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Operators: their Aluthge transforms and invariant subspaces

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This paper will be appeared in other journal. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An arbitrary operator T in $\mathcal{L}(\mathcal{H})$ has a unique polar decomposition $T=UP$, where $P=(T^{*}T)^{\frac{1}{2}}=|T|$ and U is a partial isometry with initial space the closure of the range of $|T|$ and final space the closure of the range of T. Associated with T there is a related operator $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, sometimes called the Aluthge transform of T because it was studied in [1] in the context that T is a p-hyponormal operator (to be defined below). In this note we derive some spectral connections between an arbitrary $T\in \mathcal{L}(\mathcal{H})$ and its associated Aluthge transform $\widetilde{T}$ that enable us, in particular, to generalize an invariant-subspace-theorem of Berger [2] to that context. We will also show that the hyperinvariant subspace problems for hyponormal and p-hyponormal operators are equivalent.

The following lemma is completely elementary, but sets forth basic relations between T and $\widetilde{T}$ that will be useful throughout the paper.

Lemma 1.1. *Let $T=U|T|$ (polar decomposition) be an arbitrary operator in $\mathcal{L}(\mathcal{H})$ and let $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ be its Aluthge transform. Then*

$$|T|^{\frac{1}{2}}T=\widetilde{T}|T|^{\frac{1}{2}}, \tag{1}$$

and

$$T(U|T|^{\frac{1}{2}})=(U|T|^{\frac{1}{2}})\widetilde{T}. \tag{2}$$

In particular, T is a quasiaffinity (i.e., T is one-to-one and has dense range) if and only if $|T|$ is a quasiaffinity and U is a unitary operator, so $\widetilde{T}$ is a quasiaffinity if T is. Moreover, in this case, T and $\widetilde{T}$ are quasisimilar. Furthermore, T is invertible if and only if $\widetilde{T}$ is, and in this case, T and $\widetilde{T}$ are similar.

Remark 1.2. Consider the Hilbert space $\mathcal{H}=L^{2}([0,1],\mu)$, where μ is Lebesgue measure, and let $\{e_{n}\}_{n=1}^{\infty}$ be any orthonormal basis for $\mathcal{H}$ such that e_{1} is the constant function 1. Let $U\in \mathcal{L}(\mathcal{H})$ be defined by $Ue_{n}=e_{n+1}$, $n\in \mathbf{N}$, so U is a unilateral shift, and consider $T=U(M_{x})^{2}$, where M_{x} is multiplication by the position function. Then T is clearly not a quasiaffinity, but an easy calculation shows that $\widetilde{T} = M_x U M_x$ is a quasiaffinity. Thus the corresponding implication in Lemma 1.1 only goes one way.

Our first theorem shows that there is an intimate spectral connection between (an arbitrary operator) T and its associated $\widetilde{T}$. As usual, we write $\sigma(T)$, $\sigma_p(T)$, and $\sigma_{ap}(T)$ for the spectrum, point spectrum, and approximate point spectrum of T, respectively.

Theorem 1.3. *For every T in $\mathcal{L}(\mathcal{H})$, $\sigma(T) = \sigma(\widetilde{T})$, $\sigma_{ap}(T) = \sigma_{ap}(\widetilde{T})$, $\sigma_p(T) = \sigma_p(\widetilde{T})$, $\sigma_{ap}(T^*)\backslash(0) = \sigma_{ap}((\widetilde{T})^*)\backslash(0)$, and $\sigma_p(T^*)\backslash(0) = \sigma_p((\widetilde{T})^*)\backslash(0)$.*

We remark here that the example given in Remark 1.2 shows that all the spectral equalities in Theorem 1.3 are best possible.

For an operator $A \in \mathcal{L}(\mathcal{H})$, we write, as usual, $\sigma_e(A)$, $\sigma_{le}(A)$, and $\sigma_{re}(A)$ for the essential (Calkin), left essential, and right essential spectra of A, respectively. Recall that $\lambda \in \sigma_{le}(A)$ if and only if there exists an orthonormal sequence $\{e_n\}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} \|(A-\lambda)e_n\| = 0$, or, equivalently, if and only if there exists a sequence $\{f_n\}$ of unit vectors in $\mathcal{H}$ such that $\{f_n\}$ converges weakly to zero and $\lim_{n\to\infty} \|(A-\lambda)f_n\| = 0$.

Corollary 1.4. *For any $T \in \mathcal{L}(\mathcal{H})$ with associated Aluthge transform $\widetilde{T}$, we have $\sigma_e(T) = \sigma_e(\widetilde{T})$, $\sigma_{le}(T) = \sigma_{le}(\widetilde{T})$, and $\sigma_{re}(T)\backslash(0) = \sigma_{re}(\widetilde{T})\backslash(0)$.*

We turn now to the intimate connection between the invariant subspace lattices of an arbitrary operator T and its associated $\widetilde{T}$. As usual, we write $\operatorname{Lat}(A)$ for the invariant subspace lattice of an arbitrary operator $A \in \mathcal{L}(\mathcal{H})$. If $T \in \mathcal{L}(\mathcal{H})$ is not a quasiaffinity, then $0 \in \sigma_p(T) \cup \sigma_p(T^*)$, so trivially T has a nontrivial invariant subspace. Thus when investigating the relation between $\operatorname{Lat}(T)$ and $\operatorname{Lat}(\widetilde{T})$, it suffices to consider the case that T is a quasiaffinity.

The following is an improvement of [5, Theorem 2].

Theorem 1.5. *Let $T = U|T|$ (polar decomposition) be an arbitrary quasiaffinity in $\mathcal{L}(\mathcal{H})$. Then the mapping*

$$\phi : \mathcal{N} \longrightarrow (|T|^{\frac{1}{2}}\mathcal{N})^-, \qquad \mathcal{N} \in \operatorname{Lat}(T),$$

maps $\operatorname{Lat}(T)$ *into* $\operatorname{Lat}(\widetilde{T})$, *and moreover if* $(0) \neq \mathcal{N} \neq \mathcal{H}$, *then*

$$(0) \neq \phi(\mathcal{N}) = (|T|^{\frac{1}{2}}\mathcal{N})^- \neq \mathcal{H}.$$

Moreover the mapping

$$\psi : \mathcal{M} \longrightarrow (U|T|^{\frac{1}{2}}\mathcal{M})^-, \qquad \mathcal{M} \in \operatorname{Lat}(\widetilde{T}),$$

maps $\operatorname{Lat}(\widetilde{T})$ *into* $\operatorname{Lat}(T)$, *and if* $(0) \neq \mathcal{M} \neq \mathcal{H}$, *then*

$$(0) \neq \psi(\mathcal{M}) = (U|T|^{\frac{1}{2}}\mathcal{M})^- \neq \mathcal{H}.$$

Consequently, $\mathrm{Lat}(T)$ *is nontrivial if and only if* $\mathrm{Lat}(\widetilde{T})$ *is nontrivial.*

Remark 1.6. If T in Theorem 1.5 is invertible, then T and $\widetilde{T}$ are similar (see Lemma 1.1), and thus have isomorphic invariant subspace lattices. Whether this is true for an arbitrary noninvertible quasiaffinity T, the authors have not been able to determine. Note that Theorem 1.5 implies that if one is trying to solve the invariant subspace problem for a particular quasiaffinity $T \in \mathcal{L}(\mathcal{H})$, it suffices to show that $\widetilde{T}$ has a nontrivial invariant subspace.

Definition 1.7 [1,6]. Suppose $T \in \mathcal{L}(\mathcal{H})$ and satisfies $(T^*T)^p \geq (TT^*)^p$ for some p in the interval $(0, +\infty)$. Then T is called a *p-hyponormal* operator. If $p = \frac{1}{2}$, T is sometimes called semi-hyponormal [1] and if $p = 1$, T is hyponormal.

There is a vast literature concerning p-hyponormal operators (for $0 < p < 1$) in which various special cases of Theorem 1.3 are proved (for p-hyponormal operators). For our purposes, we need only the following consequences of Löwner's inequality [8].

Remark 1.8. If $T \in \mathcal{L}(\mathcal{H})$ is a p-hyponormal operator for some p in the interval $(0, +\infty)$, then T is also q-hyponormal for every $0 < q \leq p$. In particular, an operator that is a p-hyponormal operator for some $p > 1$ is also hyponormal. Thus the interest in p-hyponormal operators has been concentrated mainly (but not exclusively; cf., for example, [4]) on those p-hyponormal operators for which $0 < p \leq 1$.

The following lemma was proved in [1] in the special case in which the partial isometry U in the polar decomposition $T = U|T|$ is a unitary operator, but the proof carries over to the general case. We introduce the proof here for conveniences.

Lemma 1.9([1]). *Suppose that* $T = U|T|$ *(polar decomposition) is an arbitrary p-hyponormal operator in* $\mathcal{L}(\mathcal{H})$ *for some* $p \in [\frac{1}{2}, 1]$. *Then its Aluthge transform* $\widetilde{T}$ *is a hyponormal operator.*

The following lemma was also proved in [1] in case $T = U|T|$ with U a unitary operator, but once again, the proof can be made to work in general.

Lemma 1.10([1]). *Suppose* $T = U|T|$ *(polar decomposition) is an arbitrary p-hyponormal operator in* $\mathcal{L}(\mathcal{H})$ *for some p in the interval* $(0, \frac{1}{2})$. *Then* $\widetilde{T}$ *is a* $(p + \frac{1}{2})$*-hyponormal operator and the Aluthge transform* $\widetilde{\widetilde{T}}$ *of* $\widetilde{T}$ *is a hyponormal operator.*

If $\mathcal{U}$ is a bounded open set in the complex plane $\mathbf{C}$, recall that a subset $\Lambda \subset \mathcal{U}$ is said to be *dominating* for $\mathcal{U}$ if every bounded function $h(z)$ holomorphic on $\mathcal{U}$ satisfies

$$\sup_{z \in \mathcal{U}} |h(z)| = \sup_{z \in \mathcal{U} \cap \Lambda} |h(z)|.$$

First we recapture the following corollary.

Corollary 1.11[5, Theorem 3] *Suppose $T = UP$ (polar decomposition) is an arbitrary p-hyponormal operator for some $p \in (0, +\infty)$, and suppose that there exists a nonempty open set $\mathcal{U}$ in $\mathbf{C}$ such that $\sigma(T) \cap \mathcal{U}$ is dominating for $\mathcal{U}$. Then T has a nontrivial invariant subspace.*

The following theorem generalizes a surprising theorem of Berger [2] for hyponormal operators to the context of p-hyponormal operators.

Theorem 1.12. *Let $T \in \mathcal{L}(\mathcal{H})$ be an arbitrary p-hyponormal operator for some $p \in (0, +\infty)$. Then there exists a positive integer K such that for all positive integers $k \geq K$, T^k has a nontrivial invariant subspace.*

Recall that a subspace $\mathcal{M}$ of $\mathcal{H}$ is a nontrivial hyperinvariant subspace for an operator $T \in \mathcal{L}(\mathcal{H})$ if $(0) \neq \mathcal{M} \neq \mathcal{H}$ and $\mathcal{M}$ is invariant under every operator in the commutant $\{S \in \mathcal{L}(\mathcal{H}) : ST = TS\}$ of T. We write Hlat(A) for the lattice of hyperinvariant subspaces of an operator $A \in \mathcal{L}(\mathcal{H})$. If $0 \neq T \in \mathcal{L}(\mathcal{H})$ and T is not a quasiaffinity, then $0 \in \sigma_p(T) \cup \sigma_p(T^*)$ and Hlat$(T) \neq \{(0), \mathcal{H}\}$ for trivial reasons. Then when investigating the relation between Hlat(T) and Hlat$(\widetilde{T})$, it suffices to that the case in which T is a quasiaffinity.

Theorem 1.13 *Let $T \in \mathcal{L}(\mathcal{H})$ be an arbitrary nonzero quasiaffinity. Then T has a nontrivial hyperinvariant subspace if and only if its Aluthge transform $\widetilde{T}$ does. Thus the hyperinvariant subspace problem for p-hyponormal operators (any $p \in (0, +\infty)$) is equivalent to the hyperinvariant subspace problem for hyponormal operators.*

Recall that $T \in \mathcal{L}(\mathcal{H})$ is a log-hyponormal operator if T is invertible and $\log(TT^*) \leq \log(T^*T)$. Note that any invertible p-hyponormal operator is log-hyponormal.

Theorem 1.14 *Suppose $T = UP$ (polar decomposition) is an arbitrary log-hyponormal operator, and suppose that there exists a nonempty open set $\mathcal{U}$ in $\mathbf{C}$ such that $\sigma(T) \cap \mathcal{U}$ is dominating for $\mathcal{U}$. Then T has a nontrivial invariant subspace.*

Recall that $T \in \mathcal{L}(\mathcal{H})$ is an ∞-hyponormal operator if T is n-hyponormal for any natural number n. Note that any ∞-hyponormal operator is p-hyponormal for any positive real number p.

We now close the paper as the following problem.

Problem 1.15 *Suppose T is an arbitrary ∞-hyponormal operator. Does T have a nontrivial invariant subspace.*

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