数理解析研究所講究録
1083 巻 1999 年 154-165

Necessary and sufficient condition for global stability of a Lotka-Volterra system with two delays

大阪府立大学工学部 齋藤 保久 (Yasuhisa Saito)

1. Introduction

We consider the following symmetrical Lotka-Volterra type predator-prey system with two delays τ_1 and τ_2

$$\begin{cases} x'(t) = x(t)[r_1 + ax(t) + \alpha x(t-\tau_1) - \beta y(t-\tau_2)] \\ y'(t) = y(t)[r_2 + ay(t) + \beta x(t-\tau_1) + \alpha y(t-\tau_2)]. \end{cases} \tag{1}$$

The initial condition of (1) is given as

$$\begin{cases} x(s) = \phi(s) \geq 0, -\tau_1 \leq s \leq 0 \ ; \phi(0) > 0 \\ y(s) = \psi(s) \geq 0, -\tau_2 \leq s \leq 0 \ ; \psi(0) > 0. \end{cases} \tag{2}$$

Here a, α, β, r_1, r_2, τ_1 and τ_2 are constants with $a < 0$, $\tau_1 \geq 0$ and $\tau_2 \geq 0$, and ϕ, ψ are continuous functions. Obviously, we can take $\beta \geq 0$ without loss of generality. We assume that (1) has a positive equilibrium (x^*, y^*), that is

$$x^* = \frac{-(a+\alpha)r_1 - \beta r_2}{(a+\alpha)^2 + \beta^2} > 0, \quad y^* = \frac{\beta r_1 - (a+\alpha)r_2}{(a+\alpha)^2 + \beta^2} > 0.$$

The positive equilibrium (x^*, y^*) is said to be globally asymptotically stable if (x^*, y^*) is stable and attracts any solution of (1) with (2). Our purpose is to seek a sharp condition for the global asymptotic stability of (x^*, y^*) for all τ_1 and τ_2, making the best use of the symmetry of (1). In this paper we give the following necessary and sufficient condition for the global asymptotic stability of (x^*, y^*) for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$,

Theorem. *The positive equilibrium (x^*, y^*) of (1) is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ if and only if*

$$\sqrt{\alpha^2 + \beta^2} \leq -a$$

holds.

Gopalsamy [2] showed that if $|\alpha| + |\beta| < -a$ holds, then the positive equilibrium (x^*, y^*) is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$. It is clear that Theorem improves the Gopalsamy's condition for (1). Recently, Lu and Wang [7] also considered the global asymptotic stability of (x^*, y^*) for (1) with $\alpha = 0$.

When the system (1) has no delay, that is $\tau_1 = \tau_2 = 0$, it is easy to see that (x^*, y^*) is globally asymptotically stable if and only if $a + \alpha < 0$ [cf. Appendix]. So we can see that the condition $\sqrt{\alpha^2 + \beta^2} \leq -a$ in Theorem reflects the delay effects.

In the proof of the sufficiency of Theorem, we use an extended LaSalle's invariance principle (also see [8] and [9] for ODE), by which our proof is more complete than that in [7].

2. Proof of Theorem

In order to consider the global asymptotic stability of the positive equilibrium (x^*, y^*) of (1), we first introduce an extention of the LaSalle's invariance principle.

For some constant $\Delta > 0$, let $C^n = C([-\Delta, 0], R^n)$. Consider the delay differential equations

$$z'(t) = f(z_t) \tag{3}$$

where $z_t \in C^n$ is defined as $z_t(\theta) = z(t+\theta)$ for $-\Delta \leq \theta \leq 0$, $f : C^n \to R^n$ is completely continuous, and solutions of (3) are continuously dependent on the initial data in C^n. The following lemma is actually a corollary of LaSalle invariance principle and the proof is omitted. (see, for example, [4, 5]).

Lemma. *Assume that for a subset G of C^n and $V : G \to R$,*

(i) V is continuous on G.

(ii) For any $\phi \in \partial G$ (the boundary of G), the limit $l(\phi)$

$$l(\phi) = \lim_{\substack{\psi \to \phi \\ \psi \in G}} V(\psi)$$

exists or is $+\infty$.

(iii) $\dot{V}_{(3)} \leq 0$ on G, where $\dot{V}_{(3)}$ is the upper right-hand derivative of V along the solution of (3).

Let $E = \{\phi \in \bar{G} \mid l(\phi) < \infty$ and $\dot{V}(\phi) = 0\}$ and M denote the largest subset in E that is invariant with respect to (3). Then every bounded solution of (3) that remains in G approches M as $t \to +\infty$.

Proof of Theorem.

(*Sufficiency.*) By using the transformation

$$\bar{x} = x - x^*, \quad \bar{y} = y - y^*,$$

the system (1) is reduced to

$$\begin{cases} x'(t) = (x^* + x(t))[ax(t) + \alpha x(t-\tau_1) - \beta y(t-\tau_2)] \\ y'(t) = (y^* + y(t))[ay(t) + \beta x(t-\tau_1) + \alpha y(t-\tau_2)] \end{cases} \tag{4}$$

where we used $x(t)$ and $y(t)$ again instead of $\bar{x}(t)$ and $\bar{y}(t)$ respectively. Define

$$G = \left\{\phi = (\phi_1, \phi_2) \in C^2 \mid \phi_i(s) + x_i^* \geq 0, \phi_i(0) + x_i^* > 0, i = 1, 2\right\}$$

where $C^2 = C([-\Delta, 0], R^2)$, $\Delta = \max\{\tau_1, \tau_2\}$ and $(x_1^*, x_2^*) = (x^*, y^*)$. We consider the functional V defined on G,

$$V(\phi) = -2a \sum_{i=1}^{2} \left\{\phi_i(0) - x_i^* \log \frac{\phi_i(0) + x_i^*}{x_i^*}\right\} + (\alpha^2 + \beta^2) \sum_{i=1}^{2} \int_{-\tau_i}^{0} \phi_i^2(\theta) d\theta. \tag{5}$$

It is clear that V is continuous on G and that

$$\lim_{\substack{\psi \to \phi \in \partial G \\ \psi \in G}} V(\psi) = +\infty.$$

Furthermore,

$$\begin{aligned} \dot{V}_{(4)}(\phi) = & -2a\left[a\phi_1(0) + \alpha\phi_1(-\tau_1) - \beta\phi_2(-\tau_2)\right]\phi_1(0) \\ & -2a\left[a\phi_2(0) + \beta\phi_1(-\tau_1) + \alpha\phi_2(-\tau_2)\right]\phi_2(0) \\ & + (\alpha^2 + \beta^2)\left\{\left[\phi_1^2(0) - \phi_1^2(-\tau_1)\right] + \left[\phi_2^2(0) - \phi_2^2(-\tau_2)\right]\right\} \\ = & -\left[a\phi_1(0) + \alpha\phi_1(-\tau_1) - \beta\phi_2(-\tau_2)\right]^2 \\ & -\left[a\phi_2(0) + \beta\phi_1(-\tau_1) + \alpha\phi_2(-\tau_2)\right]^2 \\ & -\left[a^2 - (\alpha^2 + \beta^2)\right]\left[\phi_1^2(0) + \phi_2^2(0)\right] \leq 0 \end{aligned} \tag{6}$$

on G. From (5) and (6), we see that the trivial solution of (4) is stable and that every solution is bounded.

Let

$E = \{\phi \in \bar{G} \mid l(\phi) < \infty \text{ and } \dot{V}(\phi) = 0\}$,

M : the largest subset in E that is invariant with respect to (4).

For $\phi \in M$, the solution $z_t(\phi) = (x(t+\theta), y(t+\theta))$ $(-\Delta \leq \theta \leq 0)$ of (4) through $(0, \phi)$ remains in M for $t \geq 0$ and satisfies for $t \geq 0$,

$$\dot{V}_{(4)}(z_t(\phi)) = 0.$$

Hence, for $t \geq 0$,

$$\begin{cases} ax(t) + \alpha x(t-\tau_1) - \beta y(t-\tau_2) = 0 \\ ay(t) + \beta x(t-\tau_1) + \alpha y(t-\tau_2) = 0, \end{cases} \tag{7}$$

which implies that for $t \geq 0$,

$$x'(t) = y'(t) = 0.$$

Thus, for $t \geq 0$,

$$x(t) = c_1, \quad y(t) = c_2 \tag{8}$$

for some constants c_1 and c_2. From (7) and (8), we have

$$\begin{bmatrix} a+\alpha & -\beta \\ \beta & a+\alpha \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $c_1 = c_2 = 0$ by our assumptions and thus we have

$$x(t) = y(t) = 0 \quad \text{for } t \geq 0.$$

Therefore, for any $\phi \in M$, we have

$$\phi(0) = (x(0), y(0)) = 0.$$

By Lemma, any solution $z_t = (x(t+\theta), y(t+\theta))$ tends to M. Thus

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} y(t) = 0.$$

Hence, (x^*, y^*) is globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$.

(*Necessity.*) The proof is by contradiction. Assume the assertion were false. That is, let (x^*, y^*) be globally asymptotically stable for all $\tau_1 \geq 0$ and $\tau_2 \geq 0$ and $\sqrt{\alpha^2+\beta^2} > -a$.

Linearizing (4), we have

$$\begin{cases} x'(t) = x^*[ax(t) + \alpha x(t-\tau_1) - \beta y(t-\tau_2)] \\ y'(t) = y^*[ay(t) + \beta x(t-\tau_1) + \alpha y(t-\tau_2)]. \end{cases} \tag{9}$$

Now, we will show that there exists a characteristic root λ_0 of (9) such that

$$Re(\lambda_0) > 0 \tag{10}$$

for some τ_1 and τ_2, which implies that the trivial solution of (4) is not stable (see [1, p.160, 161]).

When $\alpha \geq -a$, (x^*, y^*) is not globally asymptotically stable in case $\tau_1 = \tau_2 = 0$ [cf. Appendix]. Therefore, we have only to consider the case $\alpha < -a$.

(I) The case $0 < |\alpha| < -a$.
Let $\tau_1 = \tau_2 = \tau$, then the characteristic equation of (9) takes the form

$$\lambda^2 + p\lambda + q + (r + s\lambda)e^{-\lambda\tau} + ve^{-2\lambda\tau} = 0 \tag{11}$$

where $p = -a(x^*+y^*)$, $q = a^2x^*y^*$, $r = 2a\alpha x^*y^*$, $s = -\alpha(x^*+y^*)$ and $v = (\alpha^2+\beta^2)x^*y^*$.

When $x^* = y^*$, (11) can be factorized as

$$\left[\lambda - x^*\{a + (\alpha + i\beta)e^{-\lambda\tau}\}\right]\left[\lambda - x^*\{a + (\alpha - i\beta)e^{-\lambda\tau}\}\right] = 0. \tag{12}$$

Let us consider the equation

$$\lambda - x^*\{a + (\alpha + i\beta)e^{-\lambda\tau}\} = 0. \tag{13}$$

Set $\alpha = b\cos\theta$ and $\beta = b\sin\theta$, where b and θ are constants with $b \geq 0$. Then, we note that $b > 0$ because of $a < 0$ and $\sqrt{\alpha^2 + \beta^2} > -a$. Substituting $\lambda = iy$ into (13), we have

$$iy - x^*\left[a + b\{\cos(y\tau - \theta) - i\sin(y\tau - \theta)\}\right] = 0. \tag{14}$$

By separating the real and imaginary parts of (14), we obtain

$$\begin{cases} bx^*\cos(y\tau - \theta) = -ax^* \\ bx^*\sin(y\tau - \theta) = -y. \end{cases} \tag{15}$$

From (15), we have

$$(bx^*)^2 = (ax^*)^2 + y^2.$$

In order to solve y in (15), define the following function

$$f_1(Y) = Y + (ax^*)^2 - (bx^*)^2 \tag{16}$$

where $Y = y^2$. Then f_1 is an increasing linear function and

$$f_1(0) = x^{*2}\{a^2 - (\alpha^2 + \beta^2)\} < 0.$$

Thus, it follows that there exists a positive root Y_0 of $f_1(Y) = 0$. Substituting y_0, which satisfies $Y_0 = y_0^2$, into (15), we can get τ_0 such that (13) has a characteristic root iy_0 when $\tau = \tau_0$.

Furthermore, taking the derivative of λ with τ on (13), we have

$$\frac{d\lambda}{d\tau} = \frac{-x^*be^{i\theta}\lambda e^{-\lambda\tau}}{1 + x^*b\tau e^{i\theta}e^{-\lambda\tau}}.$$

Using (13), we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1}{-\lambda(\lambda - x^*a)} - \frac{\tau}{\lambda}.$$

Hence,

$$\operatorname{sign}\left[Re\left(\frac{d\lambda}{d\tau}\bigg|_{\lambda=iy_0,\,\tau=\tau_0}\right)\right]=\operatorname{sign}\left[Re\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\lambda=iy_0,\,\tau=\tau_0}\right)\right]$$
$$=\operatorname{sign}\left[Re\left(\frac{1}{-iy_0(iy_0-x^*a)}-\frac{\tau_0}{iy_0}\right)\right]=\operatorname{sign}\left[Re\left(\frac{1}{y_0^2+iy_0x^*a}\right)\right]>0,$$

which implies that (10) holds. Therefore, the trivial solution of (4) is not stable, that is, (x^*, y^*) is not stable near τ_0, which is a contradiction.

When $x^* \neq y^*$, (11) cannot be factorized as (12). Substituting $\lambda = iy$ into (11), we have

$$(-y^2+piy+q)e^{iy\tau}+r+siy+ve^{-iy\tau}=0. \tag{17}$$

By separating the real and imaginary parts of (17), we have

$$\begin{cases}[(-y^2+q)^2-v^2+p^2y^2]\cos(y\tau)=(r-sp)y^2-r(q-v)\\ [(-y^2+q)^2-v^2+p^2y^2]\sin(y\tau)=sy^3+[rp-s(q+v)]y\end{cases} \tag{18}$$

and thus

$$\left[(-y^2+q)^2-v^2+p^2y^2\right]^2=\left[(r-sp)y^2-r(q-v)\right]^2+\left[sy^3+[rp-s(q+v)]y\right]^2.$$

Define the following function

$$\begin{aligned}f_2(Y)=&[(-Y+q)^2-v^2+p^2Y]^2-[(r-sp)Y-r(q-v)]^2\\&-Y[sY+rp-s(q+v)]^2\end{aligned} \tag{19}$$

where $Y=y^2$, then f_2 is a quartic fuction such that $f_2 \to +\infty$ as $|Y| \to +\infty$. Since

$$f_2(0)=[a^2-(\alpha^2+\beta^2)]^2[(a+\alpha)^2+\beta^2][(a-\alpha)^2+\beta^2](x^*y^*)^4>0,$$

we cannot immediately find positive zeros of (19) and so we have to investigate f_2 in more detail. Define

$$\begin{aligned}F(Y)&=[(-Y+q)^2-v^2+p^2Y]^2\\ G(y)&=-[(r-sp)Y-r(q-v)]^2\\ H(y)&=-Y[sY+rp-s(q+v)]^2,\end{aligned}$$

then $f_2=F+G+H$. It is easy to see that positive zeroes of F, G and H are mutually different as long as $x^* \neq y^*$. Hence, the value of f_2 at the positive zero of F is negative, which, together with $f_2(0)>0$, implies that there exists a positive root of $f_2(Y)=0$. It is also clear that there exists another positive root of $f_2(Y)=0$ because $f_2 \to +\infty$ as $Y \to +\infty$. Thus, one of the two positive roots is a simple root at least.

Let Y_0 be such a simple root. Substituting y_0, which satisfies $Y_0 = y_0^2$, into (18), we can get some τ such that (11) has a characteristic root iy_0 at τ. We note that iy_0 is a simple root of (11) because Y_0 is a simple root of $f_2(Y) = 0$.

Furthermore, taking the derivative of λ with τ on (11), we have

$$\frac{d\lambda}{d\tau} = \frac{-2\lambda(\lambda^2 + p\lambda + q) - \lambda(r + s\lambda)e^{-\lambda\tau}}{2\lambda + p + 2\tau(\lambda^2 + p\lambda + q) + e^{-\lambda\tau}[s + \tau(r + s\lambda)]},$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p + se^{-\lambda\tau}}{-2\lambda(\lambda^2 + p\lambda + q) - \lambda(r + s\lambda)e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Hence, we have

$$\begin{aligned}
\text{sign}\left[Re\left(\frac{d\lambda}{d\tau}\Big|_{\lambda=iy_0}\right)\right] &= \text{sign}\left[Re\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\lambda=iy_0}\right)\right] \\
&= \text{sign}\left[Re\left(\frac{2iy_0 + p + se^{-iy_0\tau}}{-2iy_0(-y_0^2 + piy_0 + q) - iy_0(r + siy_0)e^{-iy_0\tau}} - \frac{\tau}{iy_0}\right)\right] \\
&= \text{sign}\left[Re\left\{\left(\frac{2iy_0 + p + se^{-iy_0\tau}}{-2iy_0(-y_0^2 + piy_0 + q) - iy_0(r + siy_0)e^{-iy_0\tau}}\right)^{-1}\right\}\right] \\
&= \text{sign}\left[1 + \frac{(a^2 + a\alpha\cos(y_0\tau))(x^* - y^*)^2}{(p + s\cos(y_0\tau))^2 + (2y_0 - s\sin(y_0\tau))^2}\right]. \qquad (20)
\end{aligned}$$

Since

$$(a^2 + a\alpha\cos(y_0\tau))(x^* - y^*)^2 \geq a(a + |\alpha|)(x^* - y^*)^2 > 0,$$

the last expression in (20) is positive. This implies that (10) holds, which is a contradiction.

(II) The case $\alpha = 0$.

Let $\tau_1 = \tau_2 = \tau$, then the characteristic equation of (9) takes the form

$$\lambda^2 + p\lambda + q + ve^{-2\lambda\tau} = 0. \qquad (21)$$

Substituting $\lambda = iy$ into (21), we have

$$-y^2 + piy + q + ve^{-2iy\tau} = 0. \qquad (22)$$

By separating the real and imaginary parts of (22), we have

$$\begin{cases} v\cos(2y\tau) = y^2 - q \\ v\sin(2y\tau) = py \end{cases} \qquad (23)$$

and

$$v^2 = (y^2 - q)^2 + (py)^2.$$

Define the following function

$$f_3(Y) = (Y-q)^2 + p^2Y - v^2 \tag{24}$$

where $Y = y^2$, then f_3 is a downwards convex quadratic function and

$$f_3(0) = (a^4 - \beta^4)x^{*2}y^{*2} < 0.$$

Thus, it follows that there exists a positive simple root Y_0 of $f_3(Y) = 0$. Substituting y_0, which satisfies $Y_0 = y_0^2$, into (23), we can get some τ such that (21) has a characteristic root iy_0 at τ. Here iy_0 is a simple root of (21) by the same reason as above.

Taking the derivative of λ with τ on (21), we have

$$\frac{d\lambda}{d\tau} = \frac{2v\lambda e^{-2\lambda\tau}}{2\lambda + p - 2v\tau e^{-2\lambda\tau}},$$
$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p}{2\lambda(-\lambda^2 - p\lambda - q)} - \frac{\tau}{\lambda}.$$

Hence,

$$\begin{aligned}\text{sign}\left[Re\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda=iy_0}\right)\right] &= \text{sign}\left[Re\left(\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=iy_0}\right)\right] \\ &= \text{sign}\left[Re\left(\frac{2iy_0 + p}{2iy_0(y_0^2 - piy_0 - q)} - \frac{\tau}{iy_0}\right)\right] \\ &= \text{sign}\left[Re\left(\frac{2iy_0 + p}{2y_0[py_0 + i(y_0^2 - q)]}\right)\right] \\ &= \text{sign}\left[2y_0^2 + a^2(x^{*2} + y^{*2})\right] > 0.\end{aligned}$$

This implies that (10) holds, which is a contradiction.

(III) The case $\alpha \le a$.

Let $\tau_1 = \tau$ and $\tau_2 = 0$, then the characteristic equation of (9) takes the form

$$\lambda^2 + \tilde{p}\lambda + \tilde{q} + (\tilde{r} + \tilde{s}\lambda)e^{-\lambda\tau} = 0 \tag{25}$$

where $\tilde{p} = -ax^* - (a+\alpha)y^*$, $\tilde{q} = a(a+\alpha)x^*y^*$, $\tilde{r} = [\alpha(a+\alpha) + \beta^2]x^*y^*$, $\tilde{s} = -\alpha x^*$. Let us use p, q, r and s again instead of $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ and $\tilde{s}$ respectively. Substituting $\lambda = iy$ into (25), we have

$$-y^2 + piy + q + (r + siy)e^{-iy\tau} = 0. \tag{26}$$

By separating the real and imaginary parts of (26), we have

$$\begin{cases}(r^2 + s^2y^2)\cos(y\tau) = r(y^2 - q) - spy^2 \\ (r^2 + s^2y^2)\sin(y\tau) = sy(y^2 - q) + pry\end{cases} \tag{27}$$

and

$$\left[r^2 + s^2y^2\right]^2 = \left[r(y^2 - q) - spy^2\right]^2 + \left[sy(y^2 - q) + pry\right]^2.$$

Define the following function

$$f_4(Y) = Y\left[s(Y - q) + pr\right]^2 + \left[r(Y - q) - spY\right]^2 - \left[r^2 + s^2Y\right]^2 \tag{28}$$

where $Y = y^2$, then f_4 is an upwards cubic function to the right and

$$f_4(0) = [\alpha(a + \alpha) + \beta^2]^2[(a + \alpha)^2 + \beta^2][a^2 - (\alpha^2 + \beta^2)](x^*y^*)^4 < 0.$$

Thus, there can exist some positive roots of $f_4(Y) = 0$. Now, let us show that there exists a simple root in such positive roots. We see that

$$f_4'(Y) = 3s^2Y^2 + 2\left[s^2(p^2 - 2q - s^2) + r^2\right]Y + s^2(q^2 - 2r^2) + r^2(p^2 - 2q)$$

and

$$f_4''(Y) = 6s^2Y + 2\left[s^2(p^2 - 2q - s^2) + r^2\right].$$

Let $f_4''(Y) = 0$, then

$$3s^2Y + \left[s^2(p^2 - 2q - s^2) + r^2\right] = 0,$$

and thus we have

$$\begin{aligned} -3s^2f_4'(Y) &= \left[s^2(p^2 - 2q - s^2) + r^2\right]^2 - 3s^2\left[s^2(q^2 - 2r^2) + r^2(p^2 - 2q)\right] \\ &= x^{*4}y^{*2}\left[\alpha^2(4\alpha^2 - a^2)x^{*2} + \{\alpha(a + \alpha) + \beta^2\}^2y^{*2}\right] \\ &\quad \times \left[\{\alpha(a + \alpha) + \beta^2\}^2 - \alpha^2(a + \alpha)^2\right] \\ &\quad + \alpha^4x^{*4}\left[(a^2 - \alpha^2)x^{*2} - (a + \alpha)^2y^{*2}\right]^2. \end{aligned} \tag{29}$$

Since $\alpha \leq a < 0$, (29) is positive. This prove that there exists no triple root of $f_4(Y) = 0$, which implies that there exists at least a positive simple root Y_0 of $f_4(Y) = 0$.

Substituting y_0, which satisfies $Y_0 = y_0^2$, into (27), we can get some τ such that (25) has a characteristic root iy_0 at τ. Here again iy_0 is a simple root of (25).

Taking the derivative of λ with τ on (25), we have

$$\frac{d\lambda}{d\tau} = \frac{\lambda(r + s\lambda)e^{-\lambda\tau}}{2\lambda + p + e^{-\lambda\tau}[s - \tau(r + s\lambda)]},$$

$$\begin{aligned} \left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2\lambda + p + se^{-\lambda\tau}}{\lambda(r + s\lambda)e^{-\lambda\tau}} - \frac{\tau}{\lambda} \\ &= \frac{2\lambda + p}{-\lambda(\lambda^2 + p\lambda + q)} + \frac{s}{\lambda(r + s\lambda)} - \frac{\tau}{\lambda}. \end{aligned}$$

Hence, we have

$$\begin{aligned}\text{sign}\left[Re\left(\frac{d\lambda}{d\tau}\bigg|_{\lambda=iy_0}\right)\right] &= \text{sign}\left[Re\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\lambda=iy_0}\right)\right] \\ &= \text{sign}\left[Re\left(\frac{2iy_0+p}{-iy_0(-y_0^2+piy_0+q)}+\frac{s}{iy_0(r+siy_0)}-\frac{\tau}{iy_0}\right)\right] \\ &= \text{sign}\left[\frac{s^2y_0^4+2r^2y_0^2-s^2q^2-2r^2q+p^2r^2}{[(py_0)^2+(y_0^2-q)^2][r^2+(sy_0)^2]}\right].\end{aligned} \tag{30}$$

Since

$$\begin{aligned}&-s^2q^2-2r^2q+p^2r^2 \\ &= [a^2x^{*2}+(a+\alpha)^2y^{*2}][\alpha(a+\alpha)+\beta^2]^2x^{*2}y^{*2}-a^2\alpha^2(a+\alpha)^2x^{*4}y^{*2} \\ &\geq [a^2x^{*2}+(a+\alpha)^2y^{*2}]\alpha^2(a+\alpha)^2x^{*2}y^{*2}-a^2\alpha^2(a+\alpha)^2x^{*4}y^{*2} \\ &= \alpha^2(a+\alpha)^4x^{*2}y^{*4}>0,\end{aligned}$$

the last expression in (30) is positive. This implies that (10) holds, which is a contradiction. This completes the proof.

Here, we give the following three portraits of the trajectory of (1) with (2), drawn by a computer using the Runge-Kutta method, to illustrate Theorem ($r_1 = 10$, $r_2 = -10$).

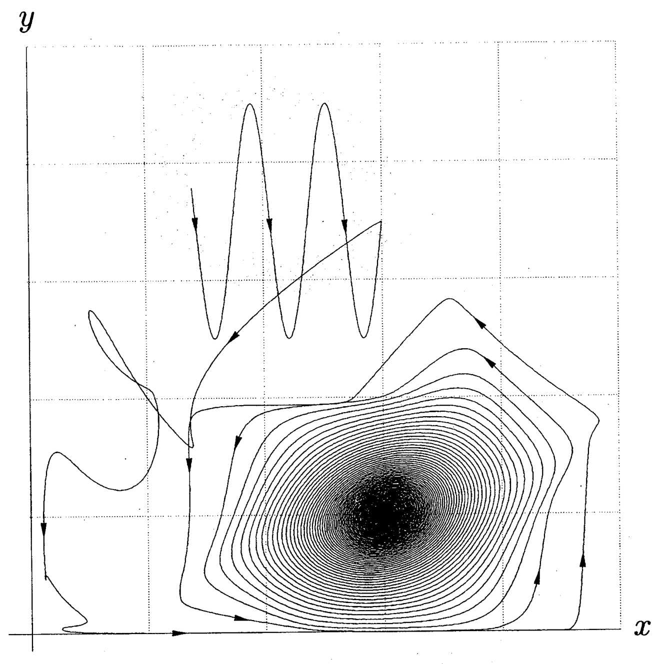

Fig.1 $a=-5,\ \alpha=3,\ \beta=3.99\quad(\sqrt{\alpha^2+\beta^2}<-a)$
$\tau_1=1,\ \tau_2=2,\ (\phi,\psi)=(3+0.8t,\ 3.5+\sin(8t))$

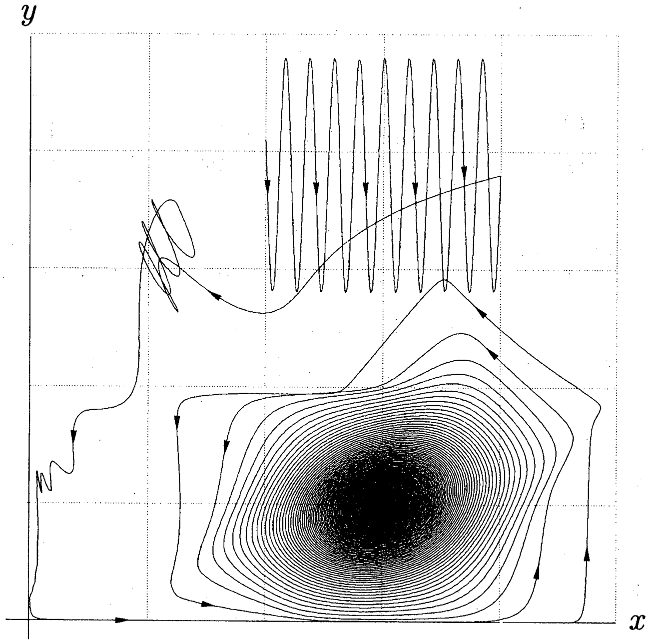

Fig.2 $a = -5,\ \alpha = 3,\ \beta = 4 \quad (\sqrt{\alpha^2 + \beta^2} = -a)$
$\tau_1 = 1,\ \tau_2 = 2,\ (\phi, \psi) = (4 + t,\ 3.8 + \sin(30t))$

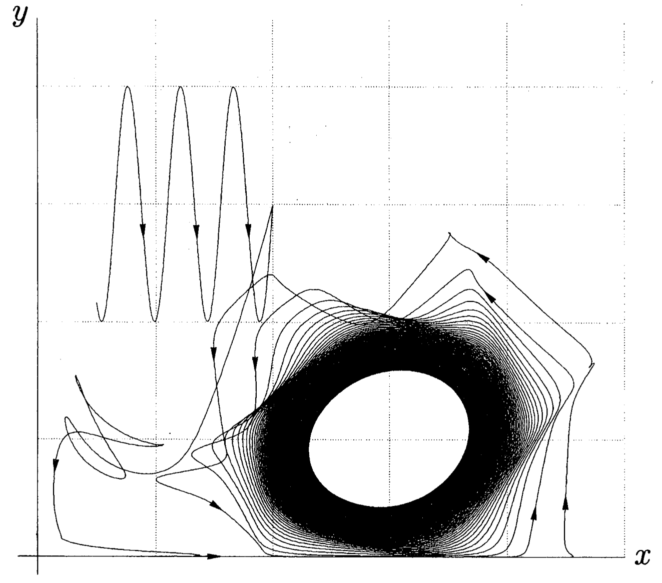

Fig.3 $a = -5,\ \alpha = 3,\ \beta = 4.01 \quad (\sqrt{\alpha^2 + \beta^2} > -a)$
$\tau_1 = 2,\ \tau_2 = 3,\ (\phi, \psi) = (2 + 0.5t,\ 3 + \sin(7t))$

3. Appendix

When $\tau_1 = \tau_2 = 0$, the system (1) become

$$\begin{cases} x'(t) = x(t)[r_1 + (a+\alpha)x(t) - \beta y(t)] \\ y'(t) = y(t)[r_2 + \beta x(t) + (a+\alpha)y(t)]. \end{cases} \tag{31}$$

By using the transformation

$$\bar{x} = x - x^*, \quad \bar{y} = y - y^*,$$

(31) is reduced to

$$\begin{cases} x'(t) = (x^* + x(t))[(a+\alpha)x(t) - \beta y(t)] \\ y'(t) = (y^* + y(t))[\beta x(t) + (a+\alpha)y(t)], \end{cases} \tag{32}$$

where we used $x(t)$ and $y(t)$ again instead of $\bar{x}(t)$ and $\bar{y}(t)$, respectively. Consider the following Liapunov function

$$V(x,y) = \left(x - x^* \log \frac{x+x^*}{x^*}\right) + \left(y - y^* \log \frac{y+y^*}{y^*}\right) \tag{33}$$

for $x > -x^*$ and $y > -y^*$, then V is positive definite. Calculating the derivative of V along the solution of (32), we have

$$\dot{V}_{(32)}(x,y) = (a+\alpha)(x^2+y^2).$$

Clearly, $\dot{V}_{(32)}$ is negative definite if and only if $a + \alpha < 0$ holds. The well-known Liapunov theorem shows that the origin $(0,0)$ is globally asymptotically stable if and only if $a + \alpha < 0$ holds.

References

[1] L. E. El'sgol'ts and S. B. Norkin, "Introduction to the Theory and Application of Differential Equations with Deviating Arguments," Academic Press, New York, 1973.

[2] K. Gopalsamy, Global asymptotic stability in Volterra's population systems, *J. Math. Biol.* **19** (1984), 157-168.

[3] J. K. Hale, "Ordinary Differential Equations," Second Edition, Robert E. Krieger Publishing Company Huntington, New York, 1980.

[4] J. K. Hale and S. M. Verduyn Lunel, "Introduction to Functional Differential Equations," Springer-Verlag, New York, 1991.

[5] Y. Kuang, "Delay Differential Equations with Applications in Population Dynamics," Academic Press, New York, 1993.

[6] S. M. Lenhart and C. C. Travis, Global stability of a biological model with time delay, *Proc. Amer. Math. Soc.* **96** (1986), 75-78.

[7] Z. Lu and W. Wang, Global stability for two-species Lotka-Volterra systems with delay, *J. Math. Anal. Appl.* **208** (1997), 277-280.

[8] H. L. Smith and P. Waltman, "The Theory of the Chemostat," Cambridge University Press, 1995

[9] G. S. K. Wolkowicz and Z. Lu, Global dynamics of a mathematical model of competition in the chemostat: general response functions and differential death rates, *SIAM J. Appl. Math.* **52** (1992), 222-233.