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UNITARY REPRESENTATIONS AND 1-COCYCLES ON THE GROUP OF DIFFEOMORPHISMS

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Abstract
We consider $\mathrm{Diff}_0(M)$, the group of all diffeomorphisms with compact supports, as an infinite dimensional Lie group and use Lie algebraic method in our analysis after preparing basic theorems. In particular in the later part of this report, we pick up 1-cocycles on $\mathrm{Diff}_0(M)$ and describe their characteristic properties together with natural representations.

1. Basic notions and theorems

1.1. Basic notion. Let M be a d-dimensional paracompact C^∞-manifold, $\mathrm{Diff}(M)$ be the set of all C^∞-diffeomorphisms g on M and

$$\mathrm{Diff}_0(M) := \{g \in \mathrm{Diff}(M) |\ \mathrm{supp} g \text{ is compact}\},$$

where $\mathrm{supp} g := Cl\{P \in M |\ g(P) \neq P\}$. We wish to regard $\mathrm{Diff}_0(M)$ as an infinite dimensional Lie group. Fortunately till now it has been known that for the case of compact manifold, $\mathrm{Diff}(M) = \mathrm{Diff}_0(M)$ is an infinite dimensional Lie group whose modelled space is an Frechét space called strong inductive limit of Hilbert spaces by some authors (cf.[16]), and for general manifolds it is possible to apply many of these results with a few modification for our purpose. After them, the Lie algebra and the exponential mapping that we should take here are the set

$$\Gamma_0(M) := \{X :\ C^\infty\text{-vector fields } X \text{ with compact support}\}$$

and the map,

$$\mathrm{Exp}(X) : \Gamma_0(M) \longmapsto \mathrm{Diff}_0(M),$$

where $\{\mathrm{Exp}(tX)\}_{t\in\mathbf{R}}$ is the 1-parameter transformation group or the integral curve generated by X.

1.2. Differential representation of unitary representation. Suppose that a unitary representation $(U, \mathcal{H})$ of $\mathrm{Diff}_0(M)$ is given. Then we have immediately by Stone's theorem, for ${}^\forall X \in \Gamma_0(M)$, ${}^\exists dU(X)$: self adjoint operator on $\mathcal{H}$ such that

$$U(\mathrm{Exp}(tX)) = \exp(\sqrt{-1}t dU(X)),$$

and the following questions arise naturally.

(1) Does the common domain of $\{dU(X)\}_{X\in\Gamma_0(M)}$ include a rich subspace such one like Gårding space ?
(2) Does $\sqrt{-1}dU$ become a linear representation under suitable restrictions of the domain of each $dU(X)$?
(3) Is the subgroup generated by $\mathrm{Exp}(X)$, $X \in \Gamma_0(M)$ dense in $\mathrm{Diff}_0(M)$?

1.3. $\mathrm{Diff}_0(M)$ as an infinite dimensional Lie group. For these questions the partial answeres obtained till now are as follows.

For (1). There is no problem for finite dimensional case, because the common domain of $\{dU(X)\}_{X\in\Gamma_0(M)}$ is nothing but the whole representation space. For the infinite dimensional case, there seems to be a way of constructing C^∞-vectors, though it is an author's conjecture, if M is compact and if the representation $(U,\mathcal{H})$ is extended to some $\mathrm{Diff}_0^k(M)$, which is the set of all C^k-diffeomporphisms with compact supports. The procedure is as follows. Use a method cloely resembling to one for unitary representations of usual locally compact Lie groups, but taking Shavgulidze measure (cf.[18]) in place of Haar measure.

For (2). It is assured by the following result which is alike to the formula derived from Campbell-Hausdorff formula.

Theorem 1.1. *Let $X,Y\in\Gamma_0(M)$ and $\{\mathrm{Exp}(tX)\}_{t\in\mathbf{R}},\{\mathrm{Exp}(tY)\}_{t\in\mathbf{R}}$ be 1-parameter subgroups of diffeomorphisms generated by X,Y, respectively. Then as n tends to $+\infty$,*

(1) $\{\mathrm{Exp}(\frac{tX}{n})\circ\mathrm{Exp}(\frac{tY}{n})\}^n$ *converges to* $\mathrm{Exp}(t(X+Y))$, *and*

(2) $\{\mathrm{Exp}(-\frac{tX}{\sqrt{n}})\circ\mathrm{Exp}(-\frac{tY}{\sqrt{n}})\circ\mathrm{Exp}(\frac{tX}{\sqrt{n}})\circ\mathrm{Exp}(\frac{tY}{\sqrt{n}})\}^n$ *converges uniformly to* $\mathrm{Exp}(-t^2[X,Y])$
together with every derivative on M and on every compact interval of t, respectively.

For the proof see [17, 22]. □

For (3). The problem (3) is also affirmative, but we must first give a topology on $\mathrm{Diff}_0(M)$. Let K be any compact subset of M. Set

$$\mathrm{Diff}(K):=\{g\in\mathrm{Diff}(M)\subseteq K\},$$

and consider on it a topology τ_K of uniform convergence of g together with every derivative. Clearly we have $\mathrm{Diff}_0(M)=\cup_K\mathrm{Diff}(K)$. So we can give the inductive limit topology τ on $\mathrm{Diff}_0(M)$, and it is noteworty that τ does not give a group topology. (cf.[24, 25]) Nevertheless $\mathrm{Diff}_0^*(M)$, the connected component of id in $\mathrm{Diff}_0(M)$, is an open normal subgroup and it is also arcwisely connected.

Now let A be an arbitrary subset of M and put

$\mathrm{Diff}_{0,A}^*(M):=\{g\in\mathrm{Diff}_0^*(M)|\ {}^\exists\{g_t\}_{0\le t\le 1}$ conti.path s.t., $g_0=\mathrm{id}$, $g_1=g$ and $g_t(P)=P$ for ${}^\forall P\in A$ and ${}^\forall t\in[0,1]\}$,

$$\Gamma_{0,A}(M):=\{X\in\Gamma_0(M)|\ X(A)=0\ \text{ for } {}^\forall A\in M\}.$$

Then as the affirmative answere of the third problem,

Theorem 1.2. *A subgroup generated by $\mathrm{Exp}(X),X\in\Gamma_{0,A}(M)$ is dense in $\mathrm{Diff}_{0,A}^*(M)$.*

For the proof see [16, 22]. □

As an immediate consequence of Theorem 1.1 and Theorem 1.2,

Theorem 1.3. *Let* $\{V_\alpha\}_{\alpha\in A}$ *be any relatively compact locally finite open covering of* M. *Then* $\mathrm{Diff}^*_{0,A}(M)$ *is generated by all local diffeomorphism groups* $\mathrm{Diff}^*_{0,A}(V_\alpha), \alpha \in A$ (*which consists of all* $g \in \mathrm{Diff}^*_{0,A}(M)$ *with* $\mathrm{supp} g \subset V_\alpha$).

Proof. Take any g from $\mathrm{Diff}^*_{0,A}(M)$. Then it is approximated by a $\mathrm{Exp}(X)$, $X \in \Gamma_{0,A}(M)$ by Theorem 1.2. Next decompose X into finitely many $X_i \in \Gamma_{0,A}(M)$, using a partition of unity subordinate to this cover. Thus each $\mathrm{Exp}(\frac{X_i}{n})$, $n \in \mathbf{N}$ belongs to our local diffeomorphism groups. Finally applying (1) in Theorem 1.1 repeatedly. This completes the proof. □

In particular in the case of $A = \emptyset$ Theorem 1.3 assures that the whole group $\mathrm{Diff}^*_0(M)$ is generated by local diffeomorphisms. It is somewhat well known, but the proof stated here rather simple. The following is also an application of these theorems.

Theorem 1.4. *There is no continuous representations of* $\mathrm{Diff}^*_0(M)$ *to* $GL(n, \mathbf{C})$ *except for trivial one.*

Proof is derived basically from Theorem 1.1 and Theorem 1.2. These results lead the above problem to a linear one. Moreover it becomes a local study by using a partition of unity, and the proof is reduced to admit the following theorem.

Theorem 1.5. *For a positive number* α, *put* $U_\alpha := \{x \in \mathbf{R}^d \mid -\alpha < x_k < \alpha \ \ (k = 1, \cdots, d)\}$, *and consider a Lie algebra* $\mathcal{G}_\alpha$ *consisting of* $\mathbf{R}^d$*-valued* C^∞*-functions* $F(x) = (f_k(x))_{1\le k\le d}$ *on* $\mathbf{R}^d$ *such that* $\mathrm{supp}F \subset U_\alpha$ *with the Lie bracket,*

$$[F, G] := \sum_{k=1}^{d} \{f_k(x)\frac{\partial G}{\partial x_k}(x) - g_k(x)\frac{\partial F}{\partial x_k}(x)\}.$$

Then there is no continuous linear representations dU *from* $\mathcal{G}_\alpha$ *to* $B(H)$ *except for trivial one, where the toplogy of* $\mathcal{G}_\alpha$ *is the usual one imposed on the space of test functions on* U_α *and* $B(H)$, *the space of all bounded operators on a complex Hilbert space* H ($\dim H$ *may be infinite*), *is eqipped with the weak operator topolology.*

Connecting with the above theorem we give another result for our later use.

Theorem 1.6. *Under the same notation as Theorem* 1.5, *put*

$$\mathcal{G}^0_\alpha := \{F = (f_k(x))_{1\le k\le d} \in \mathcal{G}_\alpha \mid F(0) = 0\}.$$

Then for any continuous linear representation dU *from* $\mathcal{G}^0_\alpha$ *to* $S(H) := \{T :$ bdd. op. on $H \mid T^* = -T\}$, *there exists a* $S \in S(H)$ *such that*

$$dU(F) = (\sum_{k=1}^{d} \frac{\partial f_k}{\partial x_k}(0))S.$$

Proofs of these theorems are rather algebraic. See [22, 23]. □

2. 1-COCYCLES ON THE GROUP OF DIFFEOMORPHISM

2.1. Five definitions for 1-cocycle. Hereafter we work on $\mathrm{Diff}_0^*(M)$ in place of $\mathrm{Diff}_0(M)$, and in a little while we denote $\mathrm{Diff}_0^*(M)$ by G. Suppose that g acts on a measure space $(X, \mathfrak{B}, \mu)$ from left as a measurable transformation, gx and that μ is $\mathrm{Diff}_0^*(M)$-quasi-invariant. That is,

$$\mu_g := \mu \circ g^{-1} \simeq \mu$$

for all $g \in \mathrm{Diff}_0^*(M)$, where μ_g is the image measure of μ under the map g.

Now we consider a $U(H)$-valued function $\theta(x, g)$ on $X \times \mathrm{Diff}_0^*(M)$, called 1-cocycle, which satisfies the following relation.

$$\forall g_1, g_2 \in \mathrm{Diff}_0^*(M), \quad \theta(x, g_1)\theta(g_1^{-1}x, g_2) = \theta(x, g_1 g_2), \tag{2.1}$$

for all $x \in M$, where H is a complex Hilbert space, and $U(H)$ is the unitary group. We give as below five definitions for regularity of 1-cocycles.

Definition 2.1. (1) *θ is said to be precontinuous, if for any fixed $x_0 \in X$ $\theta(x_0, g)$ is continuous as a function of g on $G(x_0) := \{g \in G |\ gx_0 = x_0\}$.*
(Of course if G acts transitively, the word "any" can be replaced by "some").
(2) *θ is said to be continuous, if for any fixed $x_0 \in X$ $\theta(x_0, g)$ is continuous as a function of $g \in G$.*
(3) *θ is said to be Borelian, if it is precontinuous and for any fixed $g \in G$ $\theta(x, g)$ is $\mathfrak{B}$-measurable.*
(4) *$\theta(x, g)$ is said to be strongly Borelian, if it is precontinuous and $\theta(x, g)$ is jointly measurable of both variables.*
(5) *$\theta(x, g)$ is said to be measurable, if for any fixed $g \in G$ $\theta(x, g)$ is $\mathfrak{B}$-measurable.*

Further it is sometimes expected that the following condition, a kind of continuity, is imposed in order that the natural representations corresponding to θ is continuous.

(6) $\forall h_1, h_2 \in H$, $< \theta(x, g)h_1, h_2 >_H$ converges in μ to $< h_1, h_2 >_H$ when g tends to id.

Anyway the relation between these five notions are as follows.
"Strong Borel" means "Borel", "Borel" means "Measurability" and "Precontinuity".
Also "Continuity" means "Precontinuity".

2.2. Local form of precontinuous 1-cocycles. In this subsection we consider precontinuous 1-cocycles $\overline{\theta} = \overline{\theta}(\overline{P}, g)$ on $B_M^n \times \mathrm{Diff}_0^*(M)$, where B_M^n is a space of all n-point sets of M and g acts on B_M^n in a obvious way, $\overline{P} = \{P_1, \cdots, P_n\} \longmapsto \overline{g}(\overline{P}) = \{g(P_1), \cdots, g(P_n)\}$. Since B_M^n is a quotient space of $\hat{M}^n := \{\hat{P} = (P_1, \cdots, P_n) |\ \forall i \neq j,\ P_i \neq P_j\}$ defined by an equivalence relation, we can always lift any 1-cocycle $\overline{\theta}$ to $\hat{M}^n \times \mathrm{Diff}_0^*(M)$ as a symmetric precontinuous 1-cocycle $\hat{\theta}$. Denote the diagonal action of of g on $\hat{M}^n$ by $\hat{g}$. We start at the study of local form of such $\hat{\theta}$'s. Hereafter we always assume that $\dim H < \infty$.

Theorem 2.1. (*Local form of* 1*-cocycle*)
Let $\hat{\theta}$ be precontinuous $U(H)$-valued 1-cocycle. Take any σ-finite locally Euclidean smooth measure μ on M and fix it. Then for $\forall \hat{A} \in \hat{M}^n$, there exists an open neighbourhood $V(\hat{A})$

of $\hat{M}^n$, a $U(H)$-valued map C defined on $V(\hat{A})$ and a commutative system of self-adjoint operators $\{H_k\}_k$ such that

$$\hat{\theta}(\hat{P}, g) = C(\hat{P})^{-1} \prod_{k=1}^{n} \left(\frac{d\mu_g}{d\mu}(P_k) \right)^{\sqrt{-1}H_k} C(\hat{g}^{-1}(\hat{P})), \tag{2.2}$$

provided that $(\hat{P}, g)$ satisfies the following condition $(*)$.

$(*)$ *There exists a continuous path $\{g_t\}_{0\le t\le 1}$ connecting id and g such that $\hat{g}_t^{-1}(\hat{P}) \in V(\hat{A})$ for $^\forall t \in [0,1]$.*

Moreover if $\hat{\theta}$ is continuous, we can take the C so as to be continuous.

Proof is derived basically from Theorem 1.1, Theorem 1.2 and Theorem 1.6, and further using local sections $s_{\hat{P}}$. That is $\hat{P} \longmapsto s_{\hat{P}}$ is a $\mathrm{Diff}_0^*(M)$-valued continuous map on a neighbourhood of $\hat{A}$ satisfying $s_{\hat{P}}(\hat{A}) = \hat{P}$. Finally the following relation is fundamental.

$$\hat{\theta}(\hat{P}, g) = \hat{\theta}(\hat{A}, s_{\hat{P}}^{-1})^{-1} \hat{\theta}(\hat{A}, s_{\hat{P}}^{-1} \circ g \circ s_{\hat{g}^{-1}(P)}) \hat{\theta}(\hat{A}, s_{\hat{g}^{-1}(P)}^{-1}). \qquad \square$$

Remark 2.1. *Theorem 1.3 together with Theorem 2.1 implies that in the present case, continuous 1-cocycle is strongly Borelian.*

We wish to extend the above result to a global one. So first let us observe the local behavior of the 1-cocycle changing $\hat{A}$ to another point $\hat{A}'$. Moreover in order to see the essential part and also for the brevity, we consider a case $n = 1$ and $\dim(H) = 1$. Then the local form $\theta \equiv \hat{\theta}$ is as follows.

$$\theta(P, g) = C(P)^{-1} \left(\frac{d\mu_g}{d\mu}(P) \right)^{\sqrt{-1}\lambda} C(g^{-1}(P)), \tag{2.3}$$

provided that (P, g) satisfies the condition $(*)$, where λ is a compex number with modulus 1 and C is a $\mathbf{T}^1$-valued function on $V(A)$. Suppose that $V(A) \cap V(A') \neq \emptyset$ and that the intersection is connected. Then it follows from arguments using local diffeomorphisms that

(1) $\lambda = \lambda'$ and
(2) C is equal to C' up to a multiplicative constant on $V(A) \cap V(A')$.

Thus these C's define a many valuedness function. Let us explain this situation in more detail. Assume that the third point A'' is also given and satisfies $V(A') \cap V(A'') \neq \emptyset$ and $V(A) \cap V(A'') \neq \emptyset$ such that these intersections are connected. We adjust the multiple constant so as to be first, $C = C'$ on $V(A) \cap V(A')$, and next $C' = C''$ on $V(A') \cap V(A'')$. However it may be possible that C does not coincide with C'' on $V(A) \cap V(A'')$. So the problem of "Resolution of many valuedness" arises, and it depends on a geometrical structure of M. In analytic continuiation a key to solve such a problem is a use of "Principle of monodoromy", and also in our case it works well so that the cocycle form given by (2.3) is general and global one, if assume that M is simply connected. We give it as the following general theorem.

Theorem 2.2. *Suppose that $\hat{M}^n$ is simply connected. Then for every precontinuous 1-cocycle $\hat{\theta}$ on $\hat{M}^n \times \mathrm{Diff}_0^*(M)$, there exist a $U(H)$-valued function C on $\hat{M}^n$ and a commutative system of self-adjoit operators $\{H_k\}_k$ on H such that (2.2) holds for all*

$(\hat{P}, g) \in \hat{M}^n \times \mathrm{Diff}_0^*(M)$. *Moreover if $\hat{\theta}$ is continuous, C can be taken so as to be continuous.*

More general theorem than the above one is the following.

Theorem 2.3. (*Global form of* 1*-cocycle*)
Let A be any subset of M which has no accumulation points.
(1) *Suppose that $\hat{M}_A^n$ is simply connected, where $\hat{M}_A^n := \{\hat{P} \in \hat{M}^n | \ \overline{P} \cap A = \emptyset\}$. Then for any precontinuous $U(H)$-valued 1-cocycle $\hat{\theta}$ on $\hat{M}_A^n \times \mathrm{Diff}_{0,A}^*(M)$, there exists a $U(H)$-valued map C on $\hat{M}_A^n$ and a commutative system of self-adjoint operators $\{H_k\}_k$ such that* (2.2) *holds for all $(\hat{P}, g) \in \hat{M}_A^n \times \mathrm{Diff}_{0,A}^*(M)$.*
Moreover if $\hat{\theta}$ is continuous, C can be taken so as to be continuous.
(2) *Assume that $\hat{M}_A^n$ is connected. Let $\hat{\theta}$ be given by* (2.2) *with $(C, \{H_k\}_k)$ and let $(C', \{H'_k\}_k)$ be another such pair. Then there exists a $T \in U(H)$ such that for all $\hat{P} \in \hat{M}_A^n$*

$$C'(\hat{P}) = TC(\hat{P}) \quad \text{and} \quad H'_k = TH_kT^{-1} \quad \text{for all} \ \ 1 \leq k \leq 1.$$

Proof. The proof is based on a more precise theorem which states a local form of 1-cocycles on $\hat{M}_A^n \times \mathrm{Diff}_{0,A}^*(M)$ than Theorm 2.1. For details see [23]. □

Hereafter we call $\hat{\theta}$ having the form given by (2.2) canonical 1-cocycle. The next theorem describes precontinuous 1-cocycles on $B_M^n \times \mathrm{Diff}_0^*(M)$ in the case that $\hat{M}^n$ is simply connected.

Theorem 2.4. *Suppose that $\hat{M}^n$ is connected. Then in order that a canonical* 1*-cocycle $\hat{\theta} \equiv \hat{\theta}(C, H_k)$ is symmetric, it is necessary and sufficient that there exists a unitary representation (T, H) of $\mathfrak{S}_n$, the permutation group on $\{1, \cdots, n\}$, such that*

$$\forall \hat{P}, \ \ C(\hat{P}) = T(\sigma)C(\hat{P}_\sigma) \quad \text{and} \quad H_k = T(\sigma)^{-1}H_{\sigma(k)}T(\sigma)$$

for all $1 \leq k \leq n$ and $\sigma \in \mathfrak{S}_n$, where $\hat{P}_\sigma := (P_{\sigma(1)}, \cdots, P_{\sigma(n)})$.

Proof is straightforward from the uniqueness part of Theorem 2.3. □

Considering these theorems, it is important to look for sufficient conditions for the simply connectedness of $\hat{M}^n$. One result is derived, thanks to Dimension theory, from Propositions in [3].

Theorem 2.5. *Under the assumption that a subset A of M has no accumulation points,*
(1) *if* $\dim(M) \geq 2$ *and M is connected, then so is $\hat{M}_A^n$ for every $n \in \mathrm{N}$.*
(2) *If* $\dim(M) \geq 3$ *and M is simply connected, then so is $\hat{M}_A^n$ for every $n \in \mathrm{N}$.*

Next let us state some comments on the cocycle form in the case which $\hat{M}^n$ is not simply connected. First we shall state two remarks for the case $n = 1$, and thus $\hat{M}^n$ is M itself.

Theorem 2.6. *If M is a compact connected Lie group, then the same result as in Theorem* 2.2 *holds for precontinuous* 1*-cocycles θ. Namely, every precontinuous* 1*-cocycle is canonical.*

Proof. It is due to the fact that there exists a global section, consisting of translations, on this group. For detailed discussions see [22]. □

It follows that for the case $n = 1$ simply connected condition is not necessary one. However If M is not simply connected Theorem 2.2 is no longer true as will be seen in the following example.

Example 2.1. *Consider cylinder $M := \mathbf{R} \times \mathbf{T}^1$, and denote the elements in M by (u, z), or $(u, \exp(\sqrt{-1}\theta))$. Let $g \in \mathrm{Diff}_0^*(\mathbf{R} \times \mathbf{T}^1)$ and take a continuous path $\{g_t\}_{0 \le t \le 1}$ connecting id and g. Then for each fixed $(u, z) \in \mathbf{R} \times \mathbf{T}^1$, the second component of $g_t^{-1}(u, z)$ has an continuous angular function $\theta(t, u, z)$. Put $\varphi_g(u, z) := \theta(1, u, z) - \theta(0, u, z)$. Then it is easily checked that $\varphi := \varphi_g$ does not depend on a particular choice of $\{g_t\}_{0 \le t \le 1}$. So put for any real number Ω*

$$\zeta_\Omega((u, z), g) := \exp(\sqrt{-1}\Omega\varphi(u, z)).$$

Then ζ_Ω is a continuous but non canonical 1-cocycle on $\mathrm{Diff}_0^*(\mathbf{R} \times \mathbf{T}^1)$, unless $\Omega \in \mathbf{N}$.

For the detailed discussions see [22]. Next we shall also state a few remarks in the case that M is simply connected and $\dim(M) < 3$.

The first one is that our M is equal to $\mathbf{R}^1$, thus B_M^n is simply connected and $\hat{M}^n$ consists of $n!$ connected components which are all isomorphic to B_M^n.

Theorem 2.7. *Let $M = \mathbf{R}^1$ and take an isomorphic section τ from B_M^n to $\hat{M}^n$. Then the general form of precontinuous 1-cocycles on $B_M^n \times \mathrm{Diff}_0^*(M)$ is as follows.*

$$\overline{\theta}(\overline{P}, g) = C(\overline{P})^{-1} \prod_{k=1}^{n} \left(\frac{d\mu_g}{d\mu}((\tau(\overline{P}))_k) \right)^{\sqrt{-1}H_k} C(\overline{g}^{-1}(\overline{P})), \tag{2.4}$$

where C is a $U(H)$-valued map and $\{H_k\}_k$ is a commutative system of self-adjoint operators on H.

Proof. It is derived from a similar theorem with Theorem 2.2. Of course there is a non canonical 1-cocycle $\hat{\theta}$ corresponding to the above $\overline{\theta}$ on $\hat{M}^n \times \mathrm{Diff}_0^*(M)$ even in the case $H = \mathbf{C}$. □

The second case is that $M = \mathbf{R}^2$. Here $\hat{M}^n$ is connected contrary to the previous case, however it is not simply connected for $n \ge 2$, and there exists a non canonical but symmetric 1-cocycle. A counter example closely resembling to one in the cylinder case is easily produced for example, when $n = 2$ and $H = \mathbf{C}$. We omit it.

The last case $M = \mathbf{T}^1 \equiv \mathbf{T}$ is more interesting. $B_{\mathbf{T}}^n$ and $\hat{\mathbf{T}}^n$ are non simply connected, but they are connected. Now consider a set

$$I := \{(z_1, \cdots, z_n) \in \hat{\mathbf{T}}^n | \ \arg z_1^{-1} z_k < \arg z_1^{-1} z_{k+1} \quad (k = 1, \cdots, n-1)\},$$

where the value of the argument is taken so as to be in $[0, 2\pi)$. I is a connected open set and it is a $\mathrm{Diff}_0^*(\mathbf{T})$-invariant set. Suppose that $T \in U(H)$, a commutative system of self-adjoint operators $\{H_k\}_k$ and a $U(H)$-valued map C on I are given such that they satisfy the following conditions.

$$C(z_2, z_3, \cdots, z_n, z_1) = TC(z_1, z_2, \cdots, z_n) \qquad ((z_1, z_2, \cdots, z_n) \in I). \tag{2.5}$$

$$T^n = \mathrm{id}, \tag{2.6}$$

and

$$H_k = T^{-(k-1)} H_1 T^{(k-1)} \tag{2.7}$$

Then for $\overline{P} \in B^n_{\mathbf{T}}$ we order its elements z_k $(k = 1, \cdots, n)$ in such a way that $\hat{P} := (z_1, \cdots, z_n)$ belongs to I and define

$$\overline{\theta}(\overline{P}, g) = C(\hat{P})^{-1} \prod_{k=1}^{n} \left(\frac{d\mu_g}{d\mu}(z_k) \right)^{\sqrt{-1}H_k} C(\hat{g}^{-1}(\hat{P})). \tag{2.8}$$

Although there are many, exactly n, ways of this ordering, the definition does not depend on them, and actually it gives a precontinuous 1-cocycle on $B^n_{\mathbf{T}} \times \mathrm{Diff}^*(\mathbf{T})$, and moreover it is a complete description of cocycle. That is,

Theorem 2.8. *The general form of precontinuous* 1*-cocycles on* $B^n_{\mathbf{T}} \times \mathrm{Diff}^*(\mathbf{T})$ *is given by* (2.8).

Now a question arises : Is every symmetric continuous 1-cocycle on $\hat{\mathbf{T}}^n \times \mathrm{Diff}^*(\mathbf{T})$ canonical ?
However this is negative in general as will be seen in the following example.
Let $n = 4$ and $H = \mathbf{C}^2$ and put

$$T := \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix},$$

$$H_1 = H_3 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad H_2 = H_4 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Further for any point $(z_1, z_2, z_3, z_4) \in I$, put

$$C(z_1, z_2, z_3, z_4) = \frac{1}{\sqrt{(|z_1 - z_3|^2 + |z_2 - z_4|^2)}} \begin{pmatrix} z_1 - z_3 & \overline{z_2 - z_4} \\ -(z_2 - z_4) & \overline{z_1 - z_3} \end{pmatrix}.$$

The triplet $(T, \{H_k\}_k, C)$ satisfies the above conditions, so they define a 1-cocycle. However it is not canonical, as is easily seen.

3. Natural representations on finite configuration space

3.1. Fundamental notations and a historical survey. In this section we consider natural representations connected with 1-cocycles on $B^n_M \times \mathrm{Diff}^*_0(M)$. As before let us take a smooth Euclidean and moreover probability measure μ on M and fix it. Put $\hat{\mu} \equiv \hat{\mu}^n$ for the product measure on $\hat{M}^n$ and $\overline{\mu}$ for the image measure of $\hat{\mu}$ by the natural map $\hat{M}^n \longmapsto B^n_M$. $\overline{\mu}$ is the unique, of course up to equivalence, quasi-invariant measure under the action of $\mathrm{Diff}^*_0(M)$.

Now consider natural representation $(U_{\overline{\theta}}, \mathcal{H})$ of $\mathrm{Diff}^*_0(M)$. That is,

(1) $\mathcal{H} := \mathrm{L}^2_{\overline{\mu}}(B^n_M, H)$: the set of all square summable H-valued functions w.r.t. $\overline{\mu}$,

(2) $U_{\overline{\theta}}(g) : f(\overline{P}) \in \mathrm{L}^2_{\overline{\mu}}(B^n_M, H) \longmapsto \sqrt{\dfrac{d\overline{\mu}_{\overline{g}}}{d\overline{\mu}}}(\overline{P}) \overline{\theta}(\overline{P}, g) f(\overline{g}^{-1}(\overline{P})) \in \mathrm{L}^2_{\overline{\mu}}(B^n_M, H),$

where $\overline{\theta}$ is a $U(H)$-valued measurable 1-cocycle.

Historically in the first paper of Ismagilov [7], it is shown that every unitary representations of the group $\mathrm{Diff}^*_0(\mathbf{T}^1)$ with some additional conditions are characterized as the natural representations on some spaces being analogous with such a finite configuration space or infinite one. After this natural representations over the configuration space or on the analogous one were frequently appeared in order to analyse or to construct representations of $\mathrm{Diff}_0^*(M)$ ([5, 6, 8, 9, 10, 26]). Of course there are other representations being not natural, for example [15].

3.2. Irreducibility and Equivalence.

Definition 3.1. (1) *A measurable* 1*-cocycle* $\overline{\theta}$ *is said to be irreducible, if for any* $U(H)$*-valued measurable map* $V(\overline{P})$ *there exists some complex constant* k *such that*

$$V(\overline{P}) = k\mathrm{Id}$$

for $\overline{\mu}$*-a.e.*$\overline{P}$*, povided that*

$$V(\overline{P})\overline{\theta}(\overline{P}, g) = \overline{\theta}(\overline{P}, g)V(\overline{g}^{-1}(\overline{P})) \tag{3.1}$$

for $\overline{\mu}$*-a.e.*$\overline{P}$*.*
(2) *A parallel definition for a symmetric measurable* 1*-cocycle* $\hat{\theta}$ *is given, in which* $V(\overline{P})$ *is replaced by a symmetric measurable map* $V(\hat{P})$*.*

The next theorem gives us a criterion for irreducibility of canonical cocycle.

Theorem 3.1. *Assume that* $\hat{M}^n$ *is connected and that a symmetric* 1*-cocycle* $\hat{\theta}(C, H_k)$ *has the canonical form* (2.2) *and that it is strongly Borelian. Then in order that* $\hat{\theta}$ *be irreducible, it is necessary and sufficient that the representation* (T, H) *defined in Theorem* 2.4 *and* $\{H_k\}_k$ *satisfy the following condition* (**)*.*

(**) *A unitary operator* A *on* H *is a scalar one, provided that*

$$AT(\sigma) = T(\sigma)A \quad \textit{for all} \quad \sigma \in \mathfrak{S}_n \quad \textit{and} \tag{3.2}$$

$$AH_k = H_kA \quad \textit{for all} \quad 1 \leq k \leq n. \tag{3.3}$$

(*Here the connectedness condition is necessary only for the sufficiency.*)

Proof. See [23]. □

Remark 3.1. *Theorem* 2.7 *leads to that in the case* $M = \mathbf{R}^1$ *irreducible strongly Borelian* 1*-cocycle does not exist except for* $\dim(H) = 1$*, using the similar proof with the above one. So in this case a class of natural irreducible representations is something narrow.*

Noe we go to the irreducibility and equivalence of natural representations.

Theorem 3.2. (*Irreducibility*)
Let $\overline{\theta}$ *be a strongly Borelian* 1*-cocycle on* $B_M^n \times \mathrm{Diff}_0^*(M)$ *and* $(U_{\overline{\theta}}, \mathrm{L}^2_{\overline{\mu}}(B_M^n, H))$ *be the corresponding natural representation . Then* $(U_{\overline{\theta}}, \mathrm{L}^2_{\overline{\mu}}(B_M^n, H))$ *is irreducible, if and only if so is* $\overline{\theta}$

Theorem 3.3. (*Equivalence*)
Let $\overline{\theta}_i$ $(i = 1, 2)$ be strongly Borelian 1*-cocycles and* $(U_{\overline{\theta}_i}, \mathrm{L}^2_{\overline{\mu}}(B^n_M, H_i))$ $(i = 1, 2)$ *be the corresponding natural representations. Then the representations are equivalent, if and only if the* 1*-cocycles are* 1*-cohomologous. That is, there exists a* $U(H_1, H_2)$*-valued measurable map* $V(\overline{P})$ *on* B^n_M ($U(H_1, H_2)$ *is the set of all unitary operators from* H_1 *to* H_2) *such that*

$$\overline{\theta}_1(\overline{P}, g) = V^{-1}(\overline{P})\overline{\theta}_2(\overline{P}, g)V(\overline{g}^{-1}(\overline{P})) \tag{3.4}$$

for $\overline{\mu}$*-a.e.*$\overline{P}$.

For these proofs see [23]. □

4. 1-COCYCLES ON THE INFINITE CONFIGURATION SPACE

4.1. Canonical form of 1-cocycles on the infinite configuration space. Hereafter we assume that M is non compact. Put

$\hat{M}^\infty := \{\hat{P} = (P_1, \cdots, P_n, \cdots) |\ {}^\forall i \neq j,\ P_i \neq P_j$ and $\{P_k\}_k$ has no accumulation points$\}$

and Γ_M be a space all countable, not finite, subsets of M having no accumulation points. As before Γ_M is a quotient space of $\hat{M}^\infty$ by an equivalence relation $\sim$ defined by,
$\hat{P} \sim \hat{Q}$ if and only if ${}^\exists\sigma \in \mathfrak{S}_\infty$, the permutation group on the set $\mathbf{N}$, s.t., $\hat{Q} = \hat{P}_\sigma := (P_{\sigma(1)}, \cdots, P_{\sigma(n)}, \cdots)$.
Γ_M is called infinite configuration space and its element will be denoted by $\overline{P} := \{P_1, \cdots, P_n, \cdots\}$. For further discussions we need one more equivalence relation $\approx$ on $\hat{M}^\infty$ defined by,

$$\hat{P} \approx \hat{Q} \text{ if and only if } {}^\exists N,\ \forall n \geq N,\ P_n = Q_n.$$

Denote the equivalence class to which $\hat{P}$ belongs by [P], though the notation is not exact, but it is simple.

Let $\hat{\theta}$ be precontinuous $U(H)$-valued 1-cocycle on $\hat{M}^\infty \times \mathrm{Diff}^*_0(M)$. Since under an additional assumption an orbit of $\hat{P}$ under the action of $\mathrm{Diff}^*_0(M)$ is $[P]$, it is reasonable at first to restrict $\hat{\theta}$ to [A] for each $\hat{A} \in \hat{M}^\infty$. Next combining together all the results on $\hat{M}^n_A$, $n \in \mathrm{N}$ in Theorm 2.3 by inductive limit methods, we have

Theorem 4.1. (1) *Suppose that M is simply connected and* $\dim(M) \geq 3$. *Then the general form of precontinuous* $U(H)$*-valued* 1*-cocycles on* $\hat{M}^\infty \times \mathrm{Diff}^*_0(M)$ *is as follows.*

$$\hat{\theta}(\hat{P}, g) = C(\hat{P})^{-1} \prod_{k=1}^{\infty} \left(\frac{d\mu_g}{d\mu}(P_k)\right)^{\sqrt{-1}H_k^{[P]}} C(\hat{g}^{-1}(\hat{P})), \tag{4.1}$$

where C is a $U(H)$*-valued map on* $\hat{M}^\infty$ *and* $\{H_k^{[P]}\}_k$ *is a commutative system of self-adjoint operators depending on the residue class* $[P] \in \hat{M}^\infty/\approx$ *to which* $\hat{P}$ *belongs. Moreover if* $\hat{\theta}$ *is continous, C can be taken so that* $C|\hat{M}^\infty_A$ *is continuous with respect to* τ^∞_A *for each* $A \in \hat{M}^\infty$, *where* $\hat{M}^\infty_A := \{\hat{P} \in \hat{M}^\infty |\ \hat{P} \approx \hat{A}\}$, *and* τ^∞_A *is the inductive limit topology on* $[A]$.

As before we call $\hat{\theta}$ *given by* (4.1) *canonical* 1*-cocycle.*

(2) *For the uniqueness of the above pair* $(C, \{H_k^{[P]}\}_k)$ *we assume that*
(†) *M is connected and* $\dim(M) \geq 2$.

Then if there exists an another pair $(C', \{H_k'^{[P]}\}_k)$, *there exists a* $U(H)$*-valued map* T *on* $\hat{M}^\infty/\approx$ *such that*

$$C'(\hat{P}) = T([P])C(\hat{P}) \tag{4.2}$$

for all $\hat{P} \in \hat{M}^\infty$ *and*

$$H_k'^{[P]} = T([P])H_k^{[P]}T([P])^{-1} \tag{4.3}$$

for all $1 \leq k < \infty$, *and* $\hat{P} \in \hat{M}^\infty$.

Theorem 4.2. *Let* $\hat{\theta}$ *be a canonical* 1*-cocycle on* $\hat{M}^\infty \times \mathrm{Diff}_0^*(M)$. *Then under the assumption* (†), $\hat{\theta}$ *is symmetric if and only if the pair* $(C, \{H_k^{[P]}\}_k)$ *satisfies the following conditions.*

$$C(\hat{P}) = R([P], \sigma)C(\hat{P}_\sigma) \tag{4.4}$$

for all $\hat{P} \in \hat{M}^\infty$ *and* $\sigma \in \mathfrak{S}_\infty$, *where* R *is a* 1*-cocycle on* $\hat{M}^\infty/\approx \times \mathfrak{S}_\infty$. *Namely,*

$$^\forall[P], {}^\forall\sigma, \quad R([P], \sigma)R([P_\sigma], \tau) = R([P], \sigma\tau),$$

and

$$H_k^{[P]} = R([P], \sigma)H_{\sigma^{-1}(k)}^{[P_\sigma]}R([P], \sigma)^{-1} \tag{4.5}$$

for all $1 \leq k < \infty$, $[P] \in \hat{M}^\infty/\approx$ *and* $\sigma \in \mathfrak{S}_\infty$.

Finally at the end of this section we give a criterion for the measurabilty of canonical cocycle.

Theorem 4.3. *Let* $\hat{\theta}$ *be a canonical* 1*-cocycle given by* (4.1). *Then in order that* $\hat{\theta}$ *is measurable it is necessary and sufficient that*

$$C(\hat{P})^{-1}H_k^{[P]}C(\hat{P}) \quad \textit{is measurable for each fixed } 1 \leq k < \infty, \tag{4.6}$$

and

$$C(\hat{P})^{-1}C(\hat{g}^{-1}(\hat{P})) \quad \textit{is measurable for each fixed } g \in \mathrm{Diff}_0^*(M). \tag{4.7}$$

Proof is easy. □

5. Natural representations on the infinite configuration space

5.1. Irreducibilty and Equivalence. In this subsection we consider natural representations of $\mathrm{Diff}_0^*(M)$ on Γ_M which are alike to the one on the finite configuration space. However $\mathrm{Diff}_0^*(M)$-quasi-invariant measure on $(\Gamma_M, \mathfrak{B})$, $\mathfrak{B}$ is the natural Borel field, is not uniquely determined, so we must consider also a factor of such probabitiy measures $\overline{\nu}$ on $(\Gamma_M, \mathfrak{B})$. It is known in [26] that to such a $\overline{\nu}$ there correspondes a $\mathrm{Diff}_0^*(M)$-quasi-invariant probability measure $\hat{\nu}$ on $(\hat{M}^\infty, \mathcal{C})$, $\mathcal{C}$ is also the natural Borel field on $\hat{M}^\infty$, such that

$$\hat{\nu}(E) = \sum_{\sigma \in \mathfrak{S}_\infty^0} c(\sigma)(s\overline{\nu})\sigma(E) \tag{5.1}$$

for all $E \in \mathcal{C}$, where $\mathfrak{S}_\infty^\circ := \cup_{n=1}^\infty \mathfrak{S}_n$, $c(\sigma) > 0$, $\sum_{\sigma\in\mathfrak{S}_\infty^\circ} c(\sigma) = 1$, s is a measurable section, and $(s\overline{\nu})\sigma$ is an image measure of $\overline{\nu}$ by a map, $\overline{P} \longmapsto (s(\overline{P}))_\sigma$. Note that for any symmetric measurable function f on $\hat{M}^\infty$,

$$\int_{\hat{M}^\infty} f(\hat{P})\hat{\nu}(d\hat{P}) = \int_{\Gamma_M} f(\overline{P})\overline{\nu}(\overline{P}),$$

where we use a natural identification f with the corresponding function on Γ_M.

Definition 5.1. (1) *A measurable* 1*-cocycle* $\overline{\theta}$ *on* $\Gamma_M \times \mathrm{Diff}_0^*(M)$ *is said to be* $\overline{\nu}$*-irreducible, if for any* $U(H)$*-valued measurable map* V *on* Γ_M *there exists a constant* $k \in \mathbf{C}$ *such that*

$$V(\overline{P}) = k\mathrm{Id}$$

for $\overline{\nu}$*-a.e.*$\overline{P}$*, provided that*

$$V(\overline{P})\overline{\theta}(\overline{P},g) = \overline{\theta}(\overline{P},g)V(\overline{g}^{-1}(\overline{P})) \tag{5.2}$$

for $\overline{\nu}$*-a.e.*$\overline{P}$.
(2) *A parallel definition for a symmetric measurable* 1*-cocycle* $\hat{\theta}$ *is given, in which* $V(\overline{P})$ *and* $\overline{\nu}$ *is replaced by a symmetric measurable map* $V(\hat{P})$ *and* $\hat{\nu}$*, respectively.*

Remark 5.1. (1) *Of course* $\overline{\theta}$ *is* $\overline{\nu}$*-irreducible, if the corresponding* $\hat{\theta}$ *is* $\hat{\nu}$*-irreducible and vice versa.*
(2) *If a* $\overline{\nu}$*-irreducible* 1*-cocycle exists at any rate,* $\overline{\nu}$ *must be* $\mathrm{Diff}_0^*(M)$*-ergodic.*

Theorem 5.1. *Let* $\hat{\theta} \equiv \hat{\theta}(C, H_k)$ *be a strongly Borelian canonical* 1*-cocycle. Then in order that it is* $\hat{\nu}$*-irreducible, it is necessary and sufficient that for any* $U(H)$*-valued map* $A([P])$ *defined on* $\hat{M}^\infty/\approx$ *which satisfies the conditions* (5.3) *and* (5.4) *below, there exists a constant* $k \in \mathbf{C}$ *such that*

$$A([P]) = k\mathrm{Id}$$

for $\hat{\nu}$*-a.e.*$\hat{P}$.

(5.3) A map, $\hat{P} \longmapsto C(\hat{P})^{-1}A([P])C(\hat{P})$ is measurable and it coincides with

a symmetric measurable map $V(\hat{P})$ for $\hat{\nu}$ – a.e.$\hat{P}$,
and

$$^\forall k \in \mathbf{N}, \quad A([P])H_k^{[P]} = H_k^{[P]}A([P]) \tag{5.4}$$

for $\hat{\nu}$*-a.e.*$\hat{P}$*. As before, the necessity requires no condition on* M *but for the sufficiency we assume that* $\hat{M}^n$ *is connected for every* $n \in \mathbf{N}$.

Proof. See [23]. □

Let $\overline{\nu}$ be a $\mathrm{Diff}_0^*(M)$-quasi-invariant measure on $(\Gamma_M, \mathfrak{B})$ and $\overline{\theta}$ be a measurable 1-cocycle on $\Gamma_M \times \mathrm{Diff}_0^*(M)$. Hereafter we consider natural representation $(U_{\overline{\nu},\overline{\theta}}, \mathrm{L}^2_{\overline{\nu}}(\Gamma_M, H))$ made of these factors,

$$U_{\overline{\nu},\overline{\theta}}(g) : f(\overline{P}) \in \mathrm{L}^2_{\overline{\nu}}(\Gamma_M, H) \longmapsto \sqrt{\frac{d\overline{\nu}_{\overline{g}}}{d\overline{\nu}}}(\overline{P})\overline{\theta}(\overline{P},g)f(\overline{g}^{-1}(\overline{P})) \in \mathrm{L}^2_{\overline{\nu}}(\Gamma_M, H). \tag{5.5}$$

As before there correspondes a representation on the set $\hat{\mathrm{L}}^2_{\hat{\nu}}(\hat{M}^\infty, H)$ of all square summable H-valued functions on $\hat{M}^\infty$,

$$(5.6) \quad U_{\hat{\nu},\hat{\theta}}(g) : f(\hat{P}) \in \hat{\mathrm{L}}^2_{\hat{\nu}}(\hat{M}^\infty, H) \longmapsto \sqrt{\frac{d\overline{\nu}_{\overline{g}}}{d\overline{\nu}}}(\overline{P})\hat{\theta}(\hat{P}, g) f(\hat{g}^{-1}(\hat{P})) \in \hat{\mathrm{L}}^2_{\hat{\nu}}(\hat{M}^\infty, H).$$

Theorem 5.2. (*Irreducibility*)
The natural representation given by (5.5), *where we assume that $\overline{\theta}$ is strongly Borelian and canonical, is irreducible if and only if so is $\overline{\theta}$.*

For the proof of this and next theorems also see [23]. □

Theorem 5.3. (*Equivalence*)
Let $\overline{\nu}_i$ $(i = 1, 2)$ be $\mathrm{Diff}_0^*(M)$-quasi-invariant probability measures on $(\Gamma_M, \mathfrak{B})$ and $\overline{\theta}_i$ be strongly Borelian canonical 1-cocycles on $\Gamma_M \times \mathrm{Diff}_0^*(M)$. Then $(U_{\overline{\nu}_1,\overline{\theta}_1}, L^2_{\overline{\nu}_1}(\Gamma_M, H_1))$ and $(U_{\overline{\nu}_2,\overline{\theta}_2}, L^2_{\overline{\nu}_2}(\Gamma_M, H_2))$ are equivalent if and only if

$$(5.7) \qquad \nu_1 \simeq \nu_2,$$

and

$$(5.8) \qquad \overline{\theta}_1 \text{ and } \overline{\theta}_2 \text{ are cohomologous.}$$

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