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Alexander duality theorem and Stanley-Reisner rings

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Introduction

In this article we survey [Te] and [Fr-Te].

Alexander duality theorem plays an important role in the study on a minimal free resolution of Stanley-Reisner rings. (See [Br-He$_2$], [Te-Hi$_1$], [Te-Hi$_2$], for example.) In particular, Eagon and Reiner used Alexander dual complexes and proved the following interesting theorem:

THEOREM 0.1 ([Ea-Re, Theorem 3]). *Let k be a field. and let Δ be a simplicial complex and Δ^* its Alexander dual complex. Then $k[\Delta]$ has a linear resolution if and only if $k[\Delta^*]$ is Cohen-Macaulay.*

The above result is a starting point of this article. We generalize it in the following way.

THEOREM 0.2. *Let k be a field. Let Δ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n-2$. Then*

$$\operatorname{reg} I_\Delta - \operatorname{indeg} I_\Delta = \dim k[\Delta^*] - \operatorname{depth} k[\Delta^*].$$

Note that Theorem 0.2 corresponds to Theorem 0.1 in the case that either side of the equality is 0.

Using the Auslander-Buchsbaum formula, we have the following corollary:

COROLLARY 0.3. *Let k be a field. Let Δ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n-2$. Then*

$$\operatorname{reg} I_\Delta = \operatorname{pd} k[\Delta^*].$$

Here, we use indeg $I_\Delta = \text{embdim } k[\Delta^*] - \dim k[\Delta^*]$.

It is an interesting problem to estimate regularity of homogeneous ideals. Upper bounds of regularity are studied very actively in algebraic geometry and commutative algebra, that seems to be motivated by Eisenbud-Goto Conjecture. See, for example, [Kw] and [Mi-Vo]. Here we focus on monomial ideals. We give two kind of inequalities as an application of Alexander duality.

THEOREM 0.4 ([Ho-Tr, Theorem 1.1], [Fr-Te, Theorem 3.8]). *Let I be a monomial ideal in the polynomial $A = k[x_1, x_2, \ldots, x_n]$ over a field k. Assume* $\text{codim} A/I \geq 2$. *Then we have*

$$\text{reg } I \leq \text{arith-deg} I.$$

Theorem 0.4 was first proved by Hoa and Trung. After that, Frübis-Krüger and the author proved it independently using Alexander duality.

THEOREM 0.5 (Monomial version of Eisenbud-Goto Conjecture). *Let k be a field. and let Δ be a pure simplicial complex connected in codimension 1. Then we have*

$$\text{reg } I_\Delta \leq \deg I_\Delta - \text{codim } k[\Delta] + 1.$$

As another application, we give some upper bound for the multiplicities of homogeneous k-algebras. In [Ba-Mu] and [He-Sr], among other things, the following inequality is proved :

THEOREM 0.6 ([Ba-Mu, Proposition 3.6], [He-Sr, Corollary 3.8]). *Let k be a field and let $R = k[x_1, x_2, \ldots, x_n]/I$ be a homogeneous k-algebra of codimension h_1. Then*

$$\mathrm{e}(R) \leq \binom{\text{reg } I + h_1 - 1}{h_1}.$$

We improve it as follows:

THEOREM 0.7. *Let k be a field and let $R = k[x_1, x_2, \ldots, x_n]/I$ be a homogeneous k-algebra of codimension $h_1 \geq 2$. Then*

$$\mathrm{e}(R) \leq \binom{\text{reg } I + h_1 - 1}{h_1} - \binom{\text{reg } I - \text{indeg } I + h_1 - 1}{h_1}.$$

§1. Preliminaries

We first fix notation. Let $\mathbf{N}$(resp.$\mathbf{Z}$) denote the set of nonnegative integers (resp. integers). Let $| S |$ denote the cardinality of a set S.

We recall some notation on simplicial complexes and Stanley-Reisner rings according to [St]. We refer the reader to, e.g., [Br–He], [Hi], [Ho] and [St] for the detailed information about combinatorial and algebraic background.

A *simplicial complex* Δ on the *vertex set* $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ such that (i) $\{i\} \in \Delta$ for every $1 \leq i \leq n$ and (ii) $F \in \Delta$, $G \subset F \Rightarrow G \in \Delta$. Each element F of Δ is called a *face* of Δ. We call $F \in \Delta$ an *i-face* if $| F |= i + 1$ We set $d = \max\{| F || F \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d - 1$. We call a maximal face a *facet.* We say that Δ is *pure* if every facet has the same cardinality. When Δ is pure, we call Δ *connected in codimension 1*, if for every two facets F and G, there is a sequence of facets $F = F_0, F_1, \ldots, F_p = G$ such that $| F_i \cap F_{i+1} |=| F_i | -1$ for $0 \leq i \leq p - 1$.

Let $f_i = f_i(\Delta)$, $0 \leq i \leq d - 1$, denote the number of i-faces in Δ. We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

If F is a face of Δ, then we define a subcomplex $\mathrm{link}_\Delta F$ as follows:

$$\mathrm{link}_\Delta F = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k.

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables over a field k. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, with $\{i_1, i_2, \ldots, i_r\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k.

THEOREM 1.1 (Hochster's formula on the local cohomology modules (cf. [St, Theorem 4.1])).

$$F(H^i_{\mathfrak{m}}(k[\Delta]),\ t) = \sum_{F\in\Delta} \dim_k \tilde{H}_{i-|F|-1}(\mathrm{link}_\Delta F; k)\left(\frac{t^{-1}}{1-t^{-1}}\right)^{|F|}.$$

where $H^i_{\mathfrak{m}}(k[\Delta])$ denote the i-th local cohomology module of $k[\Delta]$ with respect to the graded maximal ideal $\mathfrak{m}$.

Let A be the polynomial ring $k[x_1, x_2, \ldots, x_n]$ for a field k. Let M be a finitely generated graded A-module and let

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0$$

be a graded minimal free resolution of M over A. We call $\beta_i(M) = \sum_{j\in\mathbf{Z}} \beta_{i,j}(M)$ the i-th Betti number of M over A. We sometimes denote $\beta_i^A(M)$ for $\beta_i(M)$ to emphasize the base ring A. We define a *Castelnuovo-Mumford regularity* reg M of M by

$$\mathrm{reg}\ M = \max\ \{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

We define an *initial degree* indeg M of M by

$$\mathrm{indeg}\ M = \min\ \{i \mid M_i \neq 0\} = \min\ \{j \mid \beta_{0,j}(M) \neq 0\}.$$

THEOREM 1.2 (Hochster's formula on the Betti numbers[Ho, Theorem 5.1]).

$$\beta_{i,j}(k[\Delta]) = \sum_{F\subset[n],\ |F|=j} \dim_k \tilde{H}_{j-i-1}(\Delta_F; k),$$

where

$$\Delta_F = \{G \in \Delta \mid G \subset F\}.$$

Finally we quote some result on Gröbner bases we use later. See [Ei, Chapter 15] for complete explanation.

Let A be the polynomial ring $k[x_1, x_2, \ldots, x_n]$ for a infinite field k. Let I be a homogeneous ideal in A. We denote Gin (I) to be a *generic initial ideal* of I with respect to the reverse lexicographic order. It is well known that e$(A/\mathrm{Gin}\ (I))$ = e(A/I).

Further we have:

THEOREM 1.3 ([Ba-St]).
(1)depth $A/\mathrm{Gin}\ (I) = \mathrm{depth}\ A/I$.
(2)reg Gin $(I) = \mathrm{reg}\ I$.

§2. Alexander duality and some generalization of the Eagon-Reiner theorem

First we recall the definition of Alexander dual complexes.

Definition (cf. [Ea-Re]). For a simplicial complex Δ on the vertex set $[n]$, we define an *Alexander dual complex* Δ^* as follows:

$$\Delta^* = \{F \subset [n] : [n] \setminus F \notin \Delta\}.$$

If $\dim \Delta \leq n-3$, then Δ^* is also a simplicial complex on the vertex set $[n]$.

In the rest of the paper we always assume $\dim k[\Delta] = d$ and $\dim k[\Delta^*] = d^*$ for a fixed field k.

Now we give some generalization of the Eagon-Reiner theorem.

THEOREM 2.1. *Let Δ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n-2$. Then*

$$\text{reg } I_\Delta - \text{indeg } I_\Delta = \dim k[\Delta^*] - \text{ depth } k[\Delta^*].$$

Proof. Put depth $k[\Delta^*] = p^*$. By Hochster's formula on the local cohomology modules, we have

$$F(H^l_{\mathfrak{m}}(k[\Delta^*]),\ t) = \sum_{F\in\Delta^*} \dim_k \tilde{H}_{l-|F|-1}(\text{link}_{\Delta^*}F; k)\left(\frac{t^{-1}}{1-t^{-1}}\right)^{|F|}.$$

Hence if $l < p^*$, then $\tilde{H}_{l-|F|-1}(\text{link}_{\Delta^*}F; k) = (0)$ for all $F \in \Delta^*$. By the proof in [Ea-Re, Proposition 1], we have $\tilde{H}_{n-l-2}(\Delta_F; k) = (0)$ for all $F \subset [n]$. By Hochster's formula on the Betti numbers this means that $\beta_{i,i+n-l-1}(k[\Delta]) = 0$ for $i \geq 1$. Hence

$$\beta_{i,i+n}(I_\Delta) = \beta_{i,i+n-1}(I_\Delta) = \cdots = \beta_{i,i+n-p^*+1}(I_\Delta) = 0$$

for $i \geq 0$. Similarly, since $\tilde{H}_{n-p^*-2}(\Delta_{[n]\setminus F}; k) \cong \tilde{H}_{p^*-|F|-1}(\text{link}_{\Delta^*}F; k) \neq (0)$ for some $F \in \Delta$, we have $\beta_{i,i+n-p^*}(I_\Delta) \neq 0$ for some $i \geq 0$. Hence reg $I_\Delta = n - p^*$. By the definition of the Alexander dual complex we have indeg $I_\Delta = n - d^*$. Therefore, we have reg $I_\Delta -$ indeg $I_\Delta = d^* - p^*$. Q.E.D.

§3. On upper bounds for regularity of monomial ideals

In this section we give some upper bounds for regularity of monomial ideals.

THEOREM 3.1 ([Fr-Te, Theorem 3.1]). *Let k be a field. and let Δ be a simplicial complex. Assume* $\operatorname{codim} k[\Delta] \geq 2$. *Then we have*

$$\operatorname{reg} I_\Delta \leq \text{arith-deg} I_\Delta .$$

See, for example, [Ba-Mu] for the definition of arithmetic degree of an ideal I. Here we just remark that arithmemic degree arith-degI_Δ of a square-free monomial ideal I_Δ is the number of the facets in Δ.

Proof. Tayor resolution guarantees pd $k[\Delta^*] \leq \beta_0(I_{\Delta^*})$. Then we have

$$\operatorname{reg} I_\Delta = \operatorname{pd} k[\Delta^*] \leq \beta_0(I_{\Delta^*}) = \text{arith-deg} I_\Delta$$

by Corollary 0.3. Q.E.D.

By combinatorial argument on standard pairs, which are introduced by [St-Tr-Vo], we can show:

THEOREM 3.2 ([Fr-Te, Corollary 3.6]). *Let I be a monomial ideal of a polynomial ring. Put I^{pol} be the polarization of I. Then we have*

$$\operatorname{reg} I = \operatorname{reg} I^{\mathrm{pol}}.$$

See, for example, [St-Vo] for the definition and basic properties of the polarization of monomial ideals.

Combining Theorem 3.1 and 3.2, we have:

THEOREM 3.3 ([Ho-Tr, Theorem 1.1], [Fr-Te, Theorem 3.8]). *Let I be a monomial ideal in the polynomial $A = k[x_1, x_2, \ldots, x_n]$ over a field k. Assume $\operatorname{codim} A/I \geq 2$. Then we have*

$$\operatorname{reg} I \leq \text{arith-deg} I.$$

Next,we will prove a certain conjecture of Eisenbud (see [Ei-Po].) , which is a monomial version of Eisenbud-Goto Conjecture (see [Ei-Go]).

THEOREM 3.4. Let k be a field and let Δ be a pure simplicial complex connected in codimension 1. Then we have

$$\mathrm{reg} I_\Delta \leq \deg k[\Delta] - \mathrm{codim} k[\Delta] + 1.$$

We give a sketch of a proof, which is simplified by suggestions of Eisenbud.

Sketch of proof. Let V be the vertex set of Δ. Put $\sharp(V) = n$ and $\dim k[\Delta] = d$. We prove the theorem by induction on the number f_{d-1} of facets in Δ.

First if codim $k[\Delta] \leq 1$, then $k[\Delta]$ is a hypersurface. In this case the theorem is clear.

Suppose codim $k[\Delta] \geq 2$ and $f_{d-1} \geq 2$. Then there exists a facet $\sigma \in \Delta$ such that

$$\Delta' := \Delta \setminus \{\tau \in \Delta \mid \text{For any facet } \rho(\neq \sigma) \in \Delta;\ \ \tau \not\subset \rho\}$$

is pure and connected in codimension 1. Denote by V' the vertex set of Δ' and by f'_{d-1} the number of facets in Δ'. There are two cases.

Case(i) $V \neq V'$. Put $V \setminus V' = \{v\}$. For $W \subset V$ with $v \notin W$ we have $\Delta_W \cong \Delta'_W$. On the other hand, for $W \subset V$ with $v \in W$, We have $\tilde{H}_i(\Delta_W; k) \cong \tilde{H}_i(\Delta'_{W\setminus\{v\}}; k)$ for $i \geq 1$. Since

$$\mathrm{reg}\ I_\Delta = \max\{i+2 \mid \tilde{H}_i(\Delta_W; k) \neq 0 \text{ for some } W \subset V\},$$

we have

$$\begin{aligned} \mathrm{reg} I_\Delta &= \mathrm{reg} I_{\Delta'} \\ &\leq f'_{d-1} - (n-1-d) + 1 \\ &= f_{d-1} - (n-d) + 1. \end{aligned}$$

Case(ii) $V = V'$. We have reg I_Δ = pd $k[\Delta^*]$ by Corollary 0.3. If we prove pd $k[\Delta^*] \leq$ pd $k[(\Delta')^*] + 1$, we have

$$\begin{aligned} \mathrm{reg}\ I_\Delta &\leq \mathrm{reg}\ I_{\Delta'} + 1 \\ &\leq f'_{d-1} - (n-d) + 2 \\ &= f_{d-1} - (n-d) + 1. \end{aligned}$$

Then we have only to prove

$$\mathrm{pd}\ k[\Delta^*] \leq \mathrm{pd}\ k[(\Delta')^*] + 1.$$

Put $k[\Delta^*] = k[(\Delta')^*]/(m)$, where $m = \Pi_{x_i \in V \setminus \sigma} x_i$. If we show that

$$\mathrm{pd}\ k[(\Delta')^*] \geq \mathrm{pd}\ (I_{(\Delta')^*} + (m))/I_{(\Delta')^*},$$

then the mapping cone guarantees that

$$\mathrm{pd}\ k[\Delta^*] \leq \mathrm{pd}\ k[(\Delta')^*] + 1$$

by [E, Exercise A.3.30]. But now we have

$$\begin{aligned}(I_{(\Delta')^*} + (m))/I_{(\Delta')^*} &\cong (m)/((m) \cap I_{(\Delta')^*}) \\ &\cong (m)/((m) \cap (m_1, \ldots, m_t)) \\ &\cong (m)/(\mathrm{lcm}(m, m_1) \ldots, \mathrm{lcm}(m, m_t)) \\ &\cong A/(m'_1, \ldots, m'_t) \otimes_A (m),\end{aligned}$$

where $I_{(\Delta')^*} = (m_1, \ldots, m_t)$, $m'_i = \frac{\mathrm{lcm}(m, m_i)}{m}$, and $A = k[x_i \mid x_i \in V]$. Hence, we have only to show

$$\mathrm{pd}\ k[(\Delta')^*] \geq \mathrm{pd}\ A/(m'_1, \ldots, m'_t).$$

Now we have $k[(\Delta')^*]_m \cong A_m/(m'_1, \ldots, m'_t)A_m$. Hence we have

$$\mathrm{pd}\ k[(\Delta')^*] \geq \mathrm{pd}\ k[(\Delta')^*]_m = \mathrm{pd}\ A_m/(m'_1, \ldots, m'_t)A_m = \mathrm{pd}\ A/(m'_1, \ldots, m'_t).$$

Q.E.D.

§4. On upper bounds for multiplicities

In this section we give some upper bound for the multiplicities of homogeneous k-algebras.

First we prove the following lemma:

LEMMA 4.1.

$$\mathrm{e}(k[\Delta]) = \beta_{1,h_1}(k[\Delta^*]).$$

Proof. We have

$$h_0(\Delta) + h_1(\Delta)(1-t) + \cdots + h_d(\Delta)(1-t)^d \tag{1}$$

$$= \frac{1 - (1-t)^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)t + \cdots + h_{d^*}(\Delta^*)t^{d^*})}{t^{n-d}}. \tag{2}$$

Since indeg $I_{\Delta^*} = n - d = h_1$, we have

$$\begin{aligned}
& \beta_{1,n-d}(k[\Delta^*]) \\
= & \text{ (the coefficient of } t^{n-d} \text{ in } -(1-t)^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)t + \cdots + h_{d^*}(\Delta^*)t^{d^*})) \\
= & \text{ (the coefficient of } t^{n-d} \text{ in the numerator in (2))} \\
= & \lim_{t\to 0}(h_0(\Delta) + h_1(\Delta)(1-t) + \cdots + h_d(\Delta)(1-t)^d) \\
= & \text{ e}(k[\Delta]).
\end{aligned}$$

Q.E.D.

THEOREM 4.2. *Let $R = A/I$ be a homogeneous k-algebra of codimension $h_1 \geq 2$. Then*

$$\mathrm{e}(R) \leq \binom{\mathrm{reg}\ I + h_1 - 1}{h_1} - \binom{\mathrm{reg}\ I - \mathrm{indeg}\ I + h_1 - 1}{h_1}.$$

Proof. We may assume $|\ k\ | = \infty$. By Theorem 1.3, we have reg Gin$(I) =$ reg I and $h(A/I) = h(A/\mathrm{Gin}(I))$. Considering the polarization, we obtain a Stanley-Reisner ring $k[\Delta] = B/I_\Delta$ with $\mathrm{e}(A/I) = \mathrm{e}(k[\Delta])$ and reg $I =$ reg I_Δ. Put $p^* = \mathrm{depth}\ k[\Delta^*]$. By Theorem 2.1, we have $d^* - p^* = \mathrm{reg}\ I - (n - d^*)$, where $n = \mathrm{embdim}\ k[\Delta^*]$. Hence reg $I = n - p^*$.

Let $y_1, y_2, \ldots, y_{p^*}$ be a regular sequence in $k[\Delta^*]_1$, and let $z_1, z_2, \ldots, z_{d^*-p^*} \in (k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*}))_1$ be a system of parameters of $k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*})$. We have $k[z_1, z_2, \ldots, z_{d^*-p^*}] \subset k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*})$. Since $k[z_1, z_2, \ldots, z_{d^*-p^*}]$ is isomorphic to the polynomial ring with $d^* - p^*$ variables, we have $\dim_k(k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*}))_{h_1} \geq \binom{d^*-p^*+h_1-1}{h_1}$. By Lemma 4.1, we have

$$\begin{aligned}
\mathrm{e}(k[\Delta]) &= \beta_{1,h_1}(k[\Delta^*]) \\
&= \beta_{1,h_1}^{B/(y_1,y_2,\ldots,y_{p^*})}(k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*})) \\
&= \dim_k(B/(y_1, y_2, \ldots, y_{p^*}))_{h_1} - \dim_k(k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*}))_{h_1} \\
&\leq \binom{n - p^* + h_1 - 1}{h_1} - \binom{d^* - p^* + h_1 - 1}{h_1}.
\end{aligned}$$

Q.E.D.

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