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Unbounded C^*-seminorms and *-representations of *-algebras

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1. INTRODUCTION

Unbounded C^*-seminorms on *-algebras in the sense that they are C^*-seminorms defined on *-subalgebras have appeared in many mathematical and physical subjects (for example, locally convex *-algebras and the quantum field theory etc.). But this systematical study has not yet done sufficiently. The main purpose of this paper is to do a systematical study of unbounded C^*-seminorms and to apply it to a study of unbounded *-representations.

The paper is organized as follows: In Section 2 we construct unbounded *-representations of a *-algebra from unbounded C^*-seminorms and investigate them. Let $\mathcal{A}$ be a *-algebra. Let p be a C^*-seminorm defined on $\mathcal{A}$. Every *-representation of the Hausdorff completion of $(\mathcal{A}, p)$ gives rise to a *-representation of $\mathcal{A}$ into bounded Hilbert space operators. However, there are a number of situations in which natural C^*-seminorms are defined on *-subalgebras of $\mathcal{A}$. Then they should lead to unbounded operator representations of $\mathcal{A}$. An unbounded m^*-(resp. C^*-)seminorm is a submultiplicative * (resp. C^*-) seminorm p defined on a *-subalgebra $\mathcal{D}(p)$ of $\mathcal{A}$. Then $\mathfrak{N}_p \equiv \{x \in \mathcal{D}(p); ax \in \mathcal{D}(p), {}^\forall a \in \mathcal{A}\}$ is a left ideal of $\mathcal{A}$. It is shown that any *-representation Π_p: $\mathcal{A}_p \longrightarrow \mathfrak{B}(\mathcal{H})$ of the Hausdorff completion $\mathcal{A}_p$ of $(\mathcal{D}(p), p)$ leads to an unbounded *-representation π_p of $\mathcal{A}$ such that $\|\overline{\pi_p(x)}\| \leq p(x)$ for all $x \in \mathcal{D}(p)$. We denote by $\mathrm{Rep}(\mathcal{A}, p)$ the set of all such *-representations π_p of $\mathcal{A}$. In order to investigate representations in $\mathrm{Rep}(\mathcal{A}, p)$ in details, we introduce the notions of nondegenerate, finite, uniformly semifinite, semifinite and weakly semifinite unbounded C^*-seminorms, and show that if p is (weakly) semifinite, then there exists a strongly nondegenerate *-representation π_p in $\mathrm{Rep}(\mathcal{A}, p)$ such that $\left\|\overline{\pi_p(x)}\right\| = p(x)$ for all $x \in \mathcal{D}(p)$. Such a π_p is called well-behaved. In Section 3 we consider the converse direction of Section 2. We construct an unbounded C^*-seminorm r_π on $\mathcal{A}$ from a *-representation π of $\mathcal{A}$ and a natural representation $\pi_{r_\pi}^N$ of $\mathcal{A}$ constructed from r_π which is the restriction of the closure $\tilde{\pi}$ of π. It is shown that π is strongly nondegenerate if and only if $\pi_{r_\pi}^N$ is a well-behaved *-representation of $\mathcal{A}$. Further, it is shown that if p is a weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$ and π_p is any well-behaved *-representation, then r_{π_p} is a maximal extension of p.

2. REPRESENTATIONS INDUCED BY UNBOUNDED C^*-SEMINORMS

In this section we construct a family of *-representations of a *-algebra $\mathcal{A}$ induced by an unbounded C^*-seminorm on $\mathcal{A}$ and investigate the properties. We begin with the review of (unbounded) *-representations of $\mathcal{A}$. Throughout this section let $\mathcal{A}$ be a *-algebra. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$ and let $\mathcal{L}^\dagger(\mathcal{D})$ denote the set of all linear operators X in $\mathcal{H}$ with the domain $\mathcal{D}$ for which $X\mathcal{D} \subset \mathcal{D}$, $\mathcal{D}(X^*) \supset \mathcal{D}$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra under the usual operations and the involution $X \to X^\dagger \equiv X^*|\mathcal{D}$. A *-subalgebra of the *-algebra $\mathcal{L}^\dagger(\mathcal{D})$ is said to be an O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$. A *-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with a domain $\mathcal{D}$ is a *-homomorphism of $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D})$ and $\pi(1)$=I if $\mathcal{A}$ has identity 1, and then we write $\mathcal{D}$ and $\mathcal{H}$ by $\mathcal{D}(\pi)$ and $\mathcal{H}_\pi$, respectively. Let π_1 and π_2 be *-representations of $\mathcal{A}$. If $\mathcal{H}_{\pi_1}$ is a closed subspace of $\mathcal{H}_{\pi_2}$ and $\pi_1(x) \subset \pi_2(x)$ for each $x \in \mathcal{A}$, then π_2 is said to be an extension of π_1 and denoted by $\pi_1 \subset \pi_2$. In particular, if $\pi_1 \subset \pi_2$ and $\mathcal{H}_{\pi_1} = \mathcal{H}_{\pi_2}$, then π_2 is said to be an extension of π_1 as the same Hilbert space. Let π be a *-representation of $\mathcal{A}$. If $\mathcal{D}(\pi)$ is complete with the graph topology t_π defined by the family of seminorms $\left\{\|\bullet\|_{\pi(x)} \equiv \|\bullet\| + \|\pi(x)\bullet\|; x \in \mathcal{A}\right\}$, then π is said to be closed. It is well known that π is closed if and only if $\mathcal{D}(\pi)=\bigcap_{x\in\mathcal{A}}\mathcal{D}\left(\overline{\pi(x)}\right)$. The closure $\tilde{\pi}$ of π is defined by

$$\mathcal{D}(\tilde{\pi}) = \bigcap_{x\in\mathcal{A}}\mathcal{D}\left(\overline{\pi(x)}\right) \text{ and } \tilde{\pi}(x)\xi = \overline{\pi(x)}\xi \text{ for } x \in \mathcal{A},\ \xi \in \mathcal{D}(\tilde{\pi}).$$

Then $\tilde{\pi}$ is the smallest closed extension of π. The weak commutant $\pi(\mathcal{A})'_w$ of π is defined by

$$\pi(\mathcal{A})'_w = \left\{C \in \mathfrak{B}(\mathcal{H}_\pi); C\pi(x)\xi = \pi(x^*)^* C\xi, \ {}^\forall x \in \mathcal{A}, \ {}^\forall \xi \in \mathcal{D}(\pi)\right\},$$

where $\mathfrak{B}(\mathcal{H}_\pi)$ is the set of all bounded linear operators on $\mathcal{H}_\pi$, and it is a weakly closed $*$-invariant subspace of $\mathfrak{B}(\mathcal{H}_\pi)$, but it is not necessarily an algebra. It is known that $\pi(\mathcal{A})'_w \mathcal{D}(\pi) \subset \mathcal{D}(\pi)$ if and only if $\pi(\mathcal{A})'_w$ is a von Neumann algebra and $\overline{\pi(x)}$ is affiliated with the von Neumann algebra $\left(\pi(\mathcal{A})'_w\right)'$ for each $x \in \mathcal{A}$.

Definition 2.1. A mapping p of a subspace $\mathcal{D}(p)$ of $\mathcal{A}$ into $\mathbb{R}^+=[0,\infty)$ is said to be an unbounded (semi) norm on $\mathcal{A}$ if it is a (semi) norm on $\mathcal{D}(p)$, and p is said to be an unbounded m^*-(resp. C^*-) (semi) norm on $\mathcal{A}$ if $\mathcal{D}(p)$ is a $*$-subalgebra of $\mathcal{A}$ and p is a submultiplicative $*$-(resp. C^*-) (semi) norm on $\mathcal{D}(p)$.

If a seminorm p on a $*$-algebra $\mathcal{A}$ is a C^*-seminorm, that is, it satisfies the C^*-property $p(x^*x) = p(x)^2$, ${}^\forall x \in \mathcal{A}$, then it is a m^*- seminorm on $\mathcal{A}$, that is, $p(x^*) = p(x)$ and $p(xy) \le p(x)p(y)$ for ${}^\forall x, y \in \mathcal{A}$.

Let p be an unbounded C^*-seminorm on $\mathcal{A}$. We put

$$N_p = \{x \in \mathcal{D}(p); p(x) = 0\} \text{ and } \mathfrak{N}_p \equiv \{x \in \mathcal{D}(p); ax \in \mathcal{D}(p), \ {}^\forall a \in \mathcal{A}\}.$$

Then N_p is a $*$-ideal of $\mathcal{D}(p)$ and $\mathfrak{N}_p$ is a left ideal of $\mathcal{A}$, and the quotient $*$-algebra $\mathcal{D}(p)/N_p$ is a normed $*$-algebra with the C^*-norm $\|x + N_p\|_p \equiv p(x)$ ($x \in \mathcal{D}(p)$). We denote by $\mathcal{A}_p$ the C^*-algebra obtained by the completion of $\mathcal{D}(p)/N_p$, and denote by $\mathrm{Rep}(\mathcal{A}_p)$ the set of all $*$-representations Π_p of the C^*-algebra $\mathcal{A}_p$ on Hilbert space $\mathcal{H}_{\Pi_p}$. Put

$$\mathrm{FRep}(\mathcal{A}_p) = \{\Pi_p \in \mathrm{Rep}(\mathcal{A}_p); \Pi_p \text{ is faithful}\}$$

$$\mathrm{FNRep}(\mathcal{A}_p) = \{\Pi_p \in \mathrm{Rep}(\mathcal{A}_p); \Pi_p \text{ is faithful and nondegenerate}\}.$$

It is well known that $\mathrm{FNRep}(\mathcal{A}_p) \ne \varnothing$. For each $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$ we can define a bounded $*$-representation π_p^0 of $\mathcal{D}(p)$ on the Hilbert space $\mathcal{H}_{\Pi_p}$ by

$$\pi_p^0(x) = \Pi_p(x + N_p), \quad x \in \mathcal{D}(p).$$

The natural question arises: Can we extend the bounded $*$-representation π_p^0 of the

*-algebra $\mathcal{D}(p)$ to a (generally unbounded) *-representation of the *-algebra $\mathcal{A}$? We show that this question has affirmative answer.

Proposition 2.2. Let p be an unbounded C^*-seminorm on $\mathcal{A}$. For any $\Pi_p \in \mathrm{Rep}(\mathcal{A}_p)$, there exists a *-representation π_p of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\pi_p}$ such that $\left\|\overline{\pi_p(b)}\right\| \le p(b)$ for each $b \in \mathcal{D}(p)$. In particular, if $\Pi_p \in \mathrm{FRep}(\mathcal{A}_p)$, then $\left\|\overline{\pi_p(x)}\right\| = p(x)$ for each $x \in \mathfrak{N}_p$.

Proof. We put

$$\mathcal{D}(\pi_p) = \text{linear span of } \left\{\Pi_p(x+N_p)\xi; x \in \mathfrak{N}_p, \text{and } \xi \in \mathcal{H}_{\Pi_p}\right\}$$

$$\pi_p(a)\left(\sum_k \Pi_p(x_k+N_p)\xi_k\right) = \sum_k \Pi_p(ax_k+N_p)\xi_k \qquad \text{(finite sums)}$$

$$\text{for } a \in \mathcal{A},\ \{x_k\} \subset \mathfrak{N}_p \text{ and } \{\xi_k\} \subset \mathcal{H}_{\Pi_p}.$$

Since

$$\begin{aligned}\left(\Pi_p(ax+N_p)\xi \middle| \Pi_p(y+N_p)\eta\right) &= \left(\xi \middle| \Pi_p\left((ax+N_p)^*(y+N_p)\right)\eta\right)\\ &= \left(\xi \middle| \Pi_p(x^*a^*y+N_p)\eta\right)\\ &= \left(\xi \middle| \Pi_p(x^*+N_p)\Pi_p(a^*y+N_p)\eta\right)\\ &= \left(\Pi_p(x+N_p)\xi \middle| \Pi_p(a^*y+N_p)\eta\right)\end{aligned}$$

for each $a \in \mathcal{A}$, $x, y \in \mathfrak{N}_p$ and $\xi, \eta \in \mathcal{H}_{\Pi_p}$, it follows that $\pi_p(a)$ is a well-defined linear operator on $\mathcal{D}(\pi_p)$ for each $a \in \mathcal{A}$, so that it is easily shown that π_p is a *-representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_{\pi_p} \equiv [\mathcal{D}(\pi_p)] = \overline{\mathcal{D}(\pi_p)}^{\|\ \|}$ (the closure of $\mathcal{D}(\pi_p)$ in $\mathcal{H}_{\Pi_p}$) with domain $\mathcal{D}(\pi_p)$. Take an arbitrary $b \in \mathcal{D}(p)$. By the definition of π_p we have $\pi_p(b) = \pi_p^\circ(b) \big| \mathcal{D}(\pi_p)$, and hence

$$\left\|\overline{\pi_p(b)}\right\| \le \left\|\Pi_p(b+N_p)\right\| \le \left\|b+N_p\right\|_p = p(b).$$

Suppose $\Pi_p \in \mathrm{FRep}(\mathcal{A}_p)$ and $x \in \mathfrak{N}_p$. It is sufficient to show that $\left\|\overline{\pi_p(x)}\right\| \ge p(x)$. If $p(x)$=0, then it is obvious. Suppose $p(x) \ne 0$. We put $y = \dfrac{x}{p(x)} \in \mathfrak{N}_p$. For each $\xi \in \mathcal{H}_{\Pi_p}$ with $\|\xi\| \le 1$, we have

$$\left\|\Pi_p(y+N_p)\xi\right\| \le \left\|\Pi_p(y+N_p)\right\|\|\xi\| = p(y)\|\xi\| \le 1,$$

and so

$$\begin{aligned}\left\|\overline{\pi_p(y)}\right\| = \left\|\overline{\pi_p(y^*)}\right\| &\geq \sup\left\{\left\|\pi_p(y^*)\Pi_p(y+N_p)\xi\right\|; \xi \in \mathcal{H}_{\Pi_p} \text{ s.t. } \|\xi\| \leq 1\right\} \\ &= \sup\left\{\left\|\Pi_p(y^*y+N_p)\xi\right\|; \xi \in \mathcal{H}_{\Pi_p} \text{ s.t. } \|\xi\| \leq 1\right\} \\ &= \left\|\Pi_p(y^*y+N_p)\right\| \\ &= p(y^*y) = p(y)^2 = 1.\end{aligned}$$

Hence, we have $\left\|\overline{\pi_p(x)}\right\| \geq p(x)$. This completes the proof.

We have the following diagram:

$$\begin{tikzcd}
\mathcal{D}(p) \arrow[r] \arrow[drr, "\pi_p^0"'] \arrow[dd, phantom, "\cap"] & \mathcal{D}(p)/N_p \arrow[r, "\text{completion}"'] & \mathcal{A}_p \ (C^*\text{-algebra}) \arrow[d, "\Pi_p"] \\
& & \Pi_p(\mathcal{A}_p) \ (C^*\text{-algebra on } \mathcal{H}_{\Pi_p}) \arrow[d] \\
\mathcal{A} \arrow[rr, "\pi_p"] & & \pi_p(\mathcal{A}) \ (O^*\text{-algebra in } \mathcal{H}_{\pi_p} \subset \mathcal{H}_{\Pi_p}).
\end{tikzcd}$$

Remark: The *-representation π_p of $\mathcal{A}$ defined above by an unbounded C^*-seminorm p on $\mathcal{A}$ and an element Π_p of $\text{Rep}(\mathcal{A}_p)$ is non-zero if and only if $\mathcal{A}\mathcal{N}_p \not\subset N_p$. In what follows, we discuss several situations keeping this in mind.

Let p be an unbounded C^*-seminorm on $\mathcal{A}$. We denote by $\text{Rep}(\mathcal{A},p)$, $\text{FRep}(\mathcal{A},p)$ and $\text{FNRep}(\mathcal{A},p)$ the sets of all *-representations of $\mathcal{A}$ constructed as above by $(\mathcal{A},p)$, that is,

$$\begin{aligned}\text{Rep}(\mathcal{A},p) &= \left\{\pi_p; \Pi_p \in \text{Rep}(\mathcal{A}_p)\right\}, \\ \text{FRep}(\mathcal{A},p) &= \left\{\pi_p; \Pi_p \in \text{FRep}(\mathcal{A}_p)\right\}, \\ \text{FNRep}(\mathcal{A},p) &= \left\{\pi_p; \Pi_p \in \text{FNRep}(\mathcal{A}_p)\right\}.\end{aligned}$$

Definition 2.3. An unbounded m^*-seminorm q on $\mathcal{A}$ is said to be nondegenerate if $\mathcal{D}(q)^2$ is total in $\mathcal{D}(q)$ with respect to the seminorm q. An unbounded m^*-seminorm q on $\mathcal{A}$ is said to be finite if $\mathcal{D}(q)= \mathfrak{N}_q$; and q is said to be uniformly semifinite if there exists a net $\{u_\alpha\}$ in $\mathfrak{N}_q$ such that $u_\alpha{}^* = u_\alpha$ and $q(u_\alpha) \le 1$ for each α and $\lim_\alpha q(xu_\alpha - x)=0$ for each $x \in \mathcal{D}(q)$; and q is said to be semifinite if $\mathfrak{N}_q$ is dense in $\mathcal{D}(q)$ with respect to the seminorm q. An unbounded C^*-seminorm p on $\mathcal{A}$ is said to be weakly semifinite if $\mathrm{FRep}^{WB}(\mathcal{A}, p) \equiv \{\pi_p \in \mathrm{FRep}(\mathcal{A}, p); \mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi_p}\} \neq \varnothing$. An element π_p of $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ is said to be a well-behaved *-representation of $\mathcal{A}$ in $\mathrm{Rep}(\mathcal{A}, p)$.

Definition 2.4. A *-representation π of $\mathcal{A}$ is said to be nondegenerate if $[\pi(\mathcal{A})\mathcal{D}(\pi)]= \mathcal{H}_\pi$; and π is said to be strongly nondegenerate if there exists a left ideal $\mathcal{J}$ of $\mathcal{A}$ contained in the bounded part $\mathcal{A}_b^\pi \equiv \{x \in \mathcal{A}; \overline{\pi(x)} \in \mathfrak{B}(\mathcal{H}_\pi)\}$ of π such that $[\overline{\pi(\mathcal{J})}\mathcal{H}_\pi] = \mathcal{H}_\pi$.

Proposition 2.5. Let p be an unbounded C^*-seminorm on $\mathcal{A}$. Then the following statements hold:

(1) $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p) \subset \mathrm{FNRep}(\mathcal{A}, p)$ and every $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ satisfies the following conditions (i), (ii) and (iii):

(i) $\left[\overline{\pi_p(\mathfrak{N}_p)}\mathcal{H}_{\pi_p}\right] = \mathcal{H}_{\pi_p}$, and π_p is strongly nondegenerate.

(ii) $\|\overline{\pi_p(x)}\| = p(x)$, $\forall x \in \mathcal{D}(p)$.

(iii) $\pi_p(\mathcal{A})'_w = \overline{\pi_p(\mathcal{D}(p))}'$ and $\pi_p(\mathcal{A})'_w \mathcal{D}(\pi_p) \subset \mathcal{D}(\pi_p)$.

Conversely suppose $\pi_p \in \mathrm{FRep}(\mathcal{A}, p)$ satisfies conditions (i) and (ii) above. Then there exists an element π_p^{WB} of $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ which is a representation of π_p.

(2) Suppose p is semifinite. Then $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p) = \mathrm{FNRep}(\mathcal{A}, p)$ and $\mathfrak{N}_p{}^2$ is total in $\mathcal{D}(p)$ with respect to p, and so p is nondegenerate.

(3) Suppose p is uniformly semifinite. Then

$$\mathcal{A}_b^{\pi_p} = \mathcal{A}_b^p \equiv \{a \in \mathcal{A};\ \exists k_a > 0 \text{ s.t. } p(ax) \le k_a p(x),\ \forall x \in \mathfrak{N}_p\},$$

$$\|\overline{\pi_p(b)}\| = \sup\{p(bx);\ x \in \mathfrak{N}_p \text{ and } p(x) \le 1\},\quad \forall b \in \mathcal{A}_b^p$$

for each $\pi_p \in \mathrm{FRep}(\mathcal{A}, p)$.

(4) p is finite if and only if $\mathcal{D}(p)$ is a left ideal of $\mathcal{A}$.

3. UNBOUNDED C^*-SEMINORMS DEFINED BY *-REPRESENTATIONS

In Section 2 we constructed a family $\mathrm{Rep}(\mathcal{A}, p)$ (resp. $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$) of *-representation of $\mathcal{A}$ from an (resp. weakly semifinite) unbounded C^*-seminorm p on $\mathcal{A}$. Conversely we shall construct an unbounded C^*-seminorm r_π on $\mathcal{A}$ from a *-representation π of $\mathcal{A}$ and the natural representation $\pi^N_{r_\pi}$ of $\mathcal{A}$ constructed from r_π, and investigate the relation π and $\pi^N_{r_\pi}$. Let π be a *-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$. We put

$$\mathcal{A}_{\mathrm{b}}^{\pi} = \left\{x \in \mathcal{A}; \overline{\pi(x)} \in \mathfrak{B}(\mathcal{H}_\pi)\right\} \text{ and } \pi_{\mathrm{b}}(x) = \overline{\pi(x)}, \quad x \in \mathcal{A}_{\mathrm{b}}^{\pi}.$$

Then $\mathcal{A}_{\mathrm{b}}^{\pi}$ is a *-subalgebra of $\mathcal{A}$ and π_{b} is a bounded *-representation of $\mathcal{A}_{\mathrm{b}}^{\pi}$ on $\mathcal{H}_\pi$. We denote by $C^*(\pi)$ the C^*-algebra generated by $\pi_{\mathrm{b}}(\mathcal{A}_{\mathrm{b}}^{\pi})$. We now define an unbounded C^*-seminorm r_π on $\mathcal{A}$ as follows;

$$\mathcal{D}(r_\pi) = \mathcal{A}_{\mathrm{b}}^{\pi} \text{ and } r_\pi(x) = \|\pi_{\mathrm{b}}(x)\|, \quad x \in \mathcal{D}(r_\pi).$$

Then we put

$$\Pi(x + N_{r_\pi}) = \pi_{\mathrm{b}}(x), \quad x \in \mathcal{A}_{\mathrm{b}}^{\pi}.$$

Since $\left\|\Pi(x + N_{r_\pi})\right\| = r_\pi(x) = \left\|x + N_{r_\pi}\right\|_{r_\pi}$ for each $x \in \mathcal{A}_{\mathrm{b}}^{\pi}$, it follows that Π can be extended to a faithful *-representation $\Pi^N_{r_\pi}$ of $\mathcal{A}_{r_\pi}$ on the Hilbert space $\mathcal{H}_\pi$. The *-representation $\pi^N_{r_\pi}$ of $\mathcal{A}$ defined by $\Pi^N_{r_\pi}$ as above is called the natural representation of $\mathcal{A}$ induced by π. Since $\mathcal{H}_{\Pi^N_{r_\pi}} = \mathcal{H}_\pi$, it follows that $\mathcal{H}_{\pi^N_{r_\pi}}$ is a closed subspace of $\mathcal{H}_\pi$. We simply note the above method of the construction of $\pi^N_{r_\pi}$ by the following diagram:

$$
\begin{tikzcd}[column sep=large]
\pi \arrow[r] & \underset{\substack{\text{unbounded} \\ C^*\text{-seminorm}}}{r_\pi} \arrow[r] & \mathcal{D}(r_\pi)/N_{r_\pi} \arrow[r, "\text{completion}"'] & \underset{(C^*\text{-algebra})}{\mathcal{A}_{r_\pi}} \arrow[d, "\Pi^N_{r_\pi}"] \\
\pi^N_{r_\pi} & & & \underset{C^*\text{-algebra on } \mathcal{H}_\pi}{C^*(\pi) = \overline{\pi_{\mathrm{b}}(\mathcal{A}^\pi_{\mathrm{b}})}^{\|\ \|}} \arrow[lll]
\end{tikzcd}
$$

We have the following results for the relation of π and $\pi^N_{r_\pi}$:

Proposition 3.1. Suppose π is a *-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$. Then the following statements hold:

(1) $\pi^N_{r_\pi} \subset \tilde{\pi}$.

(2) Suppose π_{b} is nondegenerate. Then $\pi^N_{r_\pi} \in \mathrm{FNRep}(\mathcal{A}, r_\pi)$.

(3) π is strongly nondegenerate if and only if $\pi^N_{r_\pi} \in \mathrm{Rep}^{WB}(\mathcal{A}, r_\pi)$. If this is true, then $\pi^N_{r_\pi}$ is strongly nondegenerate with $\mathcal{A}^{\pi^N_{r_\pi}}_{\mathrm{b}} = \mathcal{A}^\pi_{\mathrm{b}}$, and r_π is weakly semifinite.

(4) Suppose there exists a net $\{u_\alpha\}$ in $\mathfrak{N}_{r_\pi}$ such that $s-\lim\limits_\alpha \pi(u_\alpha) = I$ and $s-\lim\limits_\alpha \pi(au_\alpha) = \pi(a)$ for each $a \in \mathcal{A}$. Then $\widetilde{\pi^N_{r_\pi}} = \tilde{\pi}$.

By Proposition 3.1 we have the following diagram:

$$
\begin{tikzcd}[column sep=huge, row sep=large]
p \arrow[r] \arrow[d, <->, dashed] & \pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p) \arrow[d, <->, dashed] \arrow[dl] \\
r_{\pi_p} \arrow[r] & \pi^N_{r_{\pi_p}} \in \mathrm{Rep}^{WB}(\mathcal{A}, r_{\pi_p}).
\end{tikzcd}
$$

We here investigate the relations of unbounded C^*-seminorms p and r_{π_p} and the *-representation π_p and $\pi_{r_{\pi_p}}$. We first define an order relation among unbounded seminorms as follows:

Definition 3.2. Let p and q be unbounded seminorms on $\mathcal{A}$. We say that p is an extention of q (or q is a restriction of p) if $\mathcal{D}(q) \subset \mathcal{D}(p)$ and $q(x) = p(x)$ for each $x \in \mathcal{D}(q)$, and then denote by $q \subset p$.

We denote by $\mathrm{C^*N}(\mathcal{A})$ the set of all unbounded C^*-seminorms on $\mathcal{A}$. Then $\mathrm{C^*N}(\mathcal{A})$ is an ordered set with the order $\subset$. For any $p \in \mathrm{C^*N}(\mathcal{A})$ we put

$$\mathrm{C^*N}(p) = \{q \in \mathrm{C^*N}(\mathcal{A}); p \subset q\}.$$

Then it follows from Zorn's lemma that $\mathrm{C^*N}(p)$ has a maximal element. We show that if p is weakly semifinite then r_{π_p} is a maximal element of $\mathrm{C^*N}(p)$.

Lemma 3.3. Let p and r be unbounded C^*-seminorms on $\mathcal{A}$. Suppose $p \subset r$. Then, for any $\pi_p \in \mathrm{Rep}(\mathcal{A}, p)$ there exists an element π_r of $\mathrm{Rep}(\mathcal{A}, r)$ such that $\pi_p \subset \pi_r$.

Proposition 3.4. Suppose p is a weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$ and $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$. Then r_{π_p} is a maximal element of $\mathrm{C^*N}(p)$ and $r_{\pi_p} = r_{\pi'_p}$ for each π_p, $\pi'_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$.

By Proposition 3.1, (3) and Proposition 3.4 we have the following

Corollary 3.5. Suppose π is a strongly nondegenerate *-representation of $\mathcal{A}$. Then r_π is maximal.

For the relation of *-representation π_p and $\pi^N_{r_{\pi_p}}$ we have the following

Proposition 3.6. Suppose p is a weakly semifinite unbounded C^*-seminorm on $\mathcal{A}$ and $\pi_p \in \mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$. Then $\pi_p \subset \pi^N_{r_{\pi_p}}$ and $\widetilde{\pi^N_{r_{\pi_p}}} = \widetilde{\pi}_p$.

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