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Problems on Geometric Structures of Projective Embeddings

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Abstract

This is a survey of our research on geometric structures of projective embeddings and includes the topics of our talks in several symposia (e.g. [23])from 1990 to 98. We clarify our main problem, which is to construct a kind of geometric composition series of projective embeddings. The concept of "geometric composition series" is an analogy in Algebraic Geometry with Jordan-Hölder series in Group theory. We present two of the candidates for the construction problem. We also give several results and new tools for approaching this problem. As a byproduct of the tools, we obtain a simplified proof for a criterion on arithmetic normality described in terms of Differential Geometry.

Keywords: Petri's Analysis, Second Fundamental Form, Syzygy, Arithmetic Normality, Infinitesimal Lifting, Geometric Shell, Geometric Composition Series, Lefschetz chain, dual Lefschetz chain, meta-Lefschetz Operators.

§0 Introduction.

In this article, we present several problems arising from the investigation on geometric structures of projective embeddings. When we use the technical term "geometric structure" of a projective embedding, it is our concern to see what kinds of intermediate ambient varieties appear for the projective subvariety defined by the given embedding.

To clarify this point more precisely, let us consider a connected complex projective manifold X of dimension $n > 0$ and an embedding $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$. Then, by an elementary fact on polynomial rings, we see that for any integer q with $n < q < N$, there

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exists a projective subvarity W of dimension q satisfying $j(X) \subset W \subset P$. In this case, we say that the variety W is an ***intermediate ambient variety*** of the subvariety $j(X)$.

On the other hand, if we suppose an additional condition on W, e.g. a variety W to be smooth along $j(X)$, namely $j(X) \subseteq Reg(W)$, then we can not assure the existence of a variety W satisfying the condition. For example, taking a Horrocks-Mumford abelian surface A in $P = \mathbb{P}^4(\mathbb{C})$ as the subvariety $j(X)$, then there is no hypersurface W with $A \subseteq Reg(W)$, which is certified by the calculation of Pic(W).

Thus we have special interest on the existence problem of intermediate ambient varieties with some additional conditions which can characterize the embedding. Then, we face an important problem, namely what conditions should be posed as the additional conditions? One of the candidates for the condition is presented in Definition 1.2. By using the concept "geometric shell", we can state our very optimistic Working Hypothesis 1.7, which claims the existence of a projective embedding with a good decomposition by geometric shells. As an approach to this working hypothesis, we summarize in §2 the results on Lefschetz operators and on meta-Lefschetz operators. We also present Main Conjecture 2.6 and clarify the relation with the former working hypothesis. As a preparation for attacking the conjectures, we newly introduce several key concepts for the infinitesimal method in §3. They often help us to remove the difficulty of higher obstructions for making the correspondence between subsheaves of the normal bundle and intermediate ambient varieties. In §4, we discuss arithmetic normality from two points of view. The first view point concern our framework and strategy for studying geometric structures of projective embeddings. From the second point of view, namely that of Differential Geometry, we explain a criterion for arithmetic normality in terms of the second fundamental form. Here we describe an outline of another proof for the criterion which is simplified by the tools in §3. This will show the power of our new tools.

In this article, we consider only the objects defined over the complex number field $\mathbb{C}$. In case of handling graded objects, we consider only homomorphims of preserving their grading otherwise mentioned. For example, "minimal free resolution" always means "graded minimal free resolution". Sometimes we state our result by using a pair of a variety and one of its embeddings instead of using the term "subvariety". That is only to emphasize the fact that we can chose the embedding suitably with fixing the variety itself in the real situation.

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§1 Working Hypothesis.

In this section, we present a key concept for considering geometric structures of embeddings and show several problems, in particular our optimistic working hypothesis. We hope that this may also bring us an insight for studying the syzygies of projective subvarieties.

Let us confirm our notation used in the sequel.

Notation 1.1 *Let us take a complex projective scheme X of dimension n and one of its embeddings $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$. The sheaf of ideals defining $j(X)$ in P and the conormal sheaf are denoted by I_X and $N^{\vee}_{X/P} = I_X/I_X^2$, respectively. Taking a $\mathbb{C}$-basis $\{Z_0, \dots, Z_N\}$ of $H^0(P, O_P(1))$. Then we put*

$$\begin{aligned}
S &:= \bigoplus_{m\geq 0} H^0(P, O_P(m)) \cong \mathbb{C}[Z_0, \dots, Z_N] \\
S_+ &:= = \bigoplus_{m>0} H^0(P, O_P(m)) \cong (Z_0, \dots, Z_N)\mathbb{C}[Z_0, \dots, Z_N] \\
\widetilde{R_X} &:= \bigoplus_{m\geq 0} H^0(X, O_X(m)) \\
\mathbb{I}_X &:= \bigoplus_{m\geq 0} H^0(P, I_X(m)) \\
R_X &:= Im[S \to \widetilde{R_X}] \cong S/\mathbb{I}_X \\
gsyz^q_X(m) &:= Tor^S_q(R_X, S/S_+)_{(m)},
\end{aligned}$$

where the subscript (m) of Tor above means taking its degree m part as a graded S-module. Obviously, the space $gsyz^q_X(m)$ represents minimal generators in degree m of the q-th syzygy of R_X as an S-module.

As a preparation, we recall the following key concepts introduced in [27].

Definition 1.2 (PG-shell and G-shell) *Let V and W be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$ which satisfy $V \subseteq W$ (In this case, the subscheme W is called simply an* intermediate ambient scheme *of V). If the natural map:*

$$\mu_q : Tor^S_q(R_W, S/S_+) \to Tor^S_q(R_V, S/S_+)$$

is injective for every $q \geq 1$, we say that W is a pregeometric shell (abv. PG-shell) *of V. Moreover, if W is a closed subvariety and the regular locus $Reg(W)$ of W contains V, we say that W is a* geometric shell (abv. G-shell) *of V. For the subscheme V, P and V itself are called* trivial PG-shell (or trivial G-shell).

Now let us see several elementary facts relating with "PG-shell".

Proposition 1.3 *Let V and W be closed subschemes of $P = \mathbb{P}^N(\mathbb{C})$ which satisfy $V \subseteq W$.*

(1.3.1) *If W is a hypersurface, then W is a PG-shell of V if and only if the equation H_W of W is a member of minimal generators of the homogeneous ideal $\mathbb{I}_V$ of V.*

(1.3.2) *Assume that the subscheme V is a complete intersection. Then the scheme W is a PG-shell of V if and only if the subscheme W is defined by a part of minimal generators of $\mathbb{I}_V$.*

(1.3.3) *Take a closed scheme Y such that $V \subseteq Y \subseteq W$. Assume that W is a PG-shell of V. Then W is also a PG-shell of Y. In particular, the subscheme W is also a PG-shell of the m-th infinitesimal neighborhood $Y = (V/W)_{(m)}$ of V in W, where $(V/W)_{(m)} = (|V|, O_W/I_V^{m+1})$.*

(1.3.4) *Fix the subscheme of $codim(V,P) \geq 2$. Then all non-trivial PG-shells of V form non empty algebraic family of finite components (N.B. The family of all non-trivial G-shells of V may be empty even if V itself is a smooth variety).*

(1.3.5) *If W is a PG-shell of V, then we have an inequality on their Castelnuovo-Mumford regularity: $CMreg(V) \geq CMreg(W)$.*

(1.3.6) *If W is a PG-shell of V, then we have an inequality on their arithmetic depth: $arith.depth(V) \leq arith.depth(W)$. In particular, if the natural restriction map $H^0(P, O_P(m)) \to H^0(V, O_V(m))$ is surjective for all integers m (i.e. $R_V = \widetilde{R_V}$), then the natural restriction map $H^0(P, O_P(m)) \to H^0(W, O_W(m))$ is also surjective for all integers m (i.e. $R_W = \widetilde{R_W}$).*

(1.3.7) *Assume that there exist r hypersurfaces $H_1, \dots H_r$ in P which form O_W-regular sequence and satisfy $V = W \cap H_1 \cap \dots \cap H_r$. If the restriction map $H^0(P, O_P(m)) \to H^0(V, O_V(m))$ is surjective for all integers m, then W is a PG-shell.*

(1.3.8) *Assume that the subscheme V is non-degenerate, namely no hyperplane contains V. If W has a 2-linear resolution, i.e. the homogeneous coordinate ring R_W of W has a minimal S-free resolution of the form : $0 \leftarrow R_W \leftarrow S \leftarrow F_1(-2) \leftarrow F_2(-3) \leftarrow \cdots \leftarrow F_p(-p-1) \leftarrow \cdots$, where $F_u(v)$ denotes $\oplus S(v)$: a direct sum of several copies of S with degree v shift, then W is a PG-shell of V.*

Proof. Directly from Definition 1.2, we see (1.3.1),(1.3.2) and (1.3.3). To get (1.3.4), using the injectivity of Tor_1^S, we notice that any PG-shell of V is defined by a part of minimal generators of the ideal $\mathbb{I}_V$. Let us take a parametrizing space T of all the intermediate ambient schemes of V which are defined scheme-theoretically by parts of the minimal generators of the ideal $\mathbb{I}_V$. Obviously the parametrizing space T is identified with a set-theoretic direct sum of Zariski open sets in several products of Grassmannian varieties. Through flattening stratifications incliding vertices of the affine cones of the membes in the family, we get an algebraic family of intermediate ambient schemes of V with constant Betti numbers, whose parametrizing space is named T again. We may assume that every component in T includes a point for a PG-shell of V. This family includes all the PG-shells of V. Looking at an induced chain homomorphism of (relatively) minimal free resolutions of $\mathbb{I}_V$ and of the family, we have only to extract the opensets of which corresponds to the "maximal" rank locus of every map in the chain homomorphism. For (1.3.5), use the Eisenbud-Goto criterion on Castelnuovo-Mumford regularity in [3]. Similarly (1.3.6) is obtained by applying the formula on depth and homological dimension. (1.3.7) is shown in [27]. On the claim (1.3.8), see Lemma1 in [2] from our point of view. ∎

The next fact is kindly told me by Prof.M.Hashimoto with answering my questions at Kinosaki Symposium. It may help us to construct G-shells in the real situation.

Proposition 1.4 *Let Y and Z be arithmetically Cohen-Macaulay projective subschemes of $P = \mathbb{P}^N(\mathbb{C})$. Assume that $X = Y \cap Z$ is of codimension $a + b$, where $a = codim(Y, P)$ and $b = codim(Z, P)$. Then both Y and Z are PG-shells of X.*

Proof. For R_Y and R_Z, take their minimal S-free resolutions: $\mathbb{F}_\bullet \to R_Y$, $\mathbb{G}_\bullet \to R_Z$, whose length are a and b, respectively. If we can show that the complex $\mathbb{F}_\bullet \otimes \mathbb{G}_\bullet$ is acyclic, then triviality of the complex $(\mathbb{F}_\bullet \otimes \mathbb{G}_\bullet) \otimes (S/S_+) \cong (\mathbb{F}_\bullet \otimes S/S_+) \otimes (\mathbb{G}_\bullet \otimes S/S_+)$ means that the complex $\mathbb{F}_\bullet \otimes \mathbb{G}_\bullet$ is a minimal S-free resolution of $R_X \cong R_Y \otimes R_Z$. Since the complexes $\mathbb{F}_\bullet$ and $\mathbb{G}_\bullet$ are naturally considered as subcomplexes and as direct summands of the complex $\mathbb{F}_\bullet \otimes \mathbb{G}_\bullet$, we see that the schemes Y and Z are PG-shells of X. The complex $\mathbb{F}_\bullet \otimes \mathbb{G}_\bullet$ has the length $a+b$, which coincides with $ht(\mathbb{I}_Y + \mathbb{I}_Z) = depth(\mathbb{I}_Y + \mathbb{I}_Z, S)$. To see the acyclicity of $\mathbb{F}_\bullet \otimes \mathbb{G}_\bullet$, we apply Buchsbaum-Eisenbud criterion for acyclicity on free complexes (cf. [1]). Thus we have only to show that for any prime ideal $\mathfrak{p} \in Spec(S)$ with $depth(\mathfrak{p}) < a + b$, $(\mathbb{F}_\bullet \otimes \mathbb{G}_\bullet)_\mathfrak{p}$ is acyclic. If $ht(\mathfrak{p}) = depth(\mathfrak{p}) < a + b$, then $\mathfrak{p} \not\supset \mathbb{I}_Y + \mathbb{I}_Z$, namely $\mathfrak{p} \not\supset \mathbb{I}_Y$ or $\mathfrak{p} \not\supset \mathbb{I}_Z$. For example, if $\mathfrak{p} \not\supset \mathbb{I}_Y$, then $(\mathbb{F}_\bullet)_\mathfrak{p} \to 0$ is split exact and therefore $(\mathbb{F}_\bullet \otimes \mathbb{G}_\bullet)_\mathfrak{p}$ is acyclic. ∎

The following example shows that all the exceptional cases in the classical Petri's Analysis can be considered as the cases of G-shells appearing.

Example 1.5 (Quadric hulls in Petri's Analysis) *Let $X = C$ be a non-hyperelliptic smooth projective curve of genus $g \geq 3$, and $j = \Phi_{|K_C|} : C \hookrightarrow P = \mathbb{P}^{g-1}(\mathbb{C})$ its canonical embedding. Taking the quadric hull W of $j(C)$, namely the closed subscheme defined by all equations of $j(C)$ with degree 2. Then, the quadric hull W coincides with $j(C)$ itself or is a non-trivial G-shell of $j(C)$.*

Proof. In fact, by classical Petri's Analysis (cf. [15], [16]), we see the exceptional cases explicitly, namely W is a Veronese surface in $\mathbb{P}^5(\mathbb{C})$ or a rational normal scroll. In both cases, W is a surface of minimal degree. Then apply (5.2)Lemma in [17] (see also [4]), we obtain that W has 2-linear resolutions, which implies that W is a G-shell of $j(C)$. ∎

Problem 1.6 *To make a foundation for studying PG-shells or G-shells, let us list several problems conjured up naturally in our mind.*

(1.6.1) *For a non-hyperelliptic curve C of genus $g = g(C) \geq 3$ and its canonical embedding $j = \Phi_{|K_C|} : C \hookrightarrow P = \mathbb{P}^{g-1}(\mathbb{C})$, classify all the PG-shells of $j(C)$.* (cf. Green Conjecture [2])

(1.6.2) *Describe the condition of "PG-shell" in terms of "generic initial ideals".*

(1.6.3) *Assume that a projective suscheme W is a PG-shell of a projective subvariety $V \subset P = \mathbb{P}^N(\mathbb{C})$. Then the subscheme W is always reduced and irreducible ? (N.B. When the subscheme W is a hypersurface, this is true.)*

(1.6.4) *Take smooth projective subvarieties V and W of positive dimension. Assume that the subvariety V is arithmetically normal. If W is a G-shell of V, then does the inequality on Δ-genus* (cf. [6])*: $\Delta(V, O_P|_V(1)) \geq \Delta(W, O_P|_W(1))$ hold in general? (If the polarized manifold $(V, O_P|_V(1))$ is a member of a ladder of the polarized manifold $(W, O_P|_W(1))$, then W is a G-shell of V and this inequality is true.)*

(1.6.5) *Take a smooth projective subvarity V, a vector bundle E on V, a section $\sigma \in \Gamma(V, E)$ which is transverse to the zero section, and its zero locus $X = Z(\sigma)$. Assume that V is a G-shell of X. Then is the bundle E always nef ?*

(1.6.6) *Take a smooth projective subvariety $V \subset P = \mathbb{P}^N(\mathbb{C})$ of dimension $n \geq 5$. Assume that V is not a hypersurface and has no non-trivial G-shell. Then $codim(V) \geq n/2$?* (Implied by Hartshorne's C.I.conjecture. cf. [10] [31]) *Moreover, for any positive integer M and a linear embedding $P = \mathbb{P}^N(\mathbb{C}) \subset Q = \mathbb{P}^{N+M}(\mathbb{C})$, if the subvariety V has no non-trivial G-shell except (multiple) projective cones, then does the Kodaira dimension of V satisfy the inequality $\kappa(V) \leq 0$?*

Now we present our working hypothesis in the most optimistic version, which suggests the direction of our research aiming.

Working Hypothesis 1.7 *Let X be a connected complex projective manifold of dimension $n > 0$. Then there exists an embedding: $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$, which satisfies the following conditions.*

(1.7.1) *There is a set of G-shells $\{W_p\}_{p=0}^k$ of $j(X)$ which satisy: $j(X) = W_0 \subset W_1 \subset \ldots \subset W_k \subseteq P$ and moreover $W_{p-1} \subset Reg(W_p)$ for $p = 1, \ldots, k$.*

(1.7.2) *For each $p = 1, \ldots, k$, there is a "nef" vector bundles E_p on W_p and a section $\sigma_p \in \Gamma(W_p, E_p)$ such that the zero locus $Z(\sigma_p)$ coinsides with W_{p-1} and $rank(E_p) = dim(W_p) - dim(W_{p-1})$.*

(1.7.3) *The subvariety W_k has a birational morphism from a projective bundle over a homogeneous space (in the sense of including abelian varieties).*

The set $\Xi = \{(W_p, E_p, \sigma_p)\}_{p=1}^{k}$ *of* $j(X)$ *and the integer* k *are called* a geometric composition series *of the embedding* j *or of the subvariety* $j(X)$ *and* the length *of the geometric composition series* Ξ*, respectively. For a given projective manifold* X*, if the embedding* j_0 *has a geometric composition series* Ξ_0 *whose length* k_0 *attains the minimum among the embeddings of* X *with geometric composition serieses, then we say that the geometric composition series* Ξ_0 *is* a absolutely minimal geometric composition series *of* X.

Remark 1.8 *To avoid confusion or to clarify what is in the author's mind, one should describe several points.*

(1.8.1) *For a vector bundle E on a projective variety V, we say that the bundle E is* nef *if the tautological line bundle $L_E = O_{P(E)/V}(1)$ is nef on the projective bundle $P(E) = \mathbb{P}(E)$ over V associated to the bundle E, namely for any curve C in $P(E)$, the intersection number satisfies the inequality: $(L_E.C) \geq 0$.*

(1.8.2) *Frankly speaking, the author confess that we might have to weaken our working hypothesis to some extent in the real situation. For example, we might have to replace the conditions:* (a) *"PG-shells" instead of "G-shells"* ; (b) *"reflexive sheaves" in stead of "vector bundles"* ; (c)*"rather mild singular locus of W_p" instead of "$Reg(W_p)$"* ; (d) *"$\kappa(W_k) \leq 0$" instaed of "a homogeneous space."*

Proposition 1.9 *Let X be a connected complex projective manifold of dimension $n \geq 2$ and $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$ an embedding. Then the following four conditions are equivalent.*

(1.9.1) *The subvariety $j(X)$ is a complete intersection.*

(1.9.2) *There is a set of intermediate ambient varieties $\{W_p\}_{p=0}^{N-n}$ of $j(X)$ which satisfies the conditions:* (a)*$dim(W_p) = n + p$* ; (b) *$j(X) = W_0 \subset W_1 \subset \ldots \subset W_{N-n} = P$* ; (c) *$W_{p-1} \subset Reg(W_p)$ for $p = 1, \ldots, N-n$.*

(1.9.3) *The embedding j has a geometric composition series $\Xi = \{(W_p, E_p, \sigma_p)\}_{p=1}^{N-n}$ of length $N - n$ with $rank(E_p) = 1$.*

(1.9.4) *The embedding j has a geometric composition series $\Xi_0 = \{(W_p, E_p, \sigma_p)\}_{p=1}^{1}$ of length 1 which satisfies $W_1 = P$ and $E_1 = \oplus_{s=1}^{N-n} O_P(m_s)$.*

Proof. The essential part is to show the equivalence between (1.9.1) and (1.9.2). Assume that (1.9.2). Starting from W_{N-n} and using that each W_p is a Cartier divisor of W_{p+1}, we show inductively that each W_p is a complete intersection and $Pic(W_p) \cong \mathbb{Z}O_{W_p}(1)$ for $p \geq 1$ by virtue of Corollary 3.2 in [11], which is still valid in the singular cases. Thus we have (1.9.1). Contrary, now we assume (1.9.1). A little care is needed to apply Bertini's theorem and to see that $W_{p-1} \subset Reg(W_p)$, which is rather a strong condition than $X \subset Reg(W_p)$. Take hypersurfaces $D_1, \dots, D_r$ of degree $d_1, \dots, d_r$, respectively such that $r = N - n$, $j(X) = D_1 \cap \dots \cap D_r$ and $d_1 \leq \dots \leq d_r$. Then consider the linear system $\Lambda_r = H^0(P, I_X(d_r))$ on $P = W_r$. Since $I_X(d_r)$ is generated by global sections, the base locus $Bs(\Lambda_r)$ coincides with X. Also by $D_r \in \Lambda_r$ satisfing $X \subset Reg(D_r)$, we find that general members are smooth. Then we put W_{r-1} to be a smooth member of Λ_r. Obviously $j(X) = D_1 \cap \dots \cap D_{r-1} \cap W_{r-1}$. As an induction hypothesis, we may assume that we have smooth complete intersection subvarieties: $W_k, W_{k+1}, \dots, W_r = P$ such that $dim(W_p) = n + p$, $j(X) = D_1 \cap \dots \cap D_p \cap W_p$ for $p = k, \dots, r$. We may assume $k \geq 2$. Then we consider a sublinear system $\Lambda_k := H^0(W_k, I_{X/W_k}(d_k)) \subset H^0(W_k, O_{W_k}(d_k))$ on the subvariety W_k. Since $I_{X/W_k}(d_k)$ is generated by the sections $D_1, \dots, D_k$, namely $(D_1, \dots, D_k) : \oplus_{q=1}^k O_{W_k}(d_k - d_q) \to I_{X/W_k}(d_k)$ is surjective, we have $Bs(\Lambda_k) = X$. By the same argument as above, we obtain a smooth member $W_{k-1} \in \Lambda_k$. Then, using the arithmetic normality of W_k, it is easy to see that W_{k-1} is also a complete intersection and $j(X) = D_1 \cap \dots \cap D_{k-1} \cap W_{k-1}$. ∎

§2 Conjectures.

In this section, we give some conjectures relating to Lefschetz operators. We expect that these conjecures give an approach to get our previous working hypothesis.

First, let us recall the definition of Lefschetz operators (cf. [19]).

Definition 2.1 (Lefschetz operator) *Let X be a complex projective scheme of dimension $n \geq 0$, $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$ an embedding, E an O_X-coherent sheaf, and $N^\vee_{X/P}$ the conormal sheaf of $j(X)$ in P, where I_X denotes the sheaf of ideals defining $j(X)$ in P. By natural restriction: $j^* : H^1(P, \Omega^1_P) \to H^1(X, \Omega^1_X)$, we have a hyperplane class $h = j^*(c_1(O_P(1))) \in H^1(X, \Omega^1_X)$, which induces a cohomological operator (depending on the embedding j):*

$$L_X : H^p(X, \Omega^q_X \otimes E) \xrightarrow{h\cup} H^{p+1}(X, \Omega^{q+1}_X \otimes E)$$

For a section $\sigma \in H^0(X, E)$, if the class $L^p_X(\sigma) \in H^p(X, \Omega^p_X \otimes E)$ is not zero and $L^{p+1}_X(\sigma)$ is zero, then we say that the section σ has the penetration order p *and denote it by $pent(\sigma) = p$. For an equation $F \in H^0(P, I_X(m))$ of $j(X)$ with degree m, we define $pent(F) = pent([F])$ by putting $E = N^\vee_{X/P}(m)$, where $[F]$ denotes the section of $H^0(X, N^\vee_{X/P}(m))$ induced by the natural restriction $I_X \to I_X/I^2_X = N^\vee_{X/P}$.*

We introduce meta-Lefschetz operators, which are difficult to control but give finer information than Lefschetz operators.

Definition 2.2 (meta-Lefschetz operator [27], [26]) *Let X be a complex projective scheme of dimension $n \geq 0$, $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$ an embedding, $N^\vee_{X/P} = I_X/I_X^2$ the conormal sheaf of $j(X)$ in P, where I_X denotes the sheaf of ideals defining $j(X)$ in P. Then we take the de Rham complex $\Omega_P^\bullet$ of P :*

$$0 \longrightarrow O_P \xrightarrow{d} \Omega_P^1 \xrightarrow{d} \Omega_P^2 \xrightarrow{d} \dots \xrightarrow{d} \Omega_P^N \longrightarrow 0$$

and the ideal order filtration (cf. [14]) $F_\nu^p\Omega_P^\bullet$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_X^{\nu+p} & \xrightarrow{d} & I_X^{\nu+p-1}\Omega_P^1 & \xrightarrow{d} & \dots & \xrightarrow{d} & \\ I_X^{\nu+1}\Omega_P^{p-1} & \xrightarrow{d} & \Omega_P^p & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega_P^N & \longrightarrow & 0. \end{array}$$

Now we fix ν and see $Gr^p_{F^\nu}(\Omega_P^\bullet) = F_\nu^p/F_\nu^{p+1}$:

$$\begin{array}{ccccccc} 0 \longrightarrow & I_X^{\nu+p}/I_X^{\nu+p+1} & \xrightarrow{\overline{d_I}} & I_X^{\nu+p-1}/I_X^{\nu+p} \otimes \Omega_P^1 & \xrightarrow{\overline{d_I}} & \dots \\ \xrightarrow{\overline{d_I}} & I_X^{\nu+1}/I_X^{\nu+2} \otimes \Omega_P^{p-1} & \xrightarrow{\overline{d_I}} & \Omega_P^p|_{X_{(\nu)}} & \longrightarrow & 0, \end{array}$$

where $X_{(\nu)} = (|X|, O_P/I_X^{\nu+1})$. Contrary to the fact that the exterior derivative d is not O_P-linear, the map $\overline{d_I}$ is O_P-linear and compatible with tensoring by $O_P(m)$. Thus we have:

$$I_X^{\nu+1}/I_X^{\nu+2}(m) \otimes \Omega_P^{p-1} \xrightarrow{\overline{d_I}} \Omega_P^p|_{X_{(\nu)}}(m)$$

and

$$H^t(X, I_X^{\nu+1}/I_X^{\nu+2}(m) \otimes \Omega_P^{p-1}) \xrightarrow{\overline{d_I}} H^t(X_{(\nu)}, \Omega_P^p|_{X_{(\nu)}}(m))$$

Next we consider a natural exact sequence $(\overline{LFT})$:

$$0 \longrightarrow I_X^{\nu+1}/I_X^{\nu+2} \otimes \Omega_P^p(m) \longrightarrow \Omega_P^p(m)|_{X_{(\nu+1)}} \longrightarrow \Omega_P^p(m)|_{X_{(\nu)}} \longrightarrow 0,$$

which induces an obstruction map:

$$\overline{\delta}_{LFT}^{(\nu)} : H^s(X_{(\nu)}, \Omega_P^p(m)|_{X_{(\nu)}}) \to H^{s+1}(X, I_X^{\nu+1}/I_X^{\nu+2} \otimes \Omega_P^p(m)).$$

Then we can define a map:

$$\widehat{L}_X^{(\nu)} = \overline{\delta}_{LFT}^{(\nu)} \circ \overline{d_I} : H^a(X, I_X^{\nu+1}/I_X^{\nu+2}(m) \otimes \Omega_P^b) \to H^{a+1}(X, I_X^{\nu+1}/I_X^{\nu+2}(m) \otimes \Omega_P^{b+1}),$$

which is called the ν*-th* meta-Lefschetz operator *with respect to the embedding* $j : X \hookrightarrow P$*. In case of* $\nu = 0$*, we denote it by* $\widehat{L}_X$ *instead of* $\widehat{L}_X^{(0)}$ *and call it simply* meta-Lefschetz operator *if there is no danger of confusion. Moreover, for the meta-Lefschetz operator* $\widehat{L}_X$*, we set*

$$\widehat{gsyz}_X^q(m) := Im[\widehat{L}_X : H^0(X, N_{X/P}^\vee(m) \otimes \Omega_P^{q-1}) \to H^1(X, N_{X/P}^\vee(m) \otimes \Omega_P^q)]$$

Fundamental properties on meta-Lefschetz operator are given as follows.

Theorem 2.3 **([26])** *Let* X *be a complex projective variety of dimension* n*,* $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$ *an embedding,* $N_{X/P}^\vee = I_X/I_X^2$ *the conormal sheaf of* $j(X)$ *in* P*, where* I_X *denotes the sheaf of ideals defining* $j(X)$ *in* P*. Take the meta-Lefschetz operator* $\widehat{L}_X$ *with respect to the embedding. Then the following properties hold.*

(2.3.1) *The meta-Lefschetz operator has naturality. In other words, for any closed subscheme* Y*, the diagram:*

$$\begin{array}{ccc} H^p(X, N_{X/P}^\vee(m) \otimes \Omega_P^q) & \xrightarrow{\widehat{L}_X} & H^{p+1}(X, N_{X/P}^\vee(m) \otimes \Omega_P^{q+1}) \\ {\scriptstyle natural}\downarrow & & \downarrow{\scriptstyle natural} \\ H^p(Y, N_{Y/P}^\vee(m) \otimes \Omega_P^q) & \xrightarrow[\widehat{L}_Y]{} & H^{p+1}(Y, N_{Y/P}^\vee(m) \otimes \Omega_P^{q+1}) \end{array}$$

is commutative.

(2.3.2) *The diagram:*

$$\begin{array}{ccc} H^p(X, N_{X/P}^\vee(m) \otimes \Omega_P^q) & \xrightarrow{\widehat{L}_X} & H^{p+1}(X, N_{X/P}^\vee(m) \otimes \Omega_P^{q+1}) \\ {\scriptstyle natural}\downarrow & & \downarrow{\scriptstyle natural} \\ H^p(X, N_{X/P}^\vee(m) \otimes \Omega_X^q) & \xrightarrow[-m\cdot L_X]{} & H^{p+1}(X, N_{X/P}^\vee(m) \otimes \Omega_X^{q+1}) \end{array}$$

is commutative,where L_X *denotes the Lefschetz operator.*

(2.3.3) *Assume that* $j(X)$ *has arithmetic depth* ≥ 2*, which includes the case that* X *is a normal projective variety of dimension* $n > 0$ *and the embedding is arithmetically normal, namely* $H^0(P, O_P(m)) \to H^0(X, O_X(m))$ *is surjective for all integers* m*. Then there is a natural one to one correspondence* $\gamma^q(m) : gsyz_X^q(m) \to \widehat{gsyz}_X^q(m)$ *as vector spaces. Here the space* $gsyz_X^q(m)$ *represents minimal generators in degree* m *of the* q*-th syzygy of* R_X*.*

(2.3.4) *For an integer k satisfying $n-1 \geq k \geq 1$, assume that the projective subvariety $j(X)$ has arithmetic depth $k+2$, or equivalently $H^s(X, O_X(u)) = 0$ for $u \in \mathbb{Z}$, $k \geq s \geq 1$, and $H^0(P, O_P(m)) \to H^0(X, O_X(m))$ is surjective for all integers m. Then the k-uple of the meta-Lefschetz operator:*

$$(\widehat{L}_X)^k : H^1(X, N^{\vee}_{X/P}(m) \otimes \Omega^q_P) \to H^{k+1}(X, N^{\vee}_{X/P}(m) \otimes \Omega^{q+k}_P)$$

is injective on the subspace $\widehat{gsyz}^q_X(m)$ for all integers m. Moreover, the map $\overline{d_I} : H^{k+1}(X, N^{\vee}_{X/P}(m) \otimes \Omega^{q+k}_P) \to H^{k+1}(X, \Omega^{q+k+1}_P \otimes O_X(m))$ is injective on the subspace $(\widehat{L}_X)^k(\widehat{gsyz}^q_X(m))$.

Returning to Lefschetz operators and make a preparation for defining Lefschetz chains and dual Lefschetz chain which play key roles in our conjectures.

Now we take the canonical map $\gamma^1(m) : gsyz^1_X(m) \to \widehat{gsyz}^1_X(m)$ in the Theorem2.3 above for $q = 1$ and consider the commutative diagram:

$$\begin{tikzcd}
\Sigma Z_t H^0(I_X(m-1)) \arrow[r] & H^0(I_X(m)) \arrow[r] \arrow[d, "natural"'] & gsyz^1_X(m) \arrow[r] \arrow[d, "\gamma^1(m)\cong"] & 0 \\
& H^0(N^{\vee}_X(m)) \arrow[r, "\widehat{L}_X"'] \arrow[d, "L_X"'] & \widehat{gsyz}^1_X(m) \arrow[r] \arrow[d, "Inclusion"] & 0 \\
& H^1(N^{\vee}_X(m) \otimes \Omega^1_X) \arrow[d, "L^{p-1}_X"'] & H^1(N^{\vee}_X(m) \otimes \Omega^1_P|_X) \arrow[l, "natural"] & \\
& H^p(N^{\vee}_X(m) \otimes \Omega^p_X), & &
\end{tikzcd}$$

where the first and the second rows are exact. Then we put:

$$J_p(m) = Ker[H^0(I_X(m)) \to H^0(N^{\vee}_X(m)) \xrightarrow{L^{p+1}_X} H^{p+1}(N^{\vee}_X(m) \otimes \Omega^{p+1}_X)].$$

From a chain of $\mathbb{C}$-vector spaces:

$$J_{-1}(m) := Im[\Sigma Z_t H^0(I_X(m-1))] \subseteq J_0(m) \subseteq J_1(m) \subseteq \cdots \subseteq J_{n-1}(m) \subseteq J_n(m) = H^0(I_X(m)),$$

we chose a finite subset $\{F_{1,s,m}, \cdots, F_{k(s),s,m}\}$ from $J_s(m)$ which forms a $\mathbb{C}$-basis of $J_s(m)/J_{s-1}(m)$ and define closed subschemes $W_p \subset P$ and $W^*_p \subset P$ by the equations $\{F_{1,s,m}, \cdots, F_{k(s),s,m} | 0 \leq s \leq p,\ m \in \mathbb{N}_0\}$ and by $\{F_{1,s,m}, \cdots, F_{k(s),s,m} | n \geq s \geq p,\ m \in \mathbb{N}_0\}$, respectively for $p = 0, \cdots, n$.

Definition 2.4 (Lefschetz chain and dual Lefschetz chain) *Under the circumstances, we obtain two chains of closed subschemes P. The one is :*

$$j(X) = W_n \subseteq W_{n-1} \subseteq \cdots \subseteq W_0 \subseteq P$$

and is called a Lefschetz chain *of $j(X)$. The other one is :*

$$j(X) = W_0^* \subseteq W_1^* \subseteq \cdots \subseteq W_n^* \subseteq P,$$

and is named a dual Lefschetz chain *of $j(X)$.*

Before claim our conjectures, we present fundamental properties of Lefschetz chains and dual Lefschetz chains.

Theorem 2.5 *Let X be a complex projective manifold of dimension $n > 0$, $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$ an arithmetically normal embedding. Take a Lefschetz chain and a dual Lefschetz chain of $j(X)$ as above and fix them. The following properties holds.*

(2.5.1) *The submanifold $j(X)$ is a complete intersection if and only if the Lefschetz chain is of the form :*

$$j(X) = W_n \subset W_{n-1} = \cdots = W_0 = P.$$

The similar equivalence holds on the dual Lefschetz chain by replacing the fom:

$$j(X) = W_0^* = W_1^* = \cdots = W_n^* \subset P.$$

(2.5.2) *Put $s = dim(Im[L_X^n : \oplus_m H^0(N_X^\vee(m)) \to \oplus_m H^n(N_X^\vee(m) \otimes \Omega_X^n)])$, then for the Lefschetz chain, the exact sequence:*

$$0 \longrightarrow Im(N_{W_{n-1}}^\vee|_X) \longrightarrow N_X^\vee \longrightarrow N_{X/W_{n-1}}^\vee \longrightarrow 0$$

always splits and $N_{X/W_{n-1}}^\vee \cong \oplus^s O_X(-m_i)$. Similarly for the dual Lefschetz chain, the exact sequence:

$$0 \to N_{W_n^*}^\vee|_X \to N_X^\vee \to N_{X/W_n^*}^\vee \to 0$$

always splits and $N_{W_n^*}^\vee|_X \cong \oplus^s O_X(-m_i)$.

(2.5.3) *Assume that the Standard Conjecture holds on the projective manifold X. For the Lefschetz chain, if $W_p \neq W_{p-1}$, then there is an integer m and a p-cycle ξ such that $h^p \cdot \xi > 0$ and $\xi \cdot c_r(N_X^\vee(m)) \equiv_{num.eq.} 0$, where $h = c_1(O_X(1))$. Also for the dual Lefschetz chain, if $W_p^* \neq W_{p+1}^*$, eactly the same holds.*

Outline of Proof. For (2.5.1) and (2.5.2), we have only to apply Serre duality. The claim (2.5.3) is obtained by using the result of [20] with a slight modification. To remove the condition "transverse to the zero section", we use Hironaka resolution for making the divisor normal crossing and study localized top Chern class instead of the zero locus of the section. See [30] for a precise argument. ∎

Now we can describe our main conjectures as follows.

Main Conjecture 2.6 *Let X be a complex projective manifold of dimension $n > 0$, $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$ an arithmetically normal embedding. Take a Lefschetz chain $\{W_p\}_{p=0}^n$ and a dual Lefschetz chain $\{W_p^*\}_{p=0}^n$ suitably. Then we expect the following properties hold by the suitable choice of the chains.*

(2.6.1) *Each W_p and W_p^* are PG-shell of $j(X)$.*

(2.6.2) *Each W_p and W_p^* are reduced along $j(X)$.*

(2.6.3) *Each W_p and W_p^* are irreducible.*

(2.6.4) *Each restricted conormal sheaf $N_{W_p/W_{p-1}}^\vee|_X$ is a vector bundle on X and is extendable to W_{p-1} as a vector bundle. Similarly, each restricted conormal sheaf $N_{W_p^*/W_{p+1}^*}^\vee|_X$ is a vector bundle and is extendable to W_{p+1}^* as a vector bundle.*

(2.6.5) *Fix the manifold X of dimension $n \geq 2$. Chose suitably the embedding j, Lefschetz chain $\{W_p\}_{p=0}^n$, and a dual Lefschetz chain $\{W_p^*\}_{p=0}^n$. Then a refinement of the Lefschetz chain or of the dual Lefschetz chain realizes the Working Hypothesis 1.7* (cf. Problem 2.9).

Proposition 2.7 *Let X be a connected complex projective manifold of dimension $n \geq 2$ and $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$ an arithmetically normal embedding. Assume that $j(X)$ is non-degenerate, namely no hyperplane in P contains $j(X)$. For an equation $F \in H^0(P, I_X(m))$ of $j(X)$, take the class $[F] \in H^0(X, N_{X/P}^\vee(m))$ induced by the equation F. Then we have:*

(2.7.1) *If $m = 2$, then the class $(\widehat{L}_X)^2([F]) \in H^2(X, N^\vee_{X/P}(2) \otimes \Omega^2_P)$ is not zero. More generally, take a non-zero class $\tau \in \widehat{gsyz}^q_X(q+1) \subseteq H^1(X, N^\vee_{X/P}(q+1) \otimes \Omega^q_P)$ which naturally corresponds to an element of minimal generators of q-th syzygy module of R_X in degree $q+1$, then $\widehat{L}_X(\tau) \in H^2(X, N^\vee_{X/P}(q+1) \otimes \Omega^{q+1}_P)$ is not zero.*

(2.7.2) *If $m = 3$ and the class $(\widehat{L}_X)^2([F]) \in H^2(X, N^\vee_{X/P}(3) \otimes \Omega^2_P)$ is zero, then $q(X) = h^1(O_X) > 0$.*

Proof. Apply Lemma 1.5 in [27]. See also [26]. ∎

Remark 2.8 *Proposition 2.7 shows that the meta-Lefschetz operator has really finer information on the syzygies of the coordinate ring than the Lefschetz operator does. For example, take $X = \mathbb{P}^n(\mathbb{C})$ $(n \geq 2)$, an embedding j=ν-th Veronesean embedding $(\nu \geq 3)$ and any equation F of $j(X)$ in degree 2, then it is easy to see that $(L_X)^2([F]) \in H^2(X, N^\vee_{X/P}(2) \otimes \Omega^2_X)$ is zero.*

The following problem is natural to be considered. We expect that proving this problem brings us a new idea for necessary refinments of Lefschetz chains and dual Lefschetz chains and helps us to solve our previous conjectures. For partial results on this problem, see [26].

Problem 2.9 (Preservation of Filtration) *Let X be a connected complex projective manifold of dimension $n \geq 1$ and $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C})$ an embedding. Consider the exact sequence: $0 \to N^\vee_{X/P} \to \Omega^1_P|_X \to \Omega^1_X \to 0$, which induces a canonical decreasing filtration $F^\bullet_q$: $\Omega^q_P|_X = F^0_q \supset F^1_q \supset \ldots \supset F^q_q \supset F^q_q = \{0\}$ such that $F^p_q/F^{p+1}_q \cong \Omega^{q-p}_X \otimes \wedge^p N^\vee_{X/P}$. Then a natural decreasing filtration on $H^{t,s}(m) = H^s(X, N^\vee_{X/P}(m) \otimes \Omega^t_P)$ is induced by putting $F^p H^{t,s}(m) = Im[H^s(X, N^\vee_{X/P}(m) \otimes F^p_t) \to H^{t,s}(m)]$. Then does it always hold that the meta-Lefschetz operator keeps the filtration, namely $\widehat{L}_X(F^p H^{t,s}(m)) \subseteq F^p H^{t+1,s+1}(m)$?*

§3 Infinitesimal Methods.

In this section, we introduce our simple tools which consist of two key concepts. These are mysteriously powerful for controling higher obstructions appearing in the study of infinitesimal neighborhoods. These are important to consider the correspondence between subbundles of the normal bundle and intermediate ambient varieties.

Definition 3.1 (Differntial Splitting) *On a complex algbraic scheme W, we consider an O_W-linear exact sequence of O_W-coherent sheaves:*

$$0 \longrightarrow G \xrightarrow{\alpha} F \xrightarrow{\beta} E \longrightarrow 0.$$

We say that this sequence splits differentially of order $\leq \mu$ *if there exists a (holomorphic $\mathbb{C}$-linear) differential operator $\nabla_\beta : E \to F$ of order $\leq \mu$ such that $\beta \circ \nabla_\beta = Id_E$, namely, the operator ∇_β gives a splitting in the category of abelian sheaves. It is easy to see that this condition is equivalent to the condition that the existence of two differential operators $\nabla_\alpha : F \to G$ and $\nabla_\beta : E \to F$ of order $\leq \mu$ which satisfy : $\beta \circ \nabla_\beta = Id_E$; $\nabla_\alpha \circ \alpha = Id_G$; and $\alpha \circ \nabla_\alpha + \nabla_\beta \circ \beta = Id_F$. When the scheme W is smooth and the sheaf E is of locally free, the condition of splitting differentially of some order is equivalent to the condition in terms of $\mathcal{D}_W$-modules that the sequence:*

$$0 \longrightarrow G \otimes \mathcal{D}_W \xrightarrow{\alpha} F \otimes \mathcal{D}_W \xrightarrow{\beta} E \otimes \mathcal{D}_W \longrightarrow 0,$$

splits in the category of right $\mathcal{D}_W$-modules, where $\mathcal{D}_W$ denotes the sheaf of holomorphic linear differential operators on W.

As showed in [24], there are many examples where differential splittings are observed. One of the typical examples is given as follows.

Example 3.2 *Let V be a complex algebraic scheme, E a vector bundle on V, $f : G = Grass(E, r) \to V$ the Grassmann bundle which parametrizes quotient r-bundles of E. Consider the universal sequence on G:*

$$0 \longrightarrow S \xrightarrow{\alpha_G} f^*E \xrightarrow{\beta_G} Q \longrightarrow 0.$$

Then this universal sequence splits differentially of order $= 1$ (Obviously it never splits $\mathcal{O}_G$-linearly).

Definition 3.3 (H^p-G.L.C.) *Let W be a noetherian scheme, X a closed subscheme of W which is defined by a sheaf of ideals I_X, E a coherent $\mathcal{O}_W$-module.*

(3.3.1) *For each non-negative integer μ, we set the μ-th infinitesimal neighborhood $X_{(\mu)}$ of X in W to be $(|X|, \mathcal{O}_W/I_X^{\mu+1})$ and the restricted sheaf $E_{(\mu)}$ of E to $X_{(\mu)}$ to be $E/I_X^{\mu+1}E$ as usual. Let ν be a non-negative integer. We say that* the H^p-global lifting criterion of the coherent sheaf E holds at the (infinitesimal) lifting level λ along $(X_{(\nu)}, X)$ *if the equality:*

$$\begin{aligned} &Im[H^p(W, E) \to H^p(X_{(\nu)}, E_{(\nu)})] \\ &= Im[H^p(X_{(\nu+\lambda)}, E_{(\nu+\lambda)}) \to H^p(X_{(\nu)}, E_{(\nu)})] \end{aligned}$$

holds in the space of $H^p(X_{(\nu)}, E_{(\nu)})$. This condition is abbreviated as "H^p-G.L.C. of E holds at level λ along $(X_{(\nu)}, X)$".

(3.3.2) *It is called that the H^p-global lifting criterion of the coherent sheaf E holds* uniformly *at the (infinitesimal) lifting level λ along X if for any positive integer ν, H^p-G.L.C. of E holds at level λ along $(X_{(\nu)}, X)$. This condition is also abbreviated as "H^p-G.L.C. of E holds* uniformly *at lifting level λ along X".*

Let us show one of the results in [25] as the simplest example for showing the powerfulness of our previous two key concepts.

Theorem 3.4 (Quotient Type) *Let W be a complex algebraic scheme. For an exact sequence of O_W-coherent sheaves:*

$$0 \to G \xrightarrow{\alpha} F \xrightarrow{\beta} E \to 0$$

connected by O_W-linear homomorphisms α and β, assume that this sequence splits differentially of order λ. If the H^p-lifting criterion on the sheaf F holds at the level μ along $(X_{(\nu)}, X)$, then the H^p-lifting criterion on the sheaf E holds at the level $\lambda + \mu$ along $(X_{(\nu)}, X)$.

Proof. It is enough to show that for any class $\phi \in H^p(X_{(\nu)}, E_{(\nu)})$ which is an image of a class of $H^p(X_{(\nu+\lambda)}, E_{(\nu+\lambda)})$, the class ϕ can be lifted to $H^p(W, E)$.

Let us consider six natural O_W-linear homomorphisms: $e : E \to E_{(\nu)}$, $\overline{e} : E_{(\lambda+\mu+\nu)} \to E_{(\nu)}$, $r : E \to E_{(\lambda+\mu+\nu)}$, $f : F \to F_{(\nu)}$, $\overline{f} : F_{(\mu+\nu)} \to F_{(\nu)}$, and $s : F \to F_{(\mu+\nu)}$, which satisfy $e = \overline{e} \circ r$ and $f = \overline{f} \circ s$. Since the differential operator $\nabla : E \to F$ is of $\mathbb{C}$-linear and of order λ, it induces a homomorphism of abelian sheaves $\overline{\nabla} : E_{(\lambda+\mu+\nu)} \to F_{(\mu+\nu)}$ which satisfies $s \circ \nabla = \overline{\nabla} \circ r$. Then, using carefully the commutativities of the maps already checked, we see that:

$$\overline{\beta} \circ \overline{f} \circ \overline{\nabla} \circ r = \overline{\beta} \circ \overline{f} \circ s \circ \nabla = \overline{\beta} \circ f \circ \nabla = e \circ \beta \circ \nabla = e \circ Id_E = \overline{e} \circ r,$$

where $\overline{\beta} : F_{(\nu)} \to E_{(\nu)}$ denotes the natural O_W-linear homomorphism induced by $\beta : F \to E$. Considering all the homomorphisms given above as the homomorphisms in the category of abelian sheaves, the surjectivity of the homomorphism r (at each stalk) implies that:

$$\overline{\beta} \circ \overline{f} \circ \overline{\nabla} = \overline{e}.$$

Now, by assumption, we can take a class $\overline{\psi} \in H^p(X_{(\lambda+\mu+\nu)}, E_{(\lambda+\mu+\nu)})$ whose image by the map $\overline{e}$ coincides with the given class ϕ of $H^p(X_{(\nu)}, E_{(\nu)})$. Then, takinig H^p of the sheaves introdeced in the above, we have the following (a partially non-commutative) diagram:

$$
\begin{tikzcd}
H^p(F) \arrow[rrr, "\beta"] \arrow[d, "Id"'] & & & H^p(E) \arrow[d, "Id"] \\
H^p(F) \arrow[dr, "s"] \arrow[ddr, "f"'] & & & H^p(E) \arrow[lll, "\nabla"'] \arrow[dl, "r"'] \arrow[ddl, "e"] \\
& H^p(F_{(\mu+\nu)}) \arrow[d, "\overline{f}"'] & H^p(E_{(\lambda+\mu+\nu)}) \arrow[l, "\overline{\nabla}"'] \arrow[d, "\overline{e}"] & \\
& H^p(F_{(\nu)}) \arrow[r, "\overline{\beta}"'] & H^p(E_{(\nu)}), &
\end{tikzcd}
$$

By the assumption that H^p-G.L.C. of the sheaf F holds at the level μ along $(X_{(\nu)}, X)$, the class $\overline{f} \circ \overline{\nabla}(\overline{\psi})$, which is the image of $\overline{\nabla}(\overline{\psi}) \in H^p(F_{(\mu+\nu)})$ by the map $\overline{f}$, can be lifted to a class $\sigma \in H^p(W, F)$, namely $f(\sigma) = \overline{f} \circ \overline{\nabla}(\overline{\psi})$. Then, putting ψ to be $\beta(\sigma)$, we see that:

$$e(\psi) = e \circ \beta(\sigma) = \overline{\beta} \circ f(\sigma) = \overline{\beta}(\overline{f} \circ \overline{\nabla}(\overline{\psi})) = \overline{e}(\overline{\psi}) = \phi,$$

which is the desired conclusion. ∎

Corollary 3.5 *Let $V \subseteq P = \mathbb{P}^N(\mathbb{C})$ be a closed subscheme and $m \geq m_0$ non-negaive integers. Assume that the restriction map $H^0(P, O_P(m_0)) \to H^0(V, O_V(m_0))$ is surjective. Then H^0-G.L.C. of $O_P(m)$ holds at level $m - m_0$ along $(X_{(0)}, X)$. In other words, any section $\sigma \in H^0(V, O_V(m))$ can be lifted to $H^0(P, O_P(m))$ if and only if the section can be lifted to $H^0(V_{(m-m_0)}, O_{V_{(m-m_0)}}(m))$.*

Proof. By the assumption, H^0-G.L.C. of $O_P(m_0)$ holds at level 0 along $(X_{(0)}, X)$. We use induction on m by starting from the case $m = m_0$. Take a positive integer $m > m_0$. We have only to apply Theorem 3.4 to the Euler sequence:

$$0 \longrightarrow \Omega^1_P(m) \longrightarrow \oplus O_P(m-1) \longrightarrow O_P(m) \longrightarrow 0,$$

which splits differentially of order 1 for positive integer m. (N.B. In case of $m = 0$, this sequence never splits even in the sense of differential splitting.) ∎

§4 Arithmetic Normality.

In this section, we discuss arithmetic normality from the two points of view. The first one is a viwpoint for clarifying our framework and strategy of studying the geometric structures of projective embeddings. The second one is a viewpoint from Differential Geometry, which presents a crtiterion for arithmetic normality in terms of Differential Geometry.

For the first viewpoint, let us review weighted objects such as "weighted projections", which relates to "arithmetic normality" as a usual "projection" does to "linear normality".

Definition 4.1 (Weighted Projection) *For $N + L + 1$-variables with weighted degree $wt.deg(Z_p) = s_p \geq 1$ $(p = 0, \dots, N)$; $wt.deg(W_q) = w_q$ $(q = 1, \dots, L)$, take a weighted polynomial ring $T = \mathbb{C}[Z_0, \dots, Z_N, W_1, \dots, W_L]$ and its polynomial subring $S = \mathbb{C}[Z_0, \dots, Z_N]$. By applying "Proj" operation, we get a rational map between the weighted projective spaces:*

$$Proj(T) = \mathbb{P}(s_0, \dots, s_N, w_1, \dots, w_L) \overset{\pi}{\dashrightarrow} Proj(S) = \mathbb{P}(s_0, \dots, s_N),$$

which is called a weighted projection *along the center $Z = \{W_1 = \dots = W_L = 0\}$.*

Definition 4.2 (Weighted Linear Degeneration) *Consider a weighted polynomial ring $S = \mathbb{C}[Z_0, \dots, Z_N]$ with $wt.deg(Z_p) = s_p$ and a closed subscheme $X \subset Proj(S) = \mathbb{P}(s_0, \dots, s_N) = P$. If there is* a weighted linear *homogeneous polynomial $F \in S$, which is degree* 1 *without weight in at least one variable, e.g. $F = Z_0 + F_1(Z_1, \dots Z_N)$, and if X is a closed subscheme of the subscheme $Proj(S/(F)) \subset P$, then we say that the subscheme* X degenerates weighted linearly. *(In this case, the subscheme X can be isomorphically projected through a suitable weighted projection.)*

Lemma 4.3 *Let X be a complex projective scheme of dimension $n \geq 0$ and $j : X \hookrightarrow P = \mathbb{P}^N(\mathbb{C}) = \mathbb{P}(1, \dots, 1)$ an embedding to a projective N-space (in a usual sense). Then there is a weighted projective space : $Q = \mathbb{P}(1^{N+1}, w_1, \dots, w_L)$ and an embedding : $\tilde{\jmath} : X \hookrightarrow Q$ which make the commutative diagram:*

$$\begin{tikzcd} & Q \arrow[d, dotted, "\pi"] \\ X \arrow[ur, "\tilde{\jmath}"] \arrow[r, "j"'] & P \end{tikzcd}$$

and satisfy the surjectivity on the natural map: $H^0(Q, O_Q(m)) \to H^0(X, O_X(m))$ for every non-negative integers m.

Since several people asked me a proof for this lemma, it may be a little worth writing down its proof here.

Proof. The idea is very simple and is only to add enough variables with suitable weighted degree. The argument goes as follows. Let us put the vector space V to be $Im[H^0(P, O_P(1)) \to H^0(X, O_X(1))]$ and the section $\sigma_t \in V$ to be the image of $Z_t \in H^0(P, O_P(1))$ for $t = 0, 1, \ldots, N$, where $N = dim(H^0(P, O_P(1))) - 1$ and $\{Z_t\}_{t=0}^N$ form a $\mathbb{C}$-basis of $H^0(P, O_P(1))$. Since the line bundle $O_X(1) = j^* O_P(1)$ is ample, there are only finitely many positive integers m such that $dimCoker[V \otimes H^0(X, O_X(m-1)) \to H^0(X, O_X(m))] = c_m \neq 0$. Set $\{m(1), \ldots, m(u)\} = \{m \in \mathbb{N} | c_m \neq 0\}$ and $L = c_{m(1)} + c_{m(2)} + \cdots + c_{m(u)}$, where $1 \leq m(1) \leq \cdots \leq m(u)$. Now we take the sections $\tau_1, \ldots, \tau_L$ such that $\tau_{c_{m(1)}+\cdots+c_{m(s-1)}+1}, \ldots, \tau_{c_{m(1)}+\cdots+c_{m(s)}} \in H^0(X, O_X(m(s)))$ induce the $\mathbb{C}$-basis of $Coker[V \otimes H^0(X, O_X(m(s)-1)) \to H^0(X, O_X(m(s)))]$ for $s = 1, \ldots, u$. Take variables W_k with $deg(W_k) = w_k$ corresponding to the section $\tau_k \in H^0(X, O_X(w_k))$ for $k = 1, \ldots, L$, namely $w_k = m(s)$ if $c_{m(1)} + \cdots + c_{m(s-1)} + 1 \leq k \leq c_{m(1)} + \cdots + c_{m(s)}$. Now we have two essentially surjective ring homomorphisms : $T = \mathbb{C}[Z_0, \ldots, Z_N, W_1, \ldots, W_L] \to \widetilde{R_X} = \oplus_m H^0(X, O_X(m))$ and $S = \mathbb{C}[Z_0, \ldots, Z_N] \to \widetilde{R_X} = \oplus_m H^0(X, O_X(m))$ by sending Z_t to σ_t and W_k to τ_k, which make a commutative diagram:

$$\begin{tikzcd} & T \arrow[dl] \\ \widetilde{R_X} & S \arrow[l] \arrow[u, "inclusion"'] \end{tikzcd}$$

Taking their "Proj", we obtain the result. (N.B. For simplicity, we constructed the ring T rather roughly and it may have dispensable variables.) ∎

Here we would like to make a discussion on a framework and a strategy for our research. Generally the weighted projective space Q has singularities and the sheaf $O_Q(m)$ is not a line bundle but only a reflexive sheaf. On the other hand, Lemma 4.3 above shows that any projective embedding is a composition of a weighted projection and an embedding into a weighted projective space which is very similar to an arithmetically normal embedding.

Hence, to study the geometric structures of projective embedding, we can divide the problem into the three problem: (a) investigate the arithmetically normal embeddings ; (b) generalize the results of (a) into the case of weighted projective spaces (e.g. Working Hypothesis in weighted version); (c) study the effects of weighted projections on the intermediate ambient varieties and on weighted G-shells ("weighted G-shell" is similarly defined by using $Tor_q^T(-, T/T_+)$ instead of $Tor_q^S(-, S/S_+)$).

Relating to the problem (c) above, we should notice the fact that even if we have a good intermediate ambient variety W with $\jmath(X) \subset W \subset Q$, the variety W may collapse by the weighted projection but the variety X itself is projected isomorphically. Thus we believe that the arithmetic normality is a natural condition as the fundamental assumption for our research in the first step, because we can ignore the difficulty arising from weighted projections.

The arithmetic normality is equivalent to H^0-G.L.C. of $O_P(m)$ holding at level 0 along $(X_{(0)}, X)$ as we used it in the proof of Corollary 3.5. Since the bundle $O_P(m)$ is a building brock for coherent sheaves, the assumption of arithmetic normality makes the higher obstruction control much more easier than without it. We might be going a bit too far, but the difficulty of higher obstruction can be sometimes explained by relating with weighted projections.

Now we proceed to the second viewpoint on arithmetic normality, namely that from Differential Geometry. Let us recall the concepts of complex differential geometry. Take a connected complex projective submanifold $X \subseteq P = \mathbb{P}^N(\mathbb{C})$ of dimension $n > 0$. By inducing a metric on X from the Fubini-Study metric on P, we consider X to be a Kähler manifold. Consider the exact sequence of induced Hermitian vector bundles:

$$0 \longrightarrow N^\vee_{X/P} \longrightarrow \Omega^1_P|_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

Then we have Hermitian connections $\nabla : \mathcal{A}^0(\Omega^1_P|_X) \to \mathcal{A}^1(\Omega^1_P|_X)$ and $\nabla_0 : \mathcal{A}^0(N^\vee_X) \to \mathcal{A}^1(N^\vee_X)$, which induce a $\mathcal{C}^\infty$-section $A = \nabla|_{N^\vee} - \nabla_0 \in \mathcal{A}^{(1,0)}(\mathcal{H}om(N^\vee_X, \Omega^1_X))$ of (1,0)-form with values in $\mathcal{H}om(N^\vee_X, \Omega^1_X)$. Instead of $N^\vee_X$, considering Ω^1_X to be a $\mathcal{C}^\infty$-subbundle of $\Omega^1_P|_X$, we have a $\mathcal{C}^\infty$-section $B \in \mathcal{A}^{(0,1)}(\mathcal{H}om(\Omega^1_X, N^\vee_X))$ of (0,1)-form with values in $\mathcal{H}om(\Omega^1_X, N^\vee_X)$.

The following properties are well-known (cf. [8],[9],[12],[13]).

Proposition 4.4 (Second Fundamental Forms) *Under the circumstances,*

(4.4.1) *B is an adjoint of $-A$. In other words, for $\xi \in \mathcal{A}^0(N^\vee_X)$ and $\eta \in \mathcal{A}^0(\Omega^1_X)$, the equality $h(A\xi, \eta) + h(\xi, B\eta) = 0$ holds, where $h(-,-)$ denotes the Hermitian metric on $\Omega^1_P|_X$.*

(4.4.2) *Since B is $\bar\partial$-closed, it defines a class $[B] \in H^1(X, \Theta_X \otimes N^\vee_X)$, which coincides with the infinitesimal ring extension class of*

$$0 \longrightarrow N^\vee_X \longrightarrow O_P/I^2_X \longrightarrow O_X \longrightarrow 0.$$

The class $\sigma_{II}(X) = [B]$ is called the second fundamental form of type (0,1) *for X.*

(4.4.3) $A \in H^0(Sym^2(\Omega^1_X) \otimes N^\vee_X)$. *This class* A *is called the* holomorphic second fundamental form *of* X *and coincides with the differential of the Gauss map induced by the embedding. Also a linear system is defined by considering it at general point of* X ([9],[13]).

Now we take a smooth irreducible divisor D on X. Then we have an exact sequence :

$$0 \longrightarrow \Theta_D \longrightarrow \Theta_X|_D \longrightarrow N_{D/X} \longrightarrow 0,$$

and a natural induced homomorphism : $r_D : H^1(X, \Theta_X \otimes N^\vee_X) \to H^1(N_{D/X} \otimes N^\vee_X|_D)$.

Using these notation, we can describe a criterion for arithmetic normality, which was first obtained in [28] by applying the view point of weighted projection. Here we explain an outline of another proof simplified by using the tools introduced in §3.

Theorem 4.5 (Hoobler-Speiser-Usa) *Let* $X \subseteq P = \mathbb{P}^N(\mathbb{C})$ *be a connected complex projective submanifold of dimension* $n \geq 2$. *Assume that* $q(X) = h^1(O_X) = 0$. *Then the following two conditions are equivalent.*

(4.5.1) X *is arithmetically normal.*

(4.5.2) *For any integer* m *and any generic smooth member* $D \in |O_X(m)|$, $r_D(\sigma_{II}(X)) = 0$.

Outline of Proof. Showing arithmetic normality is the essential part. We apply induction on m. Take a section $\tau_D \in H^0(X, O_X(m))$ defining the divisor D. It is enough to see that the section τ_D lifts to $H^0(P, O_P(m))$. Using $q(X) = 0$ and $r_D(\sigma_{II}(X)) = 0$ in $H^1(N^\vee_X|_D(m))$, a direct computation on the exact sequence:

$$0 = H^1(N^\vee_X) \xrightarrow{\tau_D} H^1(N^\vee_X(m)) \longrightarrow H^1(N^\vee_X|_D(m))$$

tells that the obstruction class: $\overline{\delta}^{(0)}_{LFT}(\tau_D) \in H^1(X, N^\vee_X(m))$ vanishes, which means that the section τ_D lifts to $H^0(X_{(1)}, O_{X_{(1)}}(m))$. Then apply Corollary 3.5. For precise calucu-lation on $\overline{\delta}^{(0)}_{LFT}(\tau_D)$, see [29]. ∎

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