数理解析研究所講究録
1079 巻 1999 年 119-130

On games in a cooperative function form[1]

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1 Introduction

Using non-cooperative games in extensive form, n-person multistage multichoice cooperative (MMC) games with the perfect information, the finite length, and the terminal payoff function, were defined in [4] and [5]. In such games any player may proceed cooperative activity during not the whole game party but just on some set of stages which are continuous by order. We recall the basic ideas of so called partial cooperation proposed in [4] and [5], which are referred for the more details.

Let $N=\{1,2,\ldots,n\}$ be the set of players. Denote the game tree with the origin x_0 by $K(x_0)$. Suppose that the structure of $K(x_0)$ satisfies the following conditions:

1) each path has equal length and includes $(T+1)n+1$ nodes, where T is a finite natural number;

2) all players make moves according with their index order;

3) when a player makes the decision on behavior, he has perfect information;

4) within one stage every player makes by one move.

The restrictions laid on $K(x_0)$ enable to introduce the following game rules. Before the game starts each player $i\in N$ must, independently from the other players, point out t^i in the set $\{0,1,\ldots,T,T+1\}$. Taking $t^i\in\{0,\ldots,T\}$ means that player i is ready to cooperate with anyone since the stage t^i. However, if the player chooses $T+1$, he is going to keep on a non-cooperative behavior during the game. After each player $i\in N$ determined himself about t^i, the combination $(t^1,\ldots,t^i,\ldots,t^n)$ of made choices is announced and becomes commonly known. Players are permitted to alter the declared options. The given preferences exactly describe behavior of players in the game. Since the initial stage until the stage t^i player $i\in N$ keeps on the individually rational behavior and doesn't collaborate with any other player. Nevertheless, on every stage $t=t^i,\ldots,T$ he has to participate in the coalition of all players who are ready to cooperate on the stage t too. Within such behavior is used, the coalition is considered as the set of the players that have whenever cooperated during the game party, and presented by vector $s=(s_1,\ldots,s_i,\ldots,s_n)$, where components are defined by $s_i=T+1-t^i$. Suppose that in according with a combination $(t^1,\ldots,t^i,\ldots,t^n)$, a path $\{x_0,\ldots,x_T\}$ is realized. Then the sum of the terminal payoffs over all players $i\in N$ with $s_i>0$ is admitted as the payoff of the coalition s.

Note that it is no matter which path is going to be played during the game, if $t^i\neq T+1$, player i will cooperate since the stage t^i in any case. Such restriction seems too strong. In this paper we try to weaken the above mentioned conditions. As we will show, it leads to a quite different concept of partial cooperation.

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[1] We thank Prof. L. Petrosjan of St.Peterburg University for his comments and suggestions on this work.
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2 The model.

Let Γ be a finite n-person non-cooperative game in extensive form with perfect information. Denote the set of players by $N = \{1, \ldots, n\}$. Let $K(x_0)$ be the game tree with the origin x_0. According with the definition of a game in extensive form, on $K(x_0)$ there exists a partition $P_0, P_1, \ldots, P_n, P_{n+1}$ of the set of game tree nodes, where $P_0 = \emptyset$ is the set of chance points, $P_1, \ldots, P_n$ are the sets of decision points of players, and P_{n+1} is the set of endpoints. The payoffs of players are specified by terminal real-valued functions $h_i: P_{n+1} \to R^1_+$, $i \in N$.

Let us call a behavior such that a player can as cooperate as play individually, a partial cooperative one. Transform Γ assuming that players may cooperate each other within some conditions. We denote the changed game Γ by $G(x_0)$. Further, if no confusion can arise, under game one means $G(x_0)$. In this section the partial cooperation rules are described.

Demand that before the game starts each player $i \in N$ must decide if he cooperates or not. If the player doesn't want to collaborate with anybody he plays whole game alone. In case the player is going to cooperate, he has to choose a combination K_i of non-intersecting subtrees $\{K(x^1), \ldots, K(x^q)\}$, with their origins $x^1, \ldots, x^q$ being in P_i. The choices have to be independent from the other game participators, but when every player made his options all decisions are announced. The combination K_i, $i \in N$, is considered as the cooperation region of player i, i.e., player i pledges himself to proceed his cooperative behavior on the decision points in $K_i \cap P_i$. On the nodes in $P_i \setminus K_i$, player i must use his individual behavior. It is important that players are prohibited to change their choices during the game. We formalize the cooperative regions of players by means of functions

$$f_i: P_i \to \{0, 1\}, \quad i \in N. \tag{2.1}$$

Definition. f_i, $i \in N$, is called *a cooperative function* of player i, if for an arbitrary taken path $\{x_0, \ldots, x', x'', \ldots, \bar{x}\}$, where $x' \in P_i$ and $\bar{x}$ is a terminal node, from $f_i(x') = 1$ it follows that $f_i(y) = 1$ for each $y \in P_i \cap \{x'', \ldots, \bar{x}\}$.

We shall say that player i keeps cooperative behavior on a node $x \in P_i$ if and only if $f_i(x) = 1$. To interpret the game process correctly, we should explain what we mean under the cooperative and individual behaviors of players, when the given game rules are used. At the same time it will be shown that a combination $f = (f_1, f_2, \ldots, f_n)$ of cooperative functions defines a coalition structure on every node of the game tree $K(x_0)$.

The cooperative behavior. Suppose that f has been defined and after several moves the game party came to a decision point $x \in P_i$ of player i. Assume that the chosen cooperative function satisfies $f_i(x) = 1$, i.e., player i cooperates on x. Let's determine the coalition whose interests are supported by player i. Consider the set

$$S^1_f(x) = \{j \in N \,|\, f_j(y) = 1, \forall y \in P_j \cap \{x_0, \ldots, x\}\}. \tag{2.2}$$

$S^1_f(x)$ includes the players who has cooperated before player i. According with the definition of the cooperative function, players in $S^1_f(x)$ will continue to cooperate on every their decision point on the rest part $K(x)$ of the game. Notice that player i belongs to $S^1_f(x)$.

There is another group of players with whom player i should coordinate his decision on x, and it is composed of the players who hasn't made move on the path $\{x_0, \ldots, x\}$ yet, but will cooperate after player i. Let such players be united into the set $S^2_f(x)$.

Definition A subtree $K(x)$ rising at x is *the trustiness region* (TR) of player $j \in N$ if for every $y \in P_j \cap K(x)$, the cooperative function $f(y) = 1$.

Hence,

$$S_f^2(x) = \{j \in N \setminus S_f^1(x) | K(x) \text{ is TR of player } j\}. \tag{2.3}$$

Saying that player $i \in N$ proceeds the cooperative behavior on a node $x \in K(x_0)$, we mean that on x player i acts in the interests of the coalition

$$S_f(x) = S_f^1(x) \cup S_f^2(x). \tag{2.4}$$

The rest players in $N \setminus S_f(x)$ are considered as individual ones on x. Since $S_f(x)$ is defined by the cooperative function f, the whole coalition structure $S_f(x)$, $\{j_1\}$, $\{j_2\}, \ldots, \{j_{|N \setminus S_f(x)|}\}$ is specified by f as well.

The individual behavior. Now suppose that $f_i(x) = 0$. Let us determine the individual behavior of players $j_1, j_2, \ldots, j_{|N \setminus S_f(x)|}$. Notice that, once players in $S_f(x)$ are organized in a coalition, they can be replaced by the united player-coalition $S_f(x)$. Thus, actually, there stays just $|N \setminus S_f(x)|$ of the game participators on the decision point x. Let $\Gamma(x)$ be a subgame of Γ starting at x. Consider $\Gamma_f(x)$ which is $\Gamma(x)$ with the changed set of players

$$N_f(x) = \{S_f(x), j_1, \ldots, j_k, \ldots, j_{|N \setminus S_f(x)|}\}. \tag{2.5}$$

Since the coalition structure consisting of $S_f(x)$, $\{j_1\}$, $\{j_2\}, \ldots, \{j_{|N \setminus S_f(x)|}\}$ is valid at least one move, starting from x we can say that the player making decision on x acts the same manner as in $\Gamma_f(x)$. Let $\Psi_i^f(x)$, $i \in N_f(x)$, be the sets of players' strategies. Denote by $\Psi^f(x) = \prod_{i \in N_f(x)} \Psi_i^f(x)$, the set of all situations in $\Gamma_f(x)$. The payoff functions

$$b_i^f \colon \Psi^f(x) \to R_+^1, \quad i \in N_f(x), \tag{2.6}$$

of the game $\Gamma_f(x)$ are defined by means of the payoff functions of the game Γ, i.e., if a path $\{x, \ldots, \bar{x}\}$, $\bar{x} \in P_{n+1}$, is corresponded to a situation $\psi^f(x) \in \Psi^f(x)$, then

$$b_i^f(\psi^f(x)) = h_i(\bar{x}), \quad i \in N \setminus \{S_f(x)\}, \tag{2.7}$$

and

$$b_{S_f(x)}^f(\psi^f(x)) = \sum_{j \in S_f(\bar{x})} h_j(\bar{x}). \tag{2.8}$$

Assume $\overline{\psi}^f(x) = (\overline{\psi}_{j_1}^f(x), \ldots, \overline{\psi}_{j_{N \setminus S_f(x)}}^f(x), \overline{\psi}_{S_f(x)}^f(x))$ to be the absolute Nash equilibrium situation in $\Gamma_f(x)$.

Saying that players $j_1, j_2, \ldots, j_{|N \setminus S_f(x)|}$ are the individual ones on x in $G(x_0)$, we mean that on every own decision point $y \in K(x) \cap P_{j_k}$ on the subtree $K(x)$, player j_k, $k = 1, \ldots, |N \setminus S_f(x)|$, acts according with and restricted to avoid the absolute Nash equilibrium situation $\overline{\psi}^f(x)$.

Example 1. Consider a partial cooperative game $G(x_0)$ with the game tree illustrated on Figure 1. The set N is composed of three players: $N = \{1, 2, 3\}$. The decision points of player 1 are represented by circles, player 2's by triangles and those of player 3 by blocks, respectively. Players' payoffs are written in the endpoints. Assume that before the game there was chosen a combination $f = (f_1, f_2, f_3)$ of the following cooperative functions: $f_1(x_0) = 0$, $f_1(x_{22}) = 0$, $f_2(x_{11}) = 1$, $f_2(x_{23}) = 0$, $f_3(x_{21}) = 1$, $f_3(x_{12}) = 0$.

Let us find the coalition structure on the node $x_{11} \in P_2$. Once player 1 doesn't cooperate on x_0, $S_f^1(x_{11})$ includes only player 2. Since $f_3(x_{21}) = 1$ and $f_1(x_{22}) = 0$, the subtree $K(x_{11})$

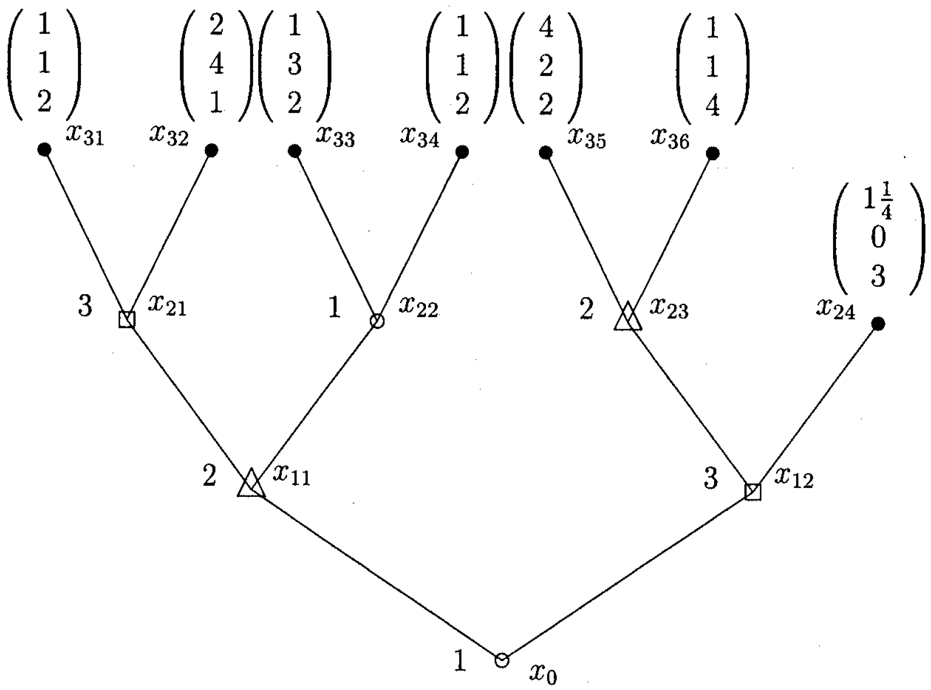

Figure 1: The game tree.

is the trustiness region of players 3 and 2 yet. Thus, $S_f(x_{11}) = \{3\}$. Hence, $S_f(x_{11}) = \{2,3\}$ and the coalition structure on x_{11} is $\{2,3\}$, $\{1\}$.

Remark 1. It is not excluded that a player plays individually even though he is on the region of his cooperative behavior.

For instance, take the combination f of the cooperative functions used in Example 1 and substitute the choice of player 1 as follows: $f_1(x_0) = 1$, $f_1(x_{22}) = 1$. Consider the set $S_f(x_0)$. Players 2 and 3 are ready to cooperate on every their personal nodes on the subtree $K(x_{11})$. Since $f_2(x_{23}) = f_3(x_{12}) = 0$, the tree $K(x_0)$ is not the trustiness region for players 2 and 3. Therefore, we obtain $S_f(x_0) = \{1\}$. According with the made interpretation of the cooperative behavior, player 1 chooses on x_0 such alternative, for $S_f(x_0)$ to get maximal payoff. However, $S_f(x_0)$ is only player 1. Thus, we can say that, he acts on x_0 as an individual player.

Remark 2. For arbitrary taken decision point x and its immediate predecessor y, let coalition $S_f(y)$ be not empty. Then, $S_f(x)$ is also not empty and, moreover, we have $S_f(y) \subset S_f(x)$.

3 The algorithm of the path construction.

In this section we investigate whether a combination f of the cooperative functions f_i, $i \in N$, defines a trajectory of the game development. Such relation between f and a game path enables to estimate each f_i, $i \in N$.

Let F_i be the set of all cooperative functions of player $i \in N$. Denote the set of all compositions of players' cooperative functions by $F = \{f = (f_1, \ldots, f_n) | f_i \in F_i, i \in N\}$. As shown before, if $f \in F$ is given, then a coalition structure on every node of the game tree can be obtained. Since it is known whose interests are prevailed on a considered decision point, we are able to find the corresponding path for all $f \in F$.

The path is determined by means of backward construction, moving from the final nodes toward the initial one. Our procedure is similar to those used in the scheme of the Nash equilibrium construction. The difference between the methods is stated in the following. Let $K(x)$ belong to a cooperation region of player i. Then, on the endpoints of $K(x)$ instead of the payoff of player i we have the payoff of a coalition which includes player i. By the Nash scheme the decisions of player i maximizing the coalition payoff can be easily determined with respect to $K(x)$. However, since the player i's payoff is not picked out from the coalition payoff, there occur difficulties on the decision points of player i between x and the root x_0, where player i plays individually. If the share of player i in the coalition payoff is known, then applying the Nash scheme again, we can find the strategy of player i on his personal nodes of the path $\{x_0, \dots, x\}$. Therefore, the definition of players' payoffs corresponding to nodes where the individual behavior is replaced by the cooperative one is the main problem considered in the algorithm.

During the explanation we will often use the following notations. Assume that x is an arbitrary node. Let the set of immediate successors of x be $Z(x)$. Denote the decision maker on x by $i(x) \in N$. We say that the decision of player $i(x)$ on x leads out at the node $\bar{x} \in Z(x)$. Finally, we propose that a combination $f = (f_1, \dots, f_n)$ of cooperative functions determines players' preferences by the rule c_f: if x is a decision point of player i, then

$$c_f(x) = \begin{cases} 1, & \text{if } f_i(x) = 1 \\ 0, & \text{if } f_i(x) = 0. \end{cases} \tag{3.1}$$

Now suppose that one of the longest path of the game tree goes through T decision points. Introduce a partition of all nodes on $T+1$ sets $X_0, X_1, \dots, X_t, \dots, X_T = \{x_0\}$, where X_t is composed of nodes which are reachable from x_0 after $T-t$ sequential moves. Denote decision points belonging to X_t by x_t, $t = 1, \dots, T$.

Running ahead, we remark that the payoffs considered by players on their decision points may not coincide with the terminal payoff functions h_i, $i \in N$. To trace the alteration of the payoff system, we will write out the terminal payoffs which are taken in account by players on nodes X_t, $t = 1, \dots, T$, by means of functions r_i^t, $i \in N$.

Assume that players $i \in N$ have arranged to proceed their behaviors according with $f \in F$. Let us find the path of the game related to the taken f.

The initial stage. Consider the set P_{n+1} of endpoints. Since no player makes move on P_{n+1}, the coalition structure on $x \in P_{n+1}$ and that on its immediate predecessor x_1, $Z(x_1) \ni x$, are the same. On the node x_1 the given f specifies coalitions $S_f(x_1)$, $\{j_1\}, \dots, \{j_{|N \setminus S_f(x_1)|}\}$. We compound the terminal payoffs $h_1(x), \dots, h_n(x)$ on x in such way the new payment structure to be in correspondence with the coalition structure on x_1. Say that, the coalition $S_f(x_1)$ gets

$$\sum_{i \in S_f(x_1)} h_i(x), \tag{3.2}$$

and an individual player j_k, $k = 1, \dots, |N \setminus S_f(x_1)|$, obtains $h_{j_k}(x)$ on the node x.

Stage 1. Shift down from the endpoints $Z(x_1)$, $x_1 \in X_1$, to their predecessors. Consider an arbitrary taken x_1. If $c_f(x_1) = 1$, player $i(x_1)$ cooperates on x_1, from which it follows that $i(x_1)$ maximizes the payoff of the coalition $S_f(x_1)$. Hence, the endpoint $\bar{x}_1 \in Z(x_1)$ has to satisfy

$$\max_{x \in Z(x_1)} \sum_{i \in S_f(x_1)} h_{i(x_1)}(x) = \sum_{i \in S_f(x_1)} h_{i(x_1)}(\bar{x}_1). \tag{3.3}$$

In case $c_f(x_1) = 0$, player $i(x_1)$ pursuits his own benefit and the node $\bar{x}_1$ is determined by

$$\max_{x \in Z(x_1)} h_{i(x_1)}(x) = h_{i(x_1)}(\bar{x}_1). \tag{3.4}$$

In the same way, we can construct trajectories rising at the rest nodes in X_1. Thus, on each subtree $K(x_1)$, $x_1 \in X_1$, there is stayed just one by one endpoint $\bar{x}_1$ that is suspected to be the final node of the constructed path of the game. Therefore, instead of considering the terminal payoff function h_i, $i \in N$, on P_{n+1}, we may deal with payoff functions $r_i^1 \colon X_1 \to R_+^1$, $i \in N$, on X_1 such that

$$r_i^1(x_1) = \begin{cases} h_i(\bar{x}_1), & \text{if } x_1 \notin P_{n+1}; \\ h_i(x_1), & \text{if } x_1 \in P_{n+1}. \end{cases} \tag{3.5}$$

Stage 2. Continue moving toward the tree root. Find the players' decisions on the nodes in X_2. As far as we specified functions r_i^1, $i \in N$, it seems that player $i(x_2)$, $x_2 \in X_2$ knows an obtained payoff for each his decision on x_2. Nevertheless, it may occur that for some set $Y(x_2)$ of nodes in $Z(x_2)$ either the payoff of the player $i(x_2)$ when $c_f(x_2) = 0$ or the payoff of the coalition $S_f(x_2)$ when $c_f(x_2) = 1$ is not determined. For example, assume that player $i(x_2)$ makes move on $K(x_2)$ twice, i.e., there exists a node $y_1 \in Z(x_2)$ such that $i(y_1)$ and $i(x_2)$ are the same player. Let $c_f(x_2) = 0$ and $c_f(y_1) = 1$. Then, whereas player $i(x_2)$ belongs to a coalition $S_f(y_1)$ on the whole subtree $K(y_1)$ he plays individually on the decision point x_2. Since the payoff of player $i(x_2)$ is not identified in the payoff $\sum_{i \in S_f(y_1)} r_i^1(\bar{y}_1)$ of the coalition $S_f(y_1)$, his payoff isn't known on $y_1 \in Z(x_2)$.

Generally speaking, the lack of information occurs when coalition structure is changed, and this alteration affects the current decision maker, i.e., there exists a node $y_1 \in Z(x_2)$ such that individually playing player $i(x_2)$ enters into multi-player coalition $S_f(y_1)$ on y_1, or coalition $S_f(x_2)$ which includes the decision maker $i(x_2)$ increases on y_1. For each node $x_2 \in X_2$ we deal with two main cases.

1) Let $Y(x_2) = \emptyset$. First, assume that $c_f(x_2) = 0$. It means that the player $i(x_2)$ doesn't cooperate on x_2 and maximizes his own payoff. Then, the path on the subtree $K(x_2)$ has to go through a node $\bar{x}_2$ specified by

$$\max_{x \in Z(x_2)} r^1_{i(x_2)}(x) = r^1_{i(x_2)}(\bar{x}_2). \tag{3.6}$$

Now assume that $c_f(x_2) = 1$. By the definition of the cooperative function, the coalition $S_f(x_2)$ may include players no grater than coalition $S_f(x_1)$ for each $x_1 \in Z(x_2)$. Therefore, since $Y(x_2) = \emptyset$, the coalitions $S_f(x_2)$ and $S_f(x_1)$ coincide. Thus, player $i(x_2)$ chooses on x_2 a branch leading to such $\bar{x}_2$ that

$$\max_{x \in Z(x_2)} \sum_{i \in S_f(x_2)} r_i^1(x) = \sum_{i \in S_f(x_2)} r_i^1(\bar{x}_2). \tag{3.7}$$

2) Now suppose that $Y(x_2) \neq \emptyset$. As we discussed above, when $c_f(x_2) = 0$ we don't know the payoff of the player $i(x_2)$ on $Y(x_2)$. On the other hand, in the case of $c_f(x_2) = 1$, we have $S_f(x_1) \setminus S_f(x_2) \neq \emptyset$. Once $S_f(x_2) \subset S_f(x_1)$, on $Y(x_2)$ the payoff of the coalition $S_f(x_2)$ is included into the payoff of the coalition $S_f(x_1)$ and thus, not defined too.

To construct path on $K(x_2)$, it is necessary for an imputation of payoff of coalition $S_f(y_1)$ to be determined for each $y_1 \in Y(x_2)$. We do it by considering a cooperative positional game

$G_f(y_1, S_f(y_1))$ on the subtree $K(y_1)$ with the set of players $S_f(y_1)$ and the characteristic function $v_f(y_1, S)$, $S \subset S_f(y_1)$, for each $y_1 \in Y(x_2)$. The explanation of the cooperative function construction will be provided later. Now, we just admit that the characteristic function can be constructed. For the sake of determination let us use the Shapley value

$$Sh^f(y_1) = (Sh^f_{k_1}(y_1), \ldots, Sh^f_{k_{|S_{y_1}|}}(y_1)) \tag{3.8}$$

as an optimal imputation of the payoff of the coalition $S_f(y_1)$. We shall say that if the choice of player $i(x_2)$ on x_2 is a branch leading to $y_1 \in Y(x_2)$, then after the game reaches the endpoint $\bar{y}_1$, the payoff of player $i(x_2)$ is to be determined by the Shapley value $Sh^f(y_1)$ and equal to $Sh^f_{i(x_2)}(y_1)$. Then, we have to correct the payoff functions r^1_i, $i \in N$. Let us describe the new payment system by means of functions $\bar{r}^1_i\colon X_1 \to R^1_+$, $i \in N$, where for $x_1 \in Z(x_2)$

$$\bar{r}^1_i(x_1) = \begin{cases} Sh^f_i(x_1), & \text{if } x_1 \in Y(x_2) \text{ and } i \in S_f(x_1); \\ r^1_i(x_1), & \text{otherwise.} \end{cases} \tag{3.9}$$

Suppose that $c_f(x_2) = 0$. Then for player $i(x_2)$ it is optimal to realize a path which goes through the decision point $\bar{x}_2 \in Z(x_2)$ satisfying

$$\max_{x \in Z(x_2)} \bar{r}^1_{i(x_2)}(x) = \bar{r}^1_{i(x_2)}(\bar{x}_2). \tag{3.10}$$

Now let $c_f(x_2) = 1$. Since player $i(x_2)$ cooperates on x_2 with coalition $S_f(x_2)$, he maximizes the coalition payoff and chooses $\bar{x}_2$ by

$$\max_{x \in Z(x_2)} \sum_{i \in S_f(x_2)} \bar{r}^1_{i(x_2)}(x) = \sum_{i \in S_f(x_2)} \bar{r}^1_{i(x_2)}(\bar{x}_2). \tag{3.11}$$

In the remainder of the second stage explanation, we remark that since for each $x_2 \in X_2$ the decision of player $i(x_2)$ on x_2 and the decision of each player $i(x_1)$ on $x_1 \in Z(x_2)$ are determined, the path which is realized on the subtree $K(x_2)$ when the game reaches x_2 is found. Hence, to construct the path on a subtree $K(x_3)$, $x_3 \in X_3$, we have to consider just the decisions of players $i(x_3)$, $x_3 \in X_3$. When $Y(x_2) \neq \emptyset$, the payoffs of players are different from those in the case of $Y(x_2) = \emptyset$. Let us define the payoffs on X_2 by functions $r^2_i\colon X_2 \to R^1_+$, $i \in N$, such that for $x_2 \in X_2$ and $i \in N$

$$r^2_i(x_2) = \begin{cases} r^1_i(\bar{x}_2), & \text{if } Y(x_2) = \emptyset; \\ \bar{r}^1_i(\bar{x}_2), & \text{if } Y(x_2) \neq \emptyset; \\ h_i(x_2), & \text{if } x_2 \in P_{n+1}. \end{cases} \tag{3.12}$$

Since the procedures on the further stages are the same, omitting explanation of every stage we deal with a stage t as an example of the general approach. So, suppose that we have reached a set of nodes X_t by continuing the moving on the game tree toward the origin x_0. Let $r^{t-1}_i\colon X_{t-1} \to R^1_+$, $i \in N$, be payoff functions obtained on the stage $t-1$ for X_{t-1}.

Stage t. We don't deal with the endpoints belonging to $X_t \cap P_{n+1}$, because they have been considered on the initial stage yet. Let us find the decisions of players on the set of non-terminal nodes $X_t \setminus P_{n+1}$. First, we discuss the case when determination of a new payment structure is not needed.

1) Assume that $Y(x_t) = \emptyset$ for all $x_t \in X_t \setminus P_{n+1}$. In this case, the functions r^{t-1}_i, $i \in N$, specify the payoff obtained at the end of the game for each player $i(x_t)$, $x_t \in X_t \setminus P_{n+1}$,

i.e., if the decision of player $i(x_t)$ leads out at a node $\bar{x}_t \in Z(x_t)$, then at the end of the game the coalition $S_f(x_t)$ will get $\sum_{i \in S_f(x_t)} r_i^{t-1}(\bar{x}_t)$, and the payoffs of individual players j_k, $k = 1, \ldots, |S_f(x_t)|$, to be $r_{j_k}^{t-1}(\bar{x}_t)$, respectively. Therefore, we can easily determine the nodes $\bar{x}_t$, where $\bar{x}_t \in Z(x_t)$ and $x_t \in X_t \setminus P_{n+1}$.

If $c_f(x_t) = 0$, then $\bar{x}_t$ has to satisfy

$$\max_{x \in Z(x_t)} r_{i(x_t)}^{t-1}(x) = r_{i(x_t)}^{t-1}(\bar{x}_t). \tag{3.13}$$

If $c_f(x_t) = 1$, then since player $i(x_t)$ belongs to the coalition $S_f(x_t)$ on x_t, the node $\bar{x}_t$ is searched by

$$\max_{x \in Z(x_t)} \sum_{i \in S_f(x_t)} r_{i(x_t)}^{t-1}(x) = \sum_{i \in S_f(x_t)} r_{i(x_t)}^{t-1}(\bar{x}_t). \tag{3.14}$$

2) Now suppose that there exists x_t such that the subset $Y(x_t) \subset Z(x_t)$ of nodes where the payoff of the coalition including player $i(x_t)$ is not defined by the functions r_i^{t-1}, $i \in N$, is not empty. Notice that since we use the terminal payoff functions, with respect to the final gains it is not important in what coalitions a player has been participated during the game. He obtains the payoff just in accordance with the coalition structures at the endpoints. Therefore, if for each successor $x_{t-1} \in Z(x_t)$ of a node x_t the share of player $i(x_t)$ in the payoff of the coalition $S_f(x)$, where x is the final point of the path rising at x_{t-1}, has been defined yet, we don't need to determine the share of player $i(x_t)$ in the payoffs $\sum_{i \in S_f(x_t)} r_i^{t-1}(x_{t-1})$ of the coalition $S_f(x_t)$ on $x_{t-1} \in Z(x_t)$.

To know the decision of player $i(x_t)$ on x_t, for $y_{t-1} \in Y(x_t)$ we consider a cooperative positional $|S_f(y_{t-1})|$–person games $G_f(y_{t-1}, S_f(y_{t-1}))$ with the characteristic functions $v_f(y_{t-1}, S)$, $S \subset S_f(y_{t-1})$. The Shapley value

$$Sh^f(y_{t-1}) = (Sh_{k_1}^f(y_{t-1}), \ldots, Sh_{k_{|S_{y_{t-1}}|}}^f(y_{t-1})), \tag{3.15}$$

is taken as an optimal imputation of payoff of coalition $S_f(y_{t-1})$. Hence, the changed payoffs on X_{t-1} are specified by functions $\bar{r}_i^{t-1}\colon X_{t-1} \to R_+^1$, $i \in N$ such that for $x_{t-1} \in Z(x_t)$

$$\bar{r}_i^{t-1}(x_{t-1}) = \begin{cases} Sh_i^f(x_{t-1}), & \text{if } x_{t-1} \in Y(x_t) \text{ and } i \in S_f(x_{t-1}); \\ r_i^{t-1}(x_{t-1}), & \text{otherwise.} \end{cases} \tag{3.16}$$

Suppose that $c_f(x_t) = 0$. Then player $i(x_t)$ chooses on x_t a branch leading out to such node $\bar{x}_t \in Z(x_t)$ that

$$\max_{x \in Z(x_t)} \bar{r}_{i(x_t)}^t(x) = \bar{r}_{i(x_t)}^t(\bar{x}_t). \tag{3.17}$$

If $c_f(x_t) = 1$, then player $i(x_t)$ cooperates on x_t with the coalition $S_f(x_t)$. Hence, $\bar{x}_t$ has to satisfy

$$\max_{x \in Z(x_t)} \sum_{i \in S_f(x_t)} \bar{r}_{i(x_t)}^t(x) = \sum_{i \in S_f(x_t)} \bar{r}_{i(x_t)}^t(\bar{x}_t). \tag{3.18}$$

Finally, since the decisions of players have been determined for every node $x_t \in X_t$, we know the game development on any subtree $K(x_t)$, $x_t \in X_t$. Besides, during the stage t we created the functions $r_i^t\colon X_t \to R_+^1$ which show the payoffs obtained by players on $x_t \in X_t$, if the game reaches x_t. The function r_i^t is defined as follows:

$$r_i^t(x_t) = \begin{cases} r_i^{t-1}(\bar{x}_t), & \text{if } Y(x_t) = \emptyset; \\ \bar{r}_i^{t-1}(\bar{x}_t), & \text{if } Y(x_t) \neq \emptyset; \\ h_i(x_t), & \text{if } x_t \in P_{n+1}, \end{cases} \tag{3.19}$$

where $x_t \in X_t$.

Continue the moving on $K(x_0)$ toward the origin x_0. By sequentially determining players' decisions on the rest sets X_τ, $\tau = t+1, \dots, T$, we can construct a path which is realized if players are ruled by the given combination $f = (f_1, \dots, f_n)$ of the cooperative functions f_i, $i \in N$. We denote the path related to f by $x(f)$.

Cooperative subgames. Now we discuss the construction of cooperative subgames $G_f(y_{t-1}, S_f(y_{t-1}))$, $y_{t-1} \in Y(x_t)$. With respect to $G_f(y_{t-1}, S_f(y_{t-1}))$ we have that, though the game tree has the information structure for n participators, the set of players contains less than n players. We demonstrate that the definition of the individual behavior made in Section 2 allows to create the characteristic function $v_f(y_{t-1}, S)$, $S \subset S_f(y_{t-1})$, of $G_f(y_{t-1}, S_f(y_{t-1}))$.

Consider the subgame $\Gamma(x_t)$ of the game Γ. Change the set of players of $\Gamma(x_t)$ in accordance with the coalition structure on x_t. Let the new set be

$$N_f(x_t) = \{S_f(x_t), j_1, \dots, j_{N \setminus S_f(x_t)}\}. \tag{3.20}$$

Denote the subgame $\Gamma(x_t)$ with the set of players $N_f(x_t)$ by $\Gamma_f(x_t)$; see Section 2. Return to the partial cooperation. By our interpretation of the individual behavior, on every node $x \in P_{j_k} \cap K(x_t)$ player j_k, $k = 1, \dots, |N \setminus S_f(x_t)|$, uses the absolute Nash equilibrium strategy $\overline{\psi}^f_{j_k}(x_t)$ and is prohibited to avoid it. Notice that such behavior is reasonable and convenient for a non-cooperating player and it doesn't seem as a clear restriction. Since the decisions of individual players are fixed, we have to consider just strategies $\psi^f_i(x_t)$, $i \in S_f(x_t)$, of cooperating players. Let

$$\Psi^f_S(x_t) = \prod_{i \in S} \Psi^f_i(x_t) \tag{3.21}$$

be the set of strategies of a coalition $S \subset S_f(x_t)$. Then, the following characteristic function $v_f(x_t, S)$ is superadditive:

$$v_f(x_t, S) = \max_{\psi^f_S(x_t)} \min_{\psi^f_{S_f(x_t) \setminus S}(x_t)} \sum_{i \in S} b^f_i(\overline{\psi}^f_{j_1}(x_t), \dots, \overline{\psi}^f_{j_{|S_f(x_t)|}}(x_t), \psi^f_S(x_t), \psi^f_{S_f(x_t) \setminus S}(x_t)), \tag{3.22}$$

where $S \subset S_f(x_t)$, $\psi^f_S(x_t) \in \Psi^f_S(x_t)$, $\psi^f_{S_f(x_t) \setminus S}(x_t) \in \Psi^f_{S_f(x_t) \setminus S}(x_t)$.

In the reminder of this section we make an illustration of the path construction.

Example 2. We continue Example 1. Let us find path $x(f)$ for a combination f of cooperative functions such that $f_1(x_0) = 1$, $f_1(x_{22}) = 1$, $f_2(x_{11}) = 1$, $f_2(x_{23}) = 0$, $f_3(x_{21}) = 0$, $f_3(x_{12}) = 1$. In this case, we have $S_f(x_{21}) = S_f(x_{11}) = S_f(x_{22}) = \{1,2\}$, $S_f(x_{23}) = S_f(x_{12}) = \{1,3\}$, $S_f(x_0) = \{1\}$. Thus, on x_{21}, player 3 doesn't cooperate and chooses the left branch to obtain 2. On x_{22}, player 1 maximizes the payoff of the coalition $\{1,2\}$ and select the left branch leading at x_{33}. On x_{23}, player 2 goes left to get 2. On x_{11}, player 2 is in the coalition with player 1, and hence, he chooses the right branch. On x_{12}, player 3 cooperates with player 1. Therefore, he plays left for coalition $\{1,3\}$ to obtain 6. Since player 1 cooperates on x_0 and both $S_f(x_{11}) \setminus S_f(x_0)$ and $S_f(x_{12}) \setminus S_f(x_0)$ are not empty, we must calculate the share of player 1 in the payoff of the coalition $\{1,2\}$ on $K(x_{11})$ and in that of the coalition $\{1,3\}$ on $K(x_{12})$, respectively. For these reasons we construct the cooperative subgames $G_f(x_{11}, \{1,2\})$ and $G_f(x_{12}, \{1,3\})$. The values of the characteristic function of $G(x_{11}, \{1,2\})$ as follows, $v_f(x_{11}, \{1\}) = 1$, $v_f(x_{11}, \{2\}) = 1$, $v_f(x_{11}, \{1,2\}) = 4$. Thus, the Shapley value of $G(x_{11}, \{1,2\})$ equals to $(2,2)$. Hence, for the node x_{11}, the vector-payoff

$(2,2,2)$ is corresponded. Consider the values of characteristic function of $G_f(x_{12},\{1,3\})$. We have $v_f(x_{12},\{1\}) = 1\frac{1}{4}$, $v_f(x_{12},\{3\}) = 3$, $v_f(x_{12},\{1,3\}) = 6$. The Shapley value in $G_f(x_{12},\{1,3\})$ is $(2\frac{1}{8}, 3\frac{7}{8})$. Then the vector-payoff $(2\frac{1}{8}, 2, 3\frac{7}{8})$ is defined on x_{12}. Because player 1 maximizes only his own payoff, he chooses the right branch to obtain $2\frac{1}{8}$. Thus, we can conclude that the path $x_f = \{x_0, x_{12}, x_{23}, x_{35}\}$ is related to the given combination f of the cooperative functions.

4 The payoff function.

In [4] and [5], the payoff function defines only the payoff of coalition of players who had ever cooperated in a game party, without consideration for the payoffs of non-cooperating players. Such interpretation of the payoff function is suitable to consider the relation between cooperative activity of a player and the payoff of the coalition including him. In this paper we try to investigate the influence of the cooperative activity of each player on the payoff of the grant coalition N.

Definition. The function $H: F \to R^1_+$, where

$$H(f) = \sum_{i \in N} h_i(x_f), \tag{4.1}$$

is called *the payoff function* of the partial cooperative game $G(x_0)$.

We treat the solution of $G(x_0)$ as a payment system which stimulates players to act in the common interests and is acceptable by every player. Let us order each F_i, $i \in N$ as follows. In the sequence $f_i^0, f_i^1, \dots f_i^{|F_i|-1}, f_i^{|F_i|}$, the function f_i^0 should be related to the lowest cooperative activity of player i and $f_i^{|F_i|}$ to the highest one, i.e., $f_i^0(x) = 0$ and $f_i^{|F_i|}(x) = 1$ for all $x \in P_i$. Suppose that $f' = (f_1^0, \dots, f_n^0)$ and $f'' = (f_1^{|F_1|}, \dots, f_n^{|F_n|})$. Introduce a non-negative payoff vector $\beta = \{\beta_{il}\}_{i \in N, l=0,\dots,|F_i|}$, where component β_{il} expresses a numerical estimation of enforce of player i for changing cooperative function f_i^{l-1} to f_i^l. The payoff vector β is *an imputation* of $G(x_0)$ if

$$\beta_{i0} = h_i(x_{f'}), \quad i \in N, \tag{4.2}$$

and

$$\sum_{i \in N} \sum_{l=1}^{|F_i|} \beta_{il} = H(f''). \tag{4.3}$$

Denote the set of imputations by $I(x_0)$. The set

$$C(x_0) = \left\{ \beta \in I(x_0) \,\middle|\, \sum_{i \in N} \sum_{l=0}^{s_i} \beta_{il} \geq H(f), \forall f = (f_1^{s_1}, \dots, f_n^{s_n}) \in F, s_i = 0, \dots, |F_i|, i \in N \right\} \tag{4.4}$$

is called *the core* of $G(x_0)$.

We shall say that H is *an admissible payoff function* if $H(f') = \min_{f \in F} H(f)$. Now we determine a sufficient condition for existence of the non-empty core in $G(x_0)$ with the admissible payoff function.

Introduce the sets $M_i = \{0, 1, \dots, |F_i|\}$, $i \in N$. Since H is admissible, the grant coalition always can obtain the payoff $H(f')$. Thus, we should distribute just the difference $H(f'') -$

$H(f')$. Define the function $w(f) = H(f) - H(f')$, $f \in F$. Let $M = \prod_{i \in N} M_i$ and $m = (|F_1| \ldots, |F_n|)$. Let us put one-to-one correspondence between F and M. We say that $f = (f_1^{s_1}, \ldots, f_n^{s_n}) \in F$ is related to $s = (s_1, \ldots, s_n) \in M$. Consider a function $u: M \to R^1_+$ satisfying $u(s) = w(f)$ if f is related to s. If $u(s)$ is additive or superadditive, we have a multichoice game given by triple (N, m, u), where N is the set of players, m is the vector describing the number of activity levels for every player, and u is the characteristic function; see [1]–[3]. Denote the core of (N, m, u) by

$$C(u) = \{\xi = \{\xi_{il}\}_{i \in N, l=0,\ldots,|F_i|}\}, \tag{4.5}$$

where $\xi_{i0} = 0$, $i \in N$,

$$\sum_{i \in N} \sum_{l=0}^{|F_i|} \xi_{il} = u(m), \tag{4.6}$$

and for all $s \in M \setminus \{m\}$

$$\sum_{i \in N} \sum_{l=0}^{s_i} \xi_{il} \geq u(s). \tag{4.7}$$

Theorem. Suppose that $G(x_0)$ has the admissible payoff function. Then $C(x_0) \neq \emptyset$ if and only if there exists (N, m, u) and $C(u) \neq \emptyset$.

Proof. Let $C(x_0) \neq \emptyset$. Define $u(s)$ as follows:

$$u(s) = \sum_{i: s_i \neq 0} \sum_{l=0}^{s_i} \beta_{il} - \sum_{i: s_i \neq 0} \beta_{i0}, \quad s \in M. \tag{4.8}$$

Then

$$u(m) = \sum_{i \in N} \sum_{l=0}^{|F_i|} \beta_{il} - \sum_{i \in N} \beta_{i0} = H(f'') - H(f') = w(f''), \tag{4.9}$$

and $u(0, \ldots, 0) = 0$. Since $u(s)$ is additive, $C(u)$ has unique imputation ξ with components $\xi_{i0} = 0$, and $\xi_{il} = \beta_{il}$ if $l \neq 0$.

Conversely, suppose that (N, m, u) exists and $C(u) \neq \emptyset$. By the definition of (N, m, u), we have

$$\sum_{i \in N} \sum_{l=0}^{s_i} \xi_{il} \geq u(s) = w(f) = H(f) - H(f'), \tag{4.10}$$

where f is related to s, and $\xi \in C(u)$. Let $\beta_{il} = \xi_{il}$ for $l \neq 0$ and $\beta_{i0} = h_i(f')$. Then

$$H(f) \leq \sum_{i \in N} \sum_{l=0}^{s_i} \beta_{il} \tag{4.11}$$

Hence $C(x_0) \neq 0$.

To conclude the paper, we find the core $C(x_0)$ of the partial cooperative game in Example 1.

Example 3. We have that $M_1 = \{0, 1, 2\}$, $M_2 = \{0, 1, 2, 3\}$ and $M_2 = \{0, 1, 2, 3\}$. Let us use the following order of the cooperative functions: $f_1^1(x_{22}) = 1$, $f_1^1(x_0) = 0$, $f_1^2(x_{22}) = 1$, $f_1^2(x_0) = 1$, $f_2^1(x_{23}) = 1$, $f_2^1(x_{11}) = 0$, $f_2^2(x_{23}) = 0$, $f_2^2(x_{11}) = 1$, $f_2^3(x_{23}) = 1$, $f_2^3(x_{11}) = 1$, $f_3^1(x_{21}) = 1$, $f_3^1(x_{12}) = 0$, $f_3^2(x_{21}) = 0$, $f_3^2(x_{12}) = 1$, $f_3^3(x_{21}) = 1$, $f_3^3(x_{12}) = 1$. The related multichoice game (N, m, u) can be constructed and by some calculations, we have that $C(u)$ consists of the imputations

$$\xi = \left(\begin{array}{c|c|c} - & 0 & 0 \\ \hline \xi_{12} & 0 & \xi_{32} \\ \hline \xi_{11} & 0 & \xi_{31} \\ \hline 0 & 0 & 0 \end{array} \right), \tag{4.12}$$

such that $\xi_{11} + \xi_{12} \geq 1$, $\xi_{31} + \xi_{32} \geq 1$, $\xi_{11} + \xi_{31} \geq 1\frac{3}{4}$ and $\xi_{11} + \xi_{12} + \xi_{31} + \xi_{32} = 8$. By the previous theorem we can see that for each $\xi \in C(u)$ the imputation

$$\beta = \left(\begin{array}{c|c|c} - & 0 & 0 \\ \hline \xi_{12} & 0 & \xi_{32} \\ \hline \xi_{11} & 0 & \xi_{31} \\ \hline 1\frac{1}{4} & 0 & 3 \end{array} \right) \tag{4.13}$$

belongs to $C(x_0)$. Notice that all components of player 2 are zero. We explain it as follows. In our example, the realization of the path $\{x_0, x_{12}, x_{23}, x_{35}\}$ leading to the maximal payoff of the grand coalition N depends on cooperative enforce of players 1 and 3 yet. When player 2 doesn't cooperate, on the node x_{23} he chooses the left branch. Therefore, to reach the endpoint x_{35} the grand coalition doesn't need in additional activity of player 2. In other words, the cooperative activity of player 2 is dummy one. Thus, the willingness of player 2 to cooperation on the decision points x_{11} and x_{23} is estimated by zero.

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