, *Generalized Goodman-Harpe-Jones construction of subfactors I*, Comm. Math. Phys. **184** (1997), 475–491.

[36] F. Xu, *Generalized Goodman-Harpe-Jones construction of subfactors II*, Comm. Math. Phys. **184**, (1997), 493–508.

[37] F. Xu, *New braided endomorphisms from conformal inclusions*, Comm. Math. Phys. **192**, (1998), 349–403.

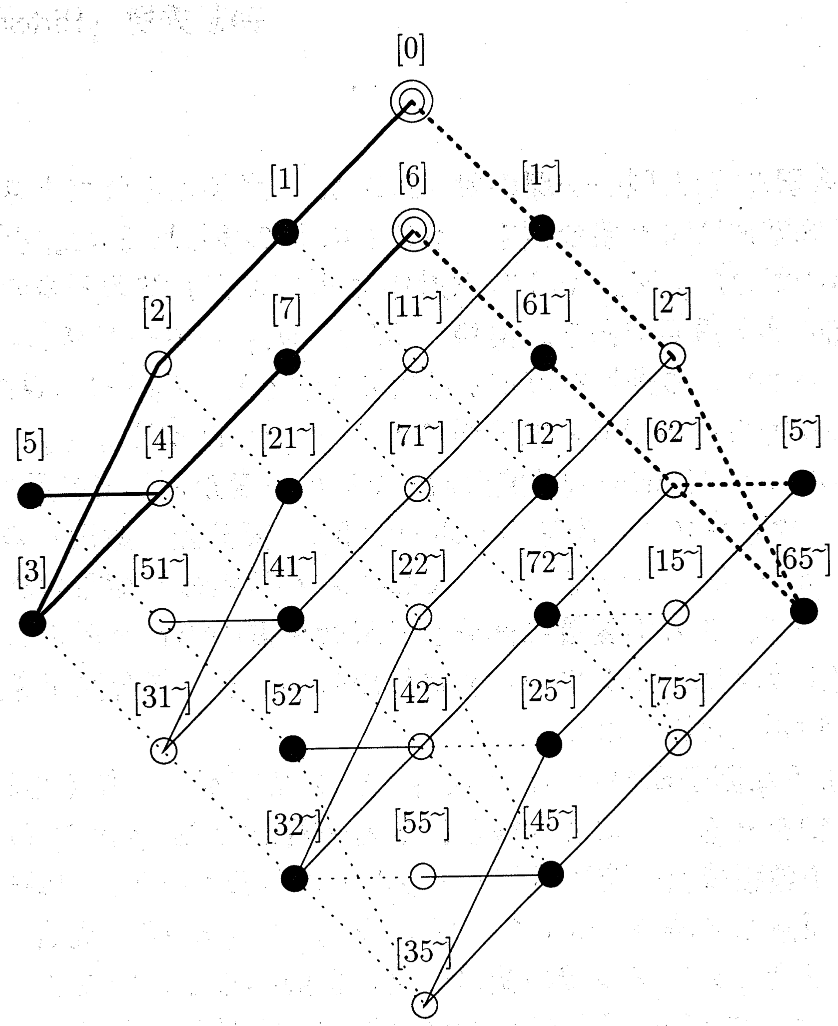


Figure 45: Chiral symmetry for the Coxeter graph E_8