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Quantum deformations of certain prehomogeneous vector spaces

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0 Notation

Let $\mathfrak{g}$ be a semisimple Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ and $W \subset GL(\mathfrak{h})$ be the root system and the Weyl group respectively. For each $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i\in I_0}$, where I_0 is an index set. Set

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}.$$

For $i \in I_0$ let $h_i \in \mathfrak{h}$, $\varpi_i \in \mathfrak{h}^*$ and $s_i \in W$ be the simple coroot, the fundamental weight, the simple reflection corresponding to i respectively. Take $e_i \in \mathfrak{g}_{\alpha_i}$, and $f_i \in \mathfrak{g}_{-\alpha_i}$, satisfying $[e_i, f_i] = h_i$. Let $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots α. We set

$$d_i = \frac{(\alpha_i, \alpha_i)}{2} \quad (i \in I_0), \qquad a_{ij} = \alpha_j(h_i) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \quad (i, j \in I_0).$$

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For a subset I of I_0 we set

$$\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z}\alpha_i, \qquad W_I = \langle s_i | i \in I \rangle,$$

$$\mathfrak{l}_I = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha), \qquad \mathfrak{n}_I^+ = \oplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_I^- = \oplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_{-\alpha},$$

$$\mathfrak{h}_I^* = \oplus_{i \in I_0 \setminus I} \mathbb{C}\varpi_i \subset \mathfrak{h}^*, \qquad \mathfrak{h}_{I,\mathbb{Z}}^* = \oplus_{i \in I_0 \setminus I} \mathbb{Z}\varpi_i \subset \mathfrak{h}^*.$$

For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the enveloping algebra of $\mathfrak{a}$.

1 Quantized enveloping algebras

The quantized enveloping algebra $U_q(\mathfrak{g})$ ([1], [7]) is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements $\{E_i, F_i, K_i, K_i^{-1}\}_{i \in I_0}$ satisfying the following relations:

$$\begin{aligned}
&K_i K_j = K_j K_i, \\
&K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
&K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \\
&K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \\
&E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0 \quad (i \neq j), \\
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0 \quad (i \neq j),
\end{aligned}$$

where $q_i = q^{d_i}$, and

$$[m]_t = \frac{t^m - t^{-m}}{t - t^{-1}}, \quad [m]_t! = \prod_{k=1}^m [k]_t, \quad \begin{bmatrix} m \\ n \end{bmatrix}_t = \frac{[m]_t!}{[n]_t![m-n]_t!} \quad (m \geq n \geq 0).$$

We define the Hopf algebra structure on $U_q(\mathfrak{g})$ as follows. The comultiplication $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is the algebra homomorphism satisfying

$$\Delta(K_i) = K_i \otimes K_i,\ \ \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i,\ \ \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i.$$

The counit $\epsilon : U_q(\mathfrak{g}) \to \mathbb{C}(q)$ is the algebra homomorphism satisfying

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0.$$

The antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra antiautomorphism satisfying

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i, \quad S(F_i) = -K_i^{-1}F_i.$$

The adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ is defined as follows. For $x, y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $(\mathrm{ad}x)(y) = \sum_k x_k^1 y S(x_k^2)$. Then $\mathrm{ad} : U_q(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$ is an algebra homomorphism.

We define subalgebras $U_q(\mathfrak{n}^\pm)$, $U_q(\mathfrak{h})$ and $U_q(\mathfrak{l}_I)$ for $I \subset I_0$ by

$$U_q(\mathfrak{n}^+) = \langle E_i | i \in I_0 \rangle, \quad U_q(\mathfrak{n}^-) = \langle F_i | i \in I_0 \rangle,$$
$$U_q(\mathfrak{h}) = \langle K_i^\pm | i \in I_0 \rangle, \quad U_q(\mathfrak{l}_I) = \langle K_i^\pm, E_j, F_j | i \in I_0, j \in I \rangle.$$

For $i \in I_0$ we define an algebra automorphism T_i of $U_q(\mathfrak{g})$ (see [8]) by

$$T_i(K_j) = K_j K_i^{-a_{ij}},$$
$$T_i(E_j) = \begin{cases} -F_iK_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)} & (i \neq j), \end{cases}$$
$$T_i(F_j) = \begin{cases} -K_i^{-1}E_i & (i = j) \\ \sum_{k=0}^{-a_{ij}} (-q_i)^{k} F_i^{(k)} F_j F_i^{(-a_{ij}-k)} & (i \neq j), \end{cases}$$

where

$$E_i^{(k)} = \frac{1}{[k]_{q_i}!} E_i^k, \quad F_i^{(k)} = \frac{1}{[k]_{q_i}!} F_i^k.$$

For $w \in W$ we choose a reduced expression $w = s_{i_1} \cdots s_{i_k}$ and set $T_w = T_{i_1} \cdots T_{i_k}$. It is known that T_w dose not depend on the choice of the reduced expression.

For $I \subset I_0$ let w_I be the longest element of W_I and set

$$U_q(\mathfrak{n}_I^-) = U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-).$$

Let w_0 be the longest element of W and take a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_r}$ of $w_I w_0$. We set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad Y_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k})$$

for $k = 1, \ldots, r$. Then it is known that $\{\beta_k | 1 \leq k \leq r\} = \Delta^+ \backslash \Delta_I$, and that $\{Y_{\beta_1}^{d_1} \cdots Y_{\beta_r}^{d_r} | d_1, \ldots d_r \in \mathbb{Z}_{\geq 0}\}$ is a basis of $U_q(\mathfrak{n}_I^-)$. This basis depends on the choice of the reduced expression of $w_I w_0$ in general.

Proposition 1.1 (ad $U_q(\mathfrak{l}_I))(U_q(\mathfrak{n}_I^-)) \subset U_q(\mathfrak{n}_I^-)$.

For $N \in \mathbb{Z}_{>0}$ we set $U_{q,N}(\mathfrak{g}) = \mathbb{C}(q^{1/N}) \otimes_{\mathbb{C}(q)} U_q(\mathfrak{g})$, and let $U_{q,N}(\mathfrak{n}^\pm)$, $U_{q,N}(\mathfrak{h})$, $U_{q,N}(\mathfrak{l}_I)$, $U_{q,N}(\mathfrak{n}_I^-)$ be the $\mathbb{C}(q)$-subalgebras of $U_{q,N}(\mathfrak{g})$ generated by $U_q(\mathfrak{n}^\pm)$, $U_q(\mathfrak{h})$, $U_q(\mathfrak{l}_I)$, $U_q(\mathfrak{n}_I^-)$ respectively.

For $\lambda \in \mathfrak{h}_I^*$ we define a $U(\mathfrak{g})$-module $M_I(\lambda)$ by

$$M_I(\lambda) = U(\mathfrak{g}) / (\sum_{h \in \mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)) + U(\mathfrak{g})\mathfrak{n}^+ + U(\mathfrak{g})(\mathfrak{l} \cap \mathfrak{n}^-)).$$

It is a highest weight module with highest weight λ and highest weight vector $m_{I,\lambda} = \bar{1}$, where $\bar{1}$ denotes the element of $M_I(\lambda)$ corresponding to $1 \in U(\mathfrak{g})$. $M_I(\lambda)$ contains a unique maximal proper submodule $K_I(\lambda)$, and $L(\lambda) = M_I(\lambda)/K_I(\lambda)$ is a unique (up to an isomorphism) irreducible highest weight module with highest weight λ.

For $\lambda \in \mathfrak{h}_{I,\mathbb{Z}}^*/N$ we define a $U(\mathfrak{g})$-module $M_I(\lambda)$ by

$$M_{I,q,N}(\lambda) = U_{q,N}(\mathfrak{g}) / (\sum_{i \in I_0} U_{q,N}(\mathfrak{g})(K_i - q_i^{\lambda(h_i)}) + \sum_{i \in I_0} U_{q,N}(\mathfrak{g}) E_i + \sum_{j \in I} U_{q,N}(\mathfrak{g}) F_j).$$

It is a highest weight module with highest weight λ and highest weight vector $m_{I,\lambda,q,N} = \bar{1}$. $M_I(\lambda)$ contains a unique maximal proper submodule $K_{I,q.N}(\lambda)$, and $L_{q,N}(\lambda) = M_{I,q,N}(\lambda)/K_{I,q,N}(\lambda)$ is a unique irreducible highest weight module with highest weight λ.

2 Main result

In the rest of this note we fix $I \subset I_0$ satisfying $\mathfrak{n}_I^+ \neq \{0\}$ and $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. This is equivalent to the following condition:

$$I = I_0 \backslash \{i_0\} \text{ with } m_{i_0} = 1,$$

where $\theta = \sum_{i \in I_0} m_i \alpha_i$ is the highest root (see [14]).

We set $\mathfrak{l} = \mathfrak{l}_I, \mathfrak{m}^\pm = \mathfrak{n}_I^\pm$ for simplicity.

Proposition 2.1 *The element $Y_\beta \in U_q(\mathfrak{m}^-)$ for $\beta \in \Delta^+ \backslash \Delta_I$ dose not depend on the choice of a reduced expression of $w_I w_0$.*

Fix a reduced expression $w_I w_0 = s_{i_1} \dots s_{i_r}$ amd set $\beta_p = s_{i_1} \dots s_{i_{p-1}}(\alpha_{i_p})$. We set

$$U_q(\mathfrak{m}^-)^m = \sum_{p_1,\dots,p_m=1}^{r} \mathbb{C}(q) Y_{\beta_{p_1}} \cdots Y_{\beta_{p_m}} \qquad (m \geq 0).$$

Lemma 2.2 *We have*

$$U_q(\mathfrak{m}^-) = \bigoplus_{m=0}^{\infty} U_q(\mathfrak{m}^-)^m,$$

$$U_q(\mathfrak{m}^-)^m = \bigoplus_{\sum_p m_p = m} \mathbb{C}(q) Y_{\beta_1}^{m_1} \cdots Y_{\beta_r}^{m_r} = \bigoplus_{\gamma \in m\alpha_{i_0} + Q_I^+} U_q(\mathfrak{m}^-)_{-\gamma}.$$

Here $U_q(\mathfrak{m}^-)_{-\gamma}$ is the weight space with respect to the adjoint action of $U_q(\mathfrak{h})$ on $U_q(\mathfrak{m}^-)$, and $Q_I^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$.

By Lemmma 2.2 we can write

$$Y_{\beta_{p_1}} Y_{\beta_{p_2}} = \sum_{\substack{s_1 \le s_2 \\ \beta_{p_1}+\beta_{p_2}=\beta_{s_1}+\beta_{s_2}}} a^{p_1,p_2}_{s_1,s_2} Y_{\beta_{s_1}} Y_{\beta_{s_2}} \qquad (a^{p_1,p_2}_{s_1,s_2} \in \mathbb{C}(q)) \tag{1}$$

for $p_1 > p_2$.

Proposition 2.3 *The $\mathbb{C}(q)$-algebra $U_q(\mathfrak{m}^-)$ is generated by the elements $\{Y_{\beta_p} | 1 \le p \le r\}$ satisfying the fundamental relations* (1) *for $p_1 > p_2$.*

By the commutativity of $\mathfrak{m}^-$, $U(\mathfrak{m}^-)$ is isomorphic to the symmetric algebra $S(\mathfrak{m}^-)$. Since $\mathfrak{m}^-$ is identified with $(\mathfrak{m}^+)^*$ via the Killing form of $\mathfrak{g}$, $S(\mathfrak{m}^-)$ is isomorphic to the algebra $\mathbb{C}[\mathfrak{m}^+]$ of polynomial functions on $\mathfrak{m}^+$. Hence we have an identification $U(\mathfrak{m}^-) = \mathbb{C}[\mathfrak{m}^+]$. We denote by $\mathbb{C}[\mathfrak{m}^+]^m$ ($m \in \mathbb{Z}_{\ge 0}$) the subspace of $\mathbb{C}[\mathfrak{m}^+]$ consisting of homogeneous elements with degree m. We set $\mathfrak{h}^*_{\mathbb{Z}}(I,+) = \{\lambda \in \mathfrak{h}^* | \lambda(h_{i_0}) \in \mathbb{Z}, \lambda(h_i) \in \mathbb{Z}_{\ge 0} \ (i \in I)\}$. For $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}(I,+)$ we denote the finite dimensional irreducible $U(\mathfrak{l})$-module (resp. $U_q(\mathfrak{l})$-module) with highest weight λ by $V(\lambda)$ (resp. $V_q(\lambda)$). We can decompose the finite dimensional $\mathfrak{l}$-module $\mathbb{C}[\mathfrak{m}^+]^m$ into a direct sum of submodules isomorphic to $V(\lambda)$ for some $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}(I,+)$. It is known that

$$\mathbb{C}[\mathfrak{m}^+] \simeq \bigoplus_{\lambda \in \Gamma^m} V(\lambda)$$

for finite subset Γ^m of $\mathfrak{h}^*_{\mathbb{Z}}(I,+)$ satisfying $\Gamma^m \cap \Gamma^{m'} = \emptyset$ for $m \ne m'$ (see [11], [12], [6]). On the other hand, since $U_q(\mathfrak{m}^-)^m$ is a finite dimensional $U_q(\mathfrak{l})$-module whose character is the same as that of $\mathbb{C}[\mathfrak{m}^+]^m$, we have

$$U_q(\mathfrak{m}^-)^m \simeq \bigoplus_{\lambda \in \Gamma^m} V_q(\lambda).$$

Let L be the algebraic group corresponding to $\mathfrak{l}$. It is known that $\mathfrak{m}^+$ consists of finitely many L-orbits, and that the orbits can be labeled by

$$\{L\text{-orbits on } \mathfrak{m}^+\} = \{C_0, C_1, \ldots, C_t\}, \quad \{0\} = C_0 \subset \overline{C_1} \subset \cdots \subset \overline{C_t} = \mathfrak{m}^+.$$

We set

$$\mathcal{I}(\overline{C_p}) = \{f \in \mathbb{C}[\mathfrak{m}^+] \mid f(\overline{C_p}) = 0\}.$$

Since $\mathcal{I}(\overline{C_p})$ is an $\mathfrak{l}$-submodule of $\mathbb{C}[\mathfrak{m}^+]$, we have

$$\mathcal{I}(\overline{C_p}) = \bigoplus_m \mathcal{I}^m(\overline{C_p}), \quad \mathcal{I}^m(\overline{C_p}) = \mathcal{I}(\overline{C_p}) \cap \mathbb{C}[\mathfrak{m}^+]^m \simeq \bigoplus_{\lambda \in \Gamma_p^m} V(\lambda)$$

for a subset Γ_p^m of Γ^m. The following facts are known (see, for example, [14]):

Proposition 2.4 *Let $p = 0, \dots, t-1$.*

(i) $\mathcal{I}^m(\overline{C_p}) = 0$ *for $m \leq p$.*

(ii) $\mathcal{I}^{p+1}(\overline{C_p})$ *is an irreducible $\mathfrak{l}$-module.*

(iii) $\mathcal{I}(\overline{C_p})$ *is generated by $\mathcal{I}^{p+1}(\overline{C_p})$ as an ideal of $\mathbb{C}[\mathfrak{m}^+]$.*

Proposition 2.5 *For $p = 0, \dots, t-1$ there exists a unique $\lambda_p \in \mathfrak{h}_I^*$ such that $K_I(\lambda_p) = \mathcal{I}(\overline{C_p}) m_{I,\lambda_p}$. Moreover,we have $\lambda_p \in \mathfrak{h}_{I,\mathbb{Z}}^*/2$.*

We set

$$\begin{aligned}
\mathcal{I}_q^m(\overline{C_p}) &= \bigoplus_{\lambda \in \Gamma_p^m} V_q(\lambda) \subset U_q(\mathfrak{m}^-)^m, \\
\mathcal{I}_q(\overline{C_p}) &= \bigoplus_m \mathcal{I}_q^m(\overline{C_p}) \subset U_q(\mathfrak{m}^-), \\
\mathcal{I}_{q,N}^m(\overline{C_p}) &= \mathbb{C}(q^{1/N}) \otimes_{\mathbb{C}(q)} \mathcal{I}_q^m(\overline{C_p}) \subset U_{q,N}(\mathfrak{m}^-)^m, \\
\mathcal{I}_{q,N}(\overline{C_p}) &= \bigoplus_m \mathcal{I}_{q,N}^m(\overline{C_p}) \subset U_{q,N}(\mathfrak{m}^-).
\end{aligned}$$

Here we identify $U_q(\mathfrak{m}^-)^m$ with $\bigoplus_{\lambda \in \Gamma^m} V_q(\lambda)$.

Proposition 2.6 ([15]) For $p = 0, \dots, t-1$ we have

$$\mathrm{ch}(L_{q,2}(\lambda_p)) = \mathrm{ch}(L(\lambda_p)), \quad K_{I,q,2}(\lambda_p) = U_{q,2}(\mathfrak{m}^-) \mathcal{I}_{q,2}^{p+1}(\overline{C_p}) m_{I,\lambda_p,q,2}.$$

By Proposition 2.6 we have the main result.

Theorem 2.7 ([15]) *We have*

$$\mathcal{I}_q(\overline{C_p}) = U_q(\mathfrak{m}^-)\mathcal{I}_q^{p+1}(\overline{C_p}) = \mathcal{I}_q^{p+1}(\overline{C_p})U_q(\mathfrak{m}^-)$$

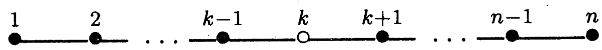
for $p = 0, \ldots, t-1$.

3 Examples

We shall give an explicit description of $\mathcal{I}_q^{p+1}(\overline{C_p})$ in each individual case. (see [16], [17])

3.1 Type A_n

We label the vertices of the Dynkin diagram as follows.



Hence we have $I_0 = \{1, \ldots, n\}$. Set $I = I_0 \backslash \{i_0\}$, where $i_0 = k$ $(k-1 \leq n-k)$. We fix a reduced expression

$$w_I w_0 = (s_k s_{k+1} \cdots s_n)(s_{k-1} s_k \cdots s_{n-1}) \cdots (s_1 s_2 \cdots s_{n-k+1}).$$

We set

$$Y_{i,j} = (-1)^{k-i}(T_k T_{k+1} \cdots T_n)(T_{k-1} T_k \cdots T_{n-1}) \cdots (T_{i+1} T_{i+2} \cdots T_{n-k+i+1}) \\ T_i T_{i+1} \cdots T_{i+j-2}(F_{i+j-1}) \\ (1 \leq i \leq k,\ 1 \leq j \leq n+1-k).$$

Set

$$\beta_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k + \cdots + \alpha_{k+j}.$$

We have $Y_{i,j} \in U_q(\mathfrak{m}^-)_{-\beta_{i,j}}$.

Then we have the following fundamental relations of $U_q(\mathfrak{m}^-)$.

$$Y_{i,j}Y_{l,m} = \begin{cases} qY_{l,m}Y_{i,j} & (i = l, j < m \text{ or } i > l, j = m) \\ Y_{l,m}Y_{i,j} & (i > l, j > m) \\ Y_{l,m}Y_{i,j} + (q - q^{-1})Y_{i,m}Y_{l,j} & (i > l, j < m). \end{cases}$$

We label $k+1$ L-orbits on $\mathfrak{m}^+$ as in Section 2. For $p = 0, 1, \ldots, k-1$ we have

$$\mathcal{I}_q^{p+1}(\overline{C_p}) = \sum \mathbb{C}(q) \begin{pmatrix} i_1 & i_2 & \cdots & i_{p+1} \\ j_1 & j_2 & \cdots & j_{p+1} \end{pmatrix}$$

where we sum over all the sequences $\{i_1, i_2, \ldots, i_{p+1}\}$, $\{j_1, j_2, \ldots, j_{p+1}\} \subset \mathbb{N}$ satisfying

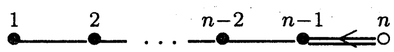
$$1 \le i_1 < i_2 < \cdots < i_{p+1} \le k, \quad 1 \le j_1 < j_2 < \cdots < j_{p+1} \le n+1-k,$$

and set

$$\begin{aligned} \begin{pmatrix} i_1 & i_2 & \cdots & i_{p+1} \\ j_1 & j_2 & \cdots & j_{p+1} \end{pmatrix} &= \sum_{\sigma \in S_{p+1}} (-q)^{l(\sigma)} Y_{i_1, j_{\sigma(1)}} Y_{i_2, j_{\sigma(2)}} \cdots Y_{i_{p+1}, j_{\sigma(p+1)}}, \\ l(\sigma) &= \sharp\{(i,j) | i < j, \sigma(i) > \sigma(j)\}. \end{aligned}$$

3.2 Type C_n

We label the vertices of the Dynkin diagram as follows.



Hence we have $I_0 = \{1, \ldots, n\}$. Set $I = I_0 \backslash \{i_0\}$, where $i_0 = n$. We fix a reduced expression

$$w_I w_0 = (s_n s_{n-1} \cdots s_1)(s_n s_{n-1} \cdots s_2) \ldots (s_n s_{n-1}) s_n.$$

We set

$$Y_{i,j} = c_{i,j}(T_nT_{n-1}\cdots T_1)(T_nT_{n-1}\cdots T_2)\cdots(T_nT_{n-1}\cdots T_{n-j}) \\ T_nT_{n-1}\cdots T_{n-j+i+1}(F_{n-j+i}) \\ (1 \le i \le j \le n),$$

where

$$c_{i,j} = \begin{cases} (q+q^{-1}) & (1 \le i = j \le n) \\ (-1)^{j-i} & (1 \le i < j \le n). \end{cases}$$

Set

$$\beta_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n.$$

We have $Y_{i,j} \in U_q(\mathfrak{m}^-)_{-\beta_{i,j}}$.

Then we have the following fundamental relations of $U_q(\mathfrak{m}^-)$.

$$Y_{i,j}Y_{l,m} = \begin{cases} q_{n-j+i}Y_{l,m}Y_{i,j} & (j = m, i > l) \\ Y_{l,m}Y_{i,j} & (j > m, i < l) \\ q_{n-m+l}Y_{l,m}Y_{i,j} & (j > m, i = l) \\ Y_{l,m}Y_{i,j} + (q-q^{-1})Y_{i,m}Y_{l,j} & (l < i < m < j) \\ qY_{l,m}Y_{i,j} + (q-q^{-1})Y_{i,m}Y_{l,j} & (l < i = m < j) \\ Y_{l,m}Y_{i,j} + (q^2-q^{-2})Y_{m,i}Y_{l,j} & (l = m < i < j) \\ Y_{l,m}Y_{i,j} + (q-q^{-1})\{Y_{l,i}Y_{m,j} - qY_{m,i}Y_{l,j}\} & (l < m < i < j) \\ Y_{l,m}Y_{i,j} + q^{-1}(q^2-q^{-2}){Y_{l,j}}^2 & (l = m < i = j) \\ Y_{l,m}Y_{i,j} + (q^2-q^{-2})Y_{l,j}Y_{m,i} & (l < m < i = j) \end{cases}$$

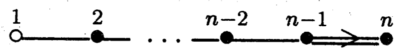
We label $n+1$ L-orbits on $\mathfrak{m}^+$ as in Section 2. For $p = 0, 1, \ldots, n-1$ the highest weight vector of $\mathcal{I}_q^{p+1}(\overline{C_p})$ is

$$\sum_{\sigma \in S_{p+1}} (-q^{-1})^{l(\sigma)} Y_{i_1, j_{\sigma(1)}} Y_{i_2, j_{\sigma(2)}} \cdots Y_{i_{p+1}, j_{\sigma(p+1)}}$$

where $i_1 = j_1 = n-p$, $i_2 = j_2 = n-p+1$, $\ldots$, $i_{p+1} = j_{p+1} = n$ and $Y_{j,i} = q^{-2}Y_{i,j}$ $(i < j)$.

3.3 Type B_n

We label the vertices of the Dynkin diagram as follows.



Hence we have $I_0 = \{1, \ldots, n\}$. Set $I = I_0 \backslash \{i_0\}$, where $i_0 = 1$. We fix a reduced expression

$$w_I w_0 = s_1 s_2 \cdots s_{n-1} s_n s_{n-1} s_{n-2} \cdots s_2 s_1.$$

We set

$$Y_i = \begin{cases} T_1 T_2 \cdots T_{i-1}(F_i) & (1 \leq i \leq n) \\ T_1 T_2 \cdots T_{n-1} T_n T_{n-1} T_{n-2} \cdots T_{2n-i+1}(F_{2n-i}) & (n+1 \leq i \leq 2n-1). \end{cases}$$

Then we have the following fundamental relations of $U_q(\mathfrak{m}^-)$.

$$Y_i Y_j = \begin{cases} q^{-2} Y_j Y_i & (i > j, i+j \neq 2n) \\ Y_j Y_i + \frac{q^{-2}-1}{q+q^{-1}} {Y_n}^2 & (i = n+1, j = n-1) \\ Y_j Y_i + (q^{-2} - q^2) \sum_{l=1}^{i-n-1} (-q^2)^{l-1} Y_{j+l} Y_{i-l} & \\ \qquad -(-q^2)^{i-n-1} \frac{q^{-2}-1}{q+q^{-1}} {Y_n}^2 & (j \leq n-2, i+j = 2n) \end{cases}$$

We label 3 L-orbits on $\mathfrak{m}^+$ as in Section 2. For $p = 0, 1$ we have

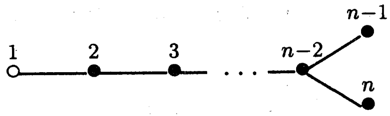
$$\mathcal{I}_q^1(\overline{C_0}) = \sum_{i=1}^{2n-1} \mathbb{C}(q) Y_i,$$
$$\mathcal{I}_q^2(\overline{C_1}) = \mathbb{C}(q)\psi$$

where $\psi = Y_n Y_n - (q+q^{-1})(1+q^{-2}) \sum_{i=1}^{n-1} (-q^{-2})^{i-1} Y_{n-i} Y_{n+i}$.

3.4 Type D_n

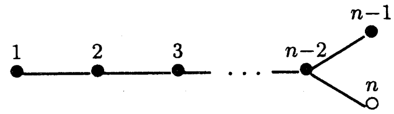
We have the following two cases.

Case 1



$$I_0 = \{1, \ldots, n\}, i_0 = 1$$

Case 2



$$I_0 = \{1, \ldots, n\}, i_0 = n$$

In case 1 we fix a reduced expression

$$w_I w_0 = s_1 s_2 \cdots s_{n-1} s_n s_{n-2} s_{n-3} \cdots s_2 s_1.$$

Set

$$Y_i = \begin{cases} T_1 T_2 \cdots T_{i-1}(F_i) & (1 \leq i \leq n) \\ T_1 T_2 \cdots T_{n-1} T_n T_{n-2} T_{n-3} \cdots T_{2n-i}(F_{2n-i-1}) & (n+1 \leq i \leq 2n-2). \end{cases}$$

Then we have the following fundamental relations of $U_q(\mathfrak{m}^-)$.

$$Y_i Y_j = \begin{cases} q^{-1} Y_j Y_i & (i > j, i+j \neq 2n-1) \\ Y_j Y_i & (i = n, j = n-1) \\ Y_j Y_i - (q - q^{-1}) \sum_{l=1}^{i-n} (-q)^{l-1} Y_{j+l} Y_{i-l} & (j \leq n-2, i+j = 2n-1) \end{cases}$$

We label 3 L-orbits on $\mathfrak{m}^+$ as in Section 2. For $p = 0, 1$ we have

$$\mathcal{I}_q^1(\overline{C_0}) = \sum_{i=1}^{2n-2} \mathbb{C}(q) Y_i,$$

$$\mathcal{I}_q^2(\overline{C_1}) = \mathbb{C}(q)\psi$$

where $\psi=\sum_{i=1}^{n-1}(-q^{-1})^{i-1}Y_{n-i}Y_{n+i-1}$.

In case 2 we fix a reduced expression

$$w_I w_0=(s_{\tau(1)}s_{n-2}\cdots s_1)(s_{\tau(2)}s_{n-2}\cdots s_2)\cdots(s_{\tau(n-2)}s_{n-2})s_{\tau(n-1)},$$

where

$$\tau(i)=\begin{cases} n & (i: odd) \\ n-1 & (i: even). \end{cases}$$

We set

$$Y_{i,j}=(-1)^{i+j-1}(T_{\tau(1)}T_{n-2}\cdots T_1)(T_{\tau(2)}T_{n-2}\cdots T_2)\cdots(T_{\tau(n-j)}T_{n-2}\cdots T_{n-j})$$
$$T_{\tau(n-j+1)}T_{n-2}\cdots T_{n-j+i+1}(F_{n-j+i})$$
$$(1\leq i<j\leq n).$$

We have $Y_{i,j}\in U_q(\mathfrak{m}^-)_{-\beta_{i,j}}$,where

$$\beta_{i,j}=\begin{cases} \alpha_i+\alpha_{i+1}+\cdots+\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n & (j\leq n-1) \\ \alpha_i+\alpha_{i+1}+\cdots+\alpha_{n-2}+\alpha_n & (j=n). \end{cases}$$

Then we have the following fundamental relations of $U_q(\mathfrak{m}^-)$.

$$Y_{i,j}Y_{l,m}=\begin{cases} qY_{l,m}Y_{i,j} & (l<i<m=j \text{ or} \\ & \ l<i=m<j \text{ or } l=i<m<j) \\ Y_{l,m}Y_{i,j} & (i<l<m<j) \\ Y_{l,m}Y_{i,j}+(q-q^{-1})Y_{i,m}Y_{l,j} & (l<i<m<j) \\ Y_{l,m}Y_{i,j} & \\ \quad +(q-q^{-1})\{Y_{l,i}Y_{m,j}-q^{-1}Y_{m,i}Y_{l,j}\} & (l<m<i<j) \end{cases}$$

We label $[n/2]+1$ L-orbits on $\mathfrak{m}^+$ as in Section 2. For $p = 0, 1, \ldots, [(n-2)/2]$ we have

$$\mathcal{I}_q^{p+1}(\overline{C_p}) = \sum \mathbb{C}(q) \begin{pmatrix} i_1 & i_2 & \ldots & i_{2p+2} \end{pmatrix}$$

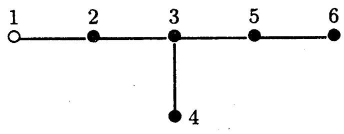
where we sum over all the sequence $\{i_1, i_2, \ldots, i_{2p+2}\} \subset \mathbb{N}$ satisfying $1 \leq i_1 < i_2 < \cdots < i_{2p+2} \leq n$, and set

$$\begin{pmatrix} i_1 & i_2 & \ldots & i_{2p+2} \end{pmatrix} = \sum_{\sigma \in \tilde{S}_{2p+2}} (-q^{-1})^{l(\sigma)} Y_{i_{\sigma(1)}, i_{\sigma(2)}} Y_{i_{\sigma(3)}, i_{\sigma(4)}} \cdots Y_{i_{\sigma(2p+1)}, i_{\sigma(2p+2)}},$$

$$\tilde{S}_{2p+2} = \{\sigma \in S_{2p+2} |\ \sigma(2k-1) < \sigma(2k+1), \sigma(2k-1) < \sigma(2k)\}.$$

3.5 Type E_6

We label the vertices of the Dynkin diagram as follows.



Hence we have $I_0 = \{1, 2, 3, 4, 5, 6\}$. Set $i_0 = 1$, $\Lambda = \{1, 2, \ldots, 16\}$. We fix a reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_3 s_2 s_1 s_6 s_5 s_3 s_2 s_4 s_3 s_5 s_6.$$

and set $Y_i = Y_{\beta_i}$ for $i \in \Lambda$ (see Section 1).

Define $\mathbf{A}(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n) \in \Lambda^8$ $(1 \leq n \leq 10)$ as follows:

$$\mathbf{A}(1) = (1, 2, 3, 4, 5, 6, 7, 8), \quad \mathbf{A}(2) = (1, 2, 3, 4, 9, 10, 11, 12),$$

$$\mathbf{A}(3) = (1, 2, 5, 6, 9, 10, 13, 14), \quad \mathbf{A}(4) = (1, 3, 5, 7, 9, 11, 13, 15),$$

$$\mathbf{A}(5) = (2, 3, 5, 8, 9, 12, 14, 15), \quad \mathbf{A}(6) = (1, 4, 6, 7, 10, 11, 13, 16),$$

$$\mathbf{A}(7) = (2, 4, 6, 8, 10, 12, 14, 16), \quad \mathbf{A}(8) = (3, 4, 7, 8, 11, 12, 15, 16),$$

$$\mathbf{A}(9) = (5, 6, 7, 8, 13, 14, 15, 16), \quad \mathbf{A}(10) = (9, 10, 11, 12, 13, 14, 15, 16).$$

For $1 \le i < j \le 16$ we have the following fundamental relations of $U_q(\mathfrak{m}^-)$.

$$Y_iY_j = \begin{cases} Y_jY_i & \text{if there exist } n \text{ such that } i = i_1^n, j = j_1^n \\ Y_{j_2^n}Y_{i_2^n} + (q - q^{-1})Y_{i_1^n}Y_{j_1^n} & \text{if there exist } n \text{ such that } i = i_2^n, j = j_2^n \\ Y_{j_m^n}Y_{i_m^n} + qY_{j_{m-1}^n}Y_{i_{m-1}^n} - q^{-1}Y_{i_{m-1}^n}Y_{j_{m-1}^n} & \\ \qquad \text{if there exist } n, m = 3, 4 \text{ such that } i = i_m^n, j = j_m^n \\ qY_jY_i & \text{otherwise.} \end{cases}$$

We label 3 L-orbits on $\mathfrak{m}^+$ as in Section 2. For $p = 0, 1$ we have

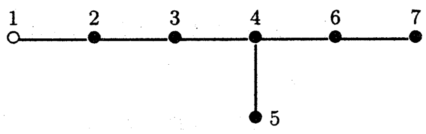
$$\mathcal{I}_q^1(\overline{C_0}) = \sum_{i=1}^{16} \mathbb{C}(q)Y_i,$$

$$\mathcal{I}_q^2(\overline{C_1}) = \sum_{n=1}^{10} \mathbb{C}(q)\psi_n$$

where $\psi_n = Y_{i_4^n}Y_{j_4^n} - qY_{i_3^n}Y_{j_3^n} + q^2Y_{i_2^n}Y_{j_2^n} - q^3Y_{i_1^n}Y_{j_1^n}$.

3.6 Type E_7

We label the vertices of the Dynkin diagram as follows.



Hence we have $I_0 = \{1, 2, 3, 4, 5, 6, 7\}$. Set $i_0 = 1$, $\Lambda = \{1, 2, \ldots, 27\}$. We fix a reduced expression

$$w_Iw_0 = s_1s_2s_3s_4s_5s_6s_4s_3s_2s_1s_7s_6s_4s_3s_5s_4s_6s_7s_2s_3s_4s_6s_5s_4s_3s_2s_1.$$

and set $Y_i = Y_{\beta_i}$ for $i \in \Lambda$ (see Section 1).

Define $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n) \in \Lambda^1 0$ $(1 \leq n \leq 27)$ as follows:

$\mathbf{B}(1) = (10, 19, 20, 21, 23, 22, 24, 25, 26, 27)$, $\mathbf{B}(2) = (9, 14, 16, 17, 23, 18, 24, 25, 26, 27)$,

$\mathbf{B}(3) = (8, 13, 15, 17, 21, 18, 22, 25, 26, 27)$, $\mathbf{B}(4) = (7, 12, 15, 16, 20, 18, 22, 24, 26, 27)$,

$\mathbf{B}(5) = (6, 11, 15, 16, 20, 17, 20, 23, 26, 27)$, $\mathbf{B}(6) = (5, 12, 13, 14, 19, 18, 22, 24, 25, 27)$,

$\mathbf{B}(7) = (4, 11, 13, 14, 19, 17, 21, 23, 25, 27)$, $\mathbf{B}(8) = (3, 11, 12, 14, 19, 16, 20, 23, 24, 27)$,

$\mathbf{B}(9) = (2, 11, 12, 13, 19, 15, 20, 21, 22, 27)$, $\mathbf{B}(10) = (1, 11, 12, 13, 14, 15, 16, 17, 18, 27)$,

$\mathbf{B}(11) = (5, 7, 8, 9, 10, 15, 22, 24, 25, 26)$, $\mathbf{B}(12) = (4, 6, 8, 9, 10, 17, 20, 23, 25, 26)$,

$\mathbf{B}(13) = (3, 6, 7, 9, 10, 16, 20, 23, 24, 26)$, $\mathbf{B}(14) = (2, 6, 7, 8, 10, 17, 21, 23, 25, 26)$,

$\mathbf{B}(15) = (3, 4, 5, 9, 10, 14, 19, 23, 24, 25)$, $\mathbf{B}(16) = (2, 4, 5, 8, 10, 13, 19, 21, 22, 25)$,

$\mathbf{B}(17) = (2, 3, 5, 7, 10, 12, 19, 20, 22, 24)$, $\mathbf{B}(18) = (2, 3, 4, 6, 10, 11, 19, 20, 21, 23)$,

$\mathbf{B}(19) = (1, 6, 7, 8, 9, 15, 16, 17, 18, 26)$, $\mathbf{B}(20) = (1, 4, 5, 8, 9, 13, 14, 17, 18, 25)$,

$\mathbf{B}(21) = (1, 3, 5, 7, 9, 12, 14, 16, 18, 24)$, $\mathbf{B}(22) = (1, 3, 4, 6, 9, 11, 14, 16, 17, 23)$,

$\mathbf{B}(23) = (1, 2, 5, 7, 8, 12, 13, 15, 18, 22)$, $\mathbf{B}(24) = (1, 2, 4, 6, 8, 11, 13, 15, 17, 21)$,

$\mathbf{B}(25) = (1, 2, 3, 6, 7, 11, 12, 15, 16, 20)$, $\mathbf{B}(26) = (1, 2, 3, 4, 5, 11, 12, 13, 14, 19)$,

$\mathbf{B}(27) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.

For $1 \leq i < j \leq 27$ we have the following fundamental relations of $U_q(\mathfrak{m}^-)$.

$$Y_i Y_j = \begin{cases} Y_j Y_i & \text{if there exist } n \text{ such that } \{i, j\} = \{i_1^n, j_1^n\} \\ Y_{j_2^n} Y_{i_2^n} + (q - q^{-1}) Y_{i_1^n} Y_{j_1^n} & \text{if there exist } n \text{ such that } i = i_2^n, j = j_2^n \\ Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} & \\ & \text{if there exist } n, m \in \{3,4,5\} \text{ such that } i = i_m^n, j = j_m^n \\ q Y_j Y_i & \text{otherwise.} \end{cases}$$

Set

$$\psi_n = Y_{i_5^n} Y_{j_5^n} - qY_{i_4^n} Y_{j_4^n} + q^2 Y_{i_3^n} Y_{j_3^n} - q^3 Y_{i_2^n} Y_{j_2^n} + q^4 Y_{i_1^n} Y_{j_1^n},$$
$$\varphi = \sum_{n \in \Lambda} (-q)^{|\beta_n|-1} Y_n \psi_n,$$

where $|\beta| = \sum_{i \in I_0} m_i$ $(\beta = \sum_{i \in I_0} m_i \alpha_i)$.

We label 4 L-orbits on $\mathfrak{m}^+$ as in Section 2. For $p = 0, 1, 2$ we have

$$\mathcal{I}_q^1(\overline{C_0}) = \sum_{i=1}^{27} \mathbb{C}(q) Y_i,$$
$$\mathcal{I}_q^2(\overline{C_1}) = \sum_{n \in \Lambda} \mathbb{C}(q) \psi_n$$
$$\mathcal{I}_q^3(\overline{C_2}) = \mathbb{C}(q) \varphi.$$

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