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Radial symmetry of self-similar solutions for semilinear heat equations

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We consider the symmetry properties of positive solutions of the equation

$$\Delta u+\frac{1}{2}x\cdot\nabla u+\frac{1}{p-1}u+u^{p}=0 \qquad \text{in } \boldsymbol{R}^{n}, \tag{1.1}$$

where $n\geq 2$ and $p>1$. This equation arises in the study of (forward) self-similar solutions of the semilinear heat equation

$$w_{t}=\Delta w+w^{p} \qquad \text{in } \boldsymbol{R}^{n}\times(0, \infty). \tag{1.2}$$

It is well known that if $w(x, t)$ satisfies (1.2), then, for $\mu>0$ the rescaled functions

$$w_{\mu}(x, t)=\mu^{2/(p-1)}w\left(\mu x, \mu^{2}t\right)$$

define a one parameter family of solutions to (1.2). A solution w is said to be self-similar, when $w_{\mu}(x, t)=w(x, t)$ for all $\mu>0$. It can be easily checked that w is a self-similar solution to (1.2) if and only if w has the form

$$w(x, t)=t^{-1/(p-1)}u\left(x/\sqrt{t}\right), \tag{1.3}$$

where u satisfies the elliptic Eq. (1.1). Moreover, if u has spherical symmetry, that is if $u=u(r)$, $r=|x|$, then u satisfies the ordinary differential equation

$$u''+\left(\frac{n-1}{r}+\frac{r}{2}\right)u'+\frac{1}{p-1}u+u^{p}=0, \quad r>0. \tag{1.4}$$

Such self-similar solutions are often used to describe the large time behavior of global solutions to the Cauchy problem, see, e.g., [11, 13, 3, 14, 5, and 15], and to show nonuniqueness of solution to (1.2) with zero initial data in a certain functional space, see [12].

First we state the result concerning the symmetry properties of the solution of (1.1).

THEOREM 1.1. *Let $u \in C^2(\boldsymbol{R}^n)$ be a positive solution of* (1.1) *such that*

$$u(x) = o(|x|^{-2/(p-1)}) \quad \text{as } |x| \to \infty. \tag{1.5}$$

Then u must be radially symmetric about the origin.

The proof of Theorem 1.1 is based on the moving planes argument. This technique was developed by Serrin [18] in PDE theory, and extended and generalized by Gidas, Ni, and Nirenberg [9, 10]. We remark that with a change of variables we are still able to prove a radial symmetry result for Eq. (1.1).

Let us consider the problem

$$\begin{cases} u'' + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right) u' + \dfrac{1}{p-1} u + |u|^{p-1} u = 0, \quad r > 0, \\ u'(0) = 0 \quad \text{and} \quad u(0) = \alpha \in \boldsymbol{R}. \end{cases} \tag{1.6}$$

The problem (1.6) has been investigated extensively in [12, 16, 20, and 2]. We denote by $u(r;\alpha)$ the unique solution of (1.6). We recall that $u(r;\alpha)$ has the following properties:
(i) $\lim_{r\to\infty} r^{2/(p-1)} u(r;\alpha) = L(\alpha)$ exists and is finite for every $\alpha \in \boldsymbol{R}$ (see [12, Theorem 5]);
(ii) if $L(\alpha) = 0$, then there exists a constant $A \neq 0$ such that

$$u(r;\alpha) = Ae^{-r^2/4} r^{2/(p-1)-n}\{1 + O(r^{-2})\} \quad \text{as } r \to \infty$$

(see [16, Theorem 1]);
(iii) if $p \geq (n+2)/(n-2)$, then $u(r;\alpha)$ is positive on $[0,\infty)$ and $L(\alpha) > 0$ for every $\alpha > 0$ (see [12, Theorem 5]);
(iv) if $(n+2)/n < p < (n+2)/(n-2)$, then there exists a unique $\alpha > 0$ such that $u(r;\alpha)$ is positive on $[0,\infty)$ and $L(\alpha) = 0$ (see [20, Theorem 1] and [2, Theorem 1.2 and Corollary 1.3]).

By virtue of Theorem 1.1 we obtain the following:

COROLLARY 1.1. (i) *Assume that $p \geq (n+2)/(n-2)$. Then there exists no positive solution u of* (1.1) *satisfying* (1.5).

(ii) *Assume that $(n+2)/n < p < (n+2)/(n-2)$. Then there exists a unique positive solution $u(x)$ satisfying* (1.5). *Moreover, the solution u is radially symmetric about the origin.*

Remark. The result (i) is differently proven by [3, Proposition 4.3] based on the Pohozaev identity.

Following the notations in [3] and [14], we define

$$L^2(K) = \{u : \boldsymbol{R}^n \to \boldsymbol{R}; \int_{R^n} |u|^2 K(x)dx < \infty\} \quad \text{and}$$

$$H^1(K) = \{u : \boldsymbol{R}^n \to \boldsymbol{R}; \int_{R^n} (|u|^2 + |\nabla u|^2)K(x)dx < \infty\},$$

where $K(x) = \exp(|x|^2/4)$. Escobedo and Kavian have shown in [3, Proposition 3.5] that if $1 < p < (n+2)/(n-2)$ and if $u \in H^1(K)$ is a solution of (1.1), then $u \in C^2(\boldsymbol{R}^n)$ and satisfies $u(x) = O(\exp(-|x|^2/8))$ as $|x| \to \infty$. As a consequence of Corollary 1.1, we obtain the following:

COROLLARY 1.2. *Assume that $(n+2)/n < p < (n+2)/(n-2)$. Then the problem*

$$\begin{cases} \Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + u^p = 0 \quad \text{in } \boldsymbol{R}^n, \\ u \in H^1(K) \quad \text{and} \quad u > 0 \text{ in } \boldsymbol{R}^n, \end{cases} \tag{1.7}$$

has a unique solution.

Let us consider the Cauchy problem

$$\begin{cases} w_t = \Delta w + w^p \quad \text{in } \boldsymbol{R}^n \times (0,\infty), \\ w(x,0) = \tau w_0 \quad \text{in } \boldsymbol{R}^n, \end{cases} \tag{1.8}$$

where $w_0 \in L^2(K) \cap L^\infty(\boldsymbol{R}^n)$, $w_0 \geq 0$, and $\tau > 0$ is a parameter. We denote by $w(x,t;\tau)$ the unique solutions of (1.8) (see [15]). Combining the result by Kawanago [15, Theorem 1] and Corollary 1.2, we obtain the following, where the asymptotic behavior of $w(\cdot,t;\tau)$ as $t \to \infty$ becomes clearer.

COROLLARY 1.3. *Assume that $(n+2)/n < p < (n+2)/(n-2)$. Then there exists a unique $\tau_0 > 0$ such that the solution $w(x,t;\tau)$ is a global solution if $\tau \in (0,\tau_0]$, and $w(x,t;\tau)$ blows up in finite time if $\tau \in (\tau_0,\infty)$. Moreover, $w(x,t;\tau_0)$ satisfies*

$$\lim_{t\to\infty} \left\| t^{1/(p-1)} w(\cdot,t;\tau_0) - u_0\left(\cdot/\sqrt{t}\right) \right\|_{L^\infty(R^n)} = 0,$$

where u_0 is a unique solution of the problem (1.7).

Next we consider the existence of nonradial solutions of (1.1). Let $p > (n+2)/n$ and let $U(r)$ be a positive solution of (1.4) satisfying

$$U'(0) = 0 \quad \text{and} \quad \lim_{r\to\infty} r^{2/(p-1)}U(r) > 0. \tag{1.9}$$

The existence of such U is obtained by [12, Theorem 5]. Define $\ell = \ell(U) > 0$ as

$$\ell = \lim_{r\to\infty} r^{2/(p-1)}U(r). \tag{1.10}$$

We investigate the Cauchy problem for Eq. (1.2) with

$$w(x,0) = w_0 \in L^1_{\text{loc}}(\boldsymbol{R}^n), \tag{1.11}$$

where

$$0 \le w_0(x) \le \ell |x|^{-2/(p-1)}, \quad w_0 \not\equiv 0, \quad x \in \boldsymbol{R}^n \setminus \{0\}. \tag{1.12}$$

Relation (1.11) is taken in the sense of $L^1_{\text{loc}}(\boldsymbol{R}^n)$, that is,

$$\int_K |w(x,t) - w_0(x)| dx \to 0 \quad \text{as} \ t \to 0$$

for any compact subset K of $\boldsymbol{R}^n$. We note that $w_0 \in L^1_{\text{loc}}(\boldsymbol{R}^n)$ if (1.12) holds with $p > (n+2)/n$.

THEOREM 1.2. *Let $p > (n+2)/n$. Assume that* (1.12) *holds, where ℓ is the constant in* (1.10). *Then there exists a positive solution $w \in C^{2,1}(\boldsymbol{R}^n \times (0,\infty))$ of* (1.2) *and* (1.11). *Assume, furthermore, that $w_0 \in C(\boldsymbol{R}^n \setminus \{0\})$, then w satisfies*

$$w(x,t) \to w_0(x) \quad \text{as } t \to 0 \quad \text{uniformly in } |x| \ge r \text{ for every } r > 0. \tag{1.13}$$

Moreover, w is self-similar if $\mu^{2/(p-1)}w_0(\mu x) = w_0(x)$ for every $\mu > 0$.

COROLLARY 1.4. *Let $p > (n+2)/n$. Assume that $A : S^{n-1} \to \boldsymbol{R}$ is continuous and satisfies*

$$0 \le A(\sigma) \le \ell, \ A \not\equiv 0, \quad \sigma \in S^{n-1}. \tag{1.14}$$

Then there exists a positive self-similar solution $w \in C^{2,1}(\boldsymbol{R}^n \times (0,\infty))$ of (1.2) *satisfying* (1.11) *and* (1.13) *with $w_0(x) = A(x/|x|)|x|^{-2/(p-1)}$.*

Recall that self-similar solutions w to (1.2) have the form (1.3) with u satisfying (1.1). Therefore, $w(\sigma, t) = r^{2/(p-1)}u(r\sigma)$ for $\sigma \in S^{n-1}$, where $r = 1/\sqrt{t}$. Then we obtain the following corollary, which shows that condition (1.5) in Theorem 1.1 is crucial.

Corollary 1.5. *Let* $p > (n+2)/n$. *Assume that* $A : S^{n-1} \to \boldsymbol{R}$ *is continuous and satisfies* (1.14). *Then there exists a positive non-radial solution* u *of* (1.1) *satisfying*

$$r^{2/(p-1)}u(r\sigma) \to A(\sigma) \quad \text{as} \quad r \to \infty \quad \text{uniformly in} \quad \sigma \in S^{n-1}.$$

Remark. (i) If $1 < p \le (n+2)/n$, no time global, non-negative, and nontrivial solution exists in (1.2) (see, e.g., [7], [19]). Therefore, (1.1) admits a positive solution only if $p > (n+2)/n$.

(ii) We find that the solution w of (1.2) and (1.11) obtained in Theorem 1.2 is a minimal solution of the integral equation

$$w(x,t) = \int_{R^n} \Gamma(x-y:t)w_0(y)dy + \int_0^t \int_{R^n} \Gamma(x-y:t-s)[w(y,s)]^p dyds,$$

where $\Gamma(x:t) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. See the proof of Theorem 1.2 below.

(iii) Galaktionov and Vazquez [8] studied the Cauchy problem (1.2) and (1.11) with singular initial values for the case $p > n/(n-2)$.

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