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# Heritage of Original Matrix from its Preconditioner in the Convergent Splitting

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**Abstract** A fundamental theory of the convergent linear iteraton is developed. All linear iterative methods of the linear system $A\mathbf{x}=\mathbf{b}$ are considered instead of each individual. The convergent iteration which is considered an inversion of $A$ is equivalently expressed in several ways. General monotonicity is introduced so as to extend the ordinary one. Propositions on the convergent iteration are represented in terms of the general monotonicity. A geometrical aspect of the general monotonicity is helpful to make an intuitive image and make simple proof of propositions. Based on the convergent splitting $A=C-R_C\left(\rho\left(C^{-1}R_C\right)<1\right)$, first necessary condition of $A$ for convergent iteration is dealt with as inherited properties of $A$ from those of $C$ and then sufficiency of the inherited properties is discussed. A necessary and sufficient condition for the convergence of regular and weak regular splittings is the i-monotonicity of $A$, and a condition for the convergent Jacobi and Gauss-Seidel iterations for an Z-matrix $A$, is as well that $A$ is i-monotone, which is identical with the ordinary monotonicity: $A^{-1}\geq O$.

## 1. Intorduction

The linear iterative method has been studied with enormous effort [1]. The main subject of the method is to supply a procedure of computaion in a short time. This paper is an attempt to reconstruct a fundamental theory to elucidate mathematical concept of the iterative method, present or latent in the numerical analysis. All the convergent iterative methods are generally considered instead of each individual. To describe precisely, the iterative method is defined and equivalent notions are introduced. The convergent iterative method will be attributed to an inversion of matrix.

One of the important concepts in the iteration is the monotonicity of matrix. Due to the inversion of the convergent iteration, the monotonicity of the ordinary use is related to such a property as $A^{-1} \geq O$. This paper introduces new concept of general monotonicity which is a canonical extension of the ordinary one. The general monotonicity is represented in a way of geometry as a relation of pyramids spanned by column vectors of matrices in the monotone relationship. This notion will make a concrete image of the monotonicity.

Discussion of the iteration is made based on the splitting $A = C - R_C$ with a nonsingular matrix $C$. First, heritage of properties from $C$ to $A$ is considered in the convergent splitting. These inherited properties of $A$ are necessary conditions for convergence of the iteration. Next, consider whether the necessary conditions would be sufficient for the convergence or not. Finally, is discussed necessary and sufficient condition for the well-known iterative methods of Jacobi and Gauss-Seidel. Even for known propositions and theorems, original or improved proofs are given throughout this paper.

## 2. Definition of Iterative Method

Let $\mathbf{R}^n$ be the $n$-dimensional real space and $f_k$ $(k \geq 1)$ be transformations of $\mathbf{R}^n$. $\varphi_0$ is the identity transformation and $\varphi_{k+1}$ is inductively defined as $\varphi_{k+1} = f_{k+1} \circ \varphi_k$ for any nonnnegative integer $k$. The transformation $f_k$ is called the $k$th iterative transformation. An iterative method is to operate $f_{k+1}$ recursively on the $k$th iterate $\mathbf{x}^k$ of $\mathbf{R}^n$ with an initial vector $\mathbf{x}^0$. Then,

$$\mathbf{x}^{k+1} = f_{k+1}\left(\mathbf{x}^k\right) = \varphi_{k+1}\left(\mathbf{x}^0\right).$$

Suppose that all $f_k$ are the same as $f_k = (H, \mathbf{d})$ $(k \geq 1)$ with an iterative matrix $H$ in $M_n(\mathbf{R})$ and $\mathbf{d}$ in $\mathbf{R}^n$, where $M_n(\mathbf{R})$ denotes a set of all matrices of order $n$ with real components. For any $\mathbf{x} \in \mathbf{R}^n$, $(H, \mathbf{d})$ is defined as

$$(H, \mathbf{d})\mathbf{x} = H\mathbf{x} + \mathbf{d} \ .$$

Then, $$\mathbf{x}^{k+1} = (H, \mathbf{d})\mathbf{x}^k = (H, \mathbf{d})^{k+1}\mathbf{x}^0 = H^{k+1}\mathbf{x}^0 + \left(I + H + \cdots + H^k\right)\mathbf{d} \ .$$

The iterative transformation $(H, \mathbf{d})$ defines a linear iterative method, which is called the iterative method associated with the iterative transformation $(H, \mathbf{d})$. The objective of this paper is to consider all the linear iterative methods, hereafter simply called the iterative methods, instead of each individual.

The iterative method in the numerical analysis is a method to solve the system of linear equations

$$A\mathbf{x} = \mathbf{b}, \quad A \in \mathrm{GL}(n;\mathbf{R}) \quad , \tag{1}$$

where $\mathrm{GL}(n;\mathbf{R})$ is the group of all general linear transformations of $\mathbf{R}^n$, identified with the group of all nonsingular matrices in $M_n(\mathbf{R})$. The vector $\mathbf{b}$ is written with the coordinate vectors $\mathbf{e}_i\,(i = 1,2,\cdots,n)$ as

$$\mathbf{b} = \sum_{i=1}^{n} b_i \mathbf{e}_i \ .$$

The solutions $\mathbf{x}_i\,(i = 1,2,\cdots,n)$ of the equations $A\mathbf{x} = \mathbf{e}_i$ yield a solution of eqn. (1) such as $\mathbf{x} = \sum_{i=1}^{n} b_i \mathbf{x}_i$ . In fact, $A\mathbf{x} = \sum_{i=1}^{n} b_i A\mathbf{x}_i = \sum_{i=1}^{n} b_i \mathbf{e}_i = \mathbf{b}$. Let $\langle \mathbf{x}_i \rangle$, $\langle \mathbf{e}_i \rangle$ be matrices with column vectors $\mathbf{x}_i$, $\mathbf{e}_i$ $(i = 1,2,\cdots,n)$, respectively. Then

$$A\langle \mathbf{x}_i \rangle = \langle \mathbf{e}_i \rangle,$$

with $\langle \mathbf{e}_i \rangle = I$ (unit matrix). Thus, $\langle \mathbf{x}_i \rangle$is the inverse matrix of $A$, i.e., $\langle \mathbf{x}_i \rangle = A^{-1}$. The iterative method is, therefore, regarded as a method to solve $A\mathbf{x} = \mathbf{e}_i\,(i = 1,2,\cdots,n)$ or $AX = I$; that is, an inversion of $A$. For a given linear system $A\mathbf{x} = \mathbf{b}$, it suffices to find a set of solutions $\mathbf{x}_i\,(i = 1,2,\cdots,n)$ satisfying $A\mathbf{x}_i = \mathbf{e}_i$, so that the iterative method is considered a method depending only on the given matrix $A \in \mathrm{GL}(n;\mathbf{R})$, regardless of $\mathbf{b}$.

Let $\rho(H)$ be the spectral radius of $H$ and $\rho(H) < 1$. The eigenvalues of $I - H$ are $1 - \lambda_i$ $(i = 1,2,\cdots,n)$ with the eigenvalues $\lambda_i$ of $H$. Then, it follows that $|1 - \lambda_i| \neq 0$ and that $I - H \in \mathrm{GL}(n;\mathbf{R})$. Let $C \overset{\text{def}}{=} A(I - H)^{-1}$. Then, $C \in \mathrm{GL}(n;\mathbf{R})$ and $H = C^{-1}R_C$, where $R_C$ is given by

$$A = C - R_C \ . \tag{2}$$

**Definition 1.** Representation (2) with $C \in \mathrm{GL}(n;\mathbf{R})$ is called the splitting of $A$.

Let define $\mathbf{x}^* \overset{\text{def}}{=} (I - H)^{-1}\mathbf{d}$. From

$$(H,\mathbf{d})\mathbf{x}^* = H(I - H)^{-1}\mathbf{d} + \mathbf{d} = \{H + (I - H)\}(I - H)^{-1}\mathbf{d} = \mathbf{x}^*,$$

it follows that

$$\mathbf{x}^k - \mathbf{x}^* = (H,\mathbf{d})\mathbf{x}^{k-1} - (H,\mathbf{d})\mathbf{x}^* = H\left(\mathbf{x}^{k-1} - \mathbf{x}^*\right) = H^k\left(\mathbf{x}^0 - \mathbf{x}^*\right).$$

Since $\rho(H) < 1$ leads to convergence of $H^k$ to $O$, $\mathbf{x}^k$ converges to $\mathbf{x}^*$ independently of $\mathbf{x}^0$, as $k$ tends to infinity. The vector $\mathbf{d}$ is taken so as to satisfy $A\mathbf{x}^* = \mathbf{e}_i$ . Then, $\mathbf{d} = C^{-1}\mathbf{e}_i$ .

**Definition 2.** Splitting (2) is said to be convergent, if the iterative method associated with the iterative matrix $H = C^{-1}R_C$ is convergent.

**Proposition 3.** *Splitting (2) is a convergent splitting, if and only if* $\rho\left(C^{-1}R_C\right) < 1$.

**Proof**. The sufficiency of $\rho\left(C^{-1}R_C\right) < 1$ is mensioned above. Conversely, the convergent iterative method makes $\mathbf{x}^{k+1} = H\mathbf{x}^k + \mathbf{d}$ converge to a vector, say $\tilde{\mathbf{x}}$, for any $\mathbf{x}^0$. Let $\lambda$ be any eigenvalue of $H$ and $\mathbf{u}$ be its corresponding eigenvector. Then, $H^k\mathbf{u} = \lambda^k\mathbf{u}$. By setting $\mathbf{x}^0 = \mathbf{u} + \tilde{\mathbf{x}}$, $H^k\mathbf{u} = H^k(\mathbf{x}^0 - \tilde{\mathbf{x}}) = \mathbf{x}^k - \tilde{\mathbf{x}}$ tends to $\mathbf{0}$ as $k \to \infty$. Then, $|\lambda| < 1$. Thus, $\rho(H) < 1$. □

Consider the set defined as

$$\mathscr{R}_{\mathrm{CV}} \stackrel{\mathrm{def}}{=} \left\{ C \in \mathrm{GL}(n;\mathbf{R}) \,\middle|\, \rho\left(C^{-1}R_C\right) < 1 \right\}.$$

A matrix $C$ in $\mathscr{R}_{\mathrm{CV}}$ gives a convergent splitting of $A$. By Prop. 3, a convergent splitting is the splitting $A = C - R_C$ with $\rho(C^{-1}R_C) < 1$. Then, $C$ belongs to $\mathscr{R}_{\mathrm{CV}}$. Therefore, $\mathscr{R}_{\mathrm{CV}}$ is equivalent to the set of all the convergent splittings of $A$. By Def. 2, the set $\mathscr{CT}$ of all the convergent iterative methods of the linear system $A\mathbf{x} = \mathbf{b}$ is furthermore equivalent to $\mathscr{R}_{\mathrm{CV}}$; that is, $\mathscr{CT} \sim \mathscr{R}_{\mathrm{CV}}$, where $\sim$ denotes equivalence of the two sets or implies existence of a bijection from the one set to the other.

**Definition 4.**

$$\mathscr{CH} \stackrel{\mathrm{def}}{=} \left\{ H \in M_n(\mathbf{R}) \,\middle|\, \rho(H) < 1 \right\}$$

$$\mathscr{CF} \stackrel{\mathrm{def}}{=} \left\{ (I - H)^{-1} \,\middle|\, \rho(H) < 1,\ H \in M_n(\mathbf{R}) \right\}.$$

**Proposition 5.**

$$\begin{aligned} \mathscr{CF} &= \left\{ F_H \in \mathrm{GL}(n;\mathbf{R}) \,\middle|\, F_H = \left(\mathbf{f}_i^H\right),\ \mathbf{f}_i^H = (H,\mathbf{e}_i)\mathbf{f}_i^H,\ \rho(H) < 1,\ H \in M_n(\mathbf{R}) \right\} \\ &= \left\{ F_H \in \mathrm{GL}(n;\mathbf{R}) \,\middle|\, F_H = \left(\mathbf{f}_i^H\right),\ \mathbf{f}_i^H = \lim_{k\to\infty}(H,\mathbf{e}_i)^k\mathbf{f}_i^H,\ \rho(H) < 1,\ H \in M_n(\mathbf{R}) \right\}. \end{aligned}$$

**Proof.** The first equality is shown from

$$(I - H)\mathbf{f}_i^H = \mathbf{e}_i \quad \Leftrightarrow \quad \mathbf{f}_i^H = (H,\mathbf{e}_i)\mathbf{f}_i^H ,$$

and the second from

$$\lim_{k\to\infty}(H,\mathbf{e}_i)^k\mathbf{f}_i^H = \lim_{k\to\infty}\left\{ H^k\mathbf{f}_i^H + \left(I + H + \cdots + H^{k-1}\right)\mathbf{e}_i \right\} = (I - H)^{-1}\mathbf{e}_i \ . \qquad \square$$

$\mathscr{CF}$ is the set of all matrices whose column vectors are fixed points of the convergent iterative transformations $(H,\mathbf{e}_i)$ $(i = 1,2,\cdots,n)$.

**Proposition 6.**

$$\mathscr{CT} \sim \mathscr{R}_{\mathrm{CV}} \sim \mathscr{CH} \sim \mathscr{CF}.$$

**Proof.** $\mathscr{CT} \sim \mathscr{R}_{\mathrm{CV}}$ is shown above. The rest equivalences are derived by the following bijections $\Phi$, $\Psi$.

$$\begin{array}{ccccc} \mathscr{R}_{\mathrm{CV}} & \xrightarrow{\Phi} & \mathscr{CH} & \xrightarrow{\Psi} & \mathscr{CF} \\ C & \mapsto & H & \mapsto & F_H . \end{array}$$

Here, $\Phi$, $\Psi$ are mappings satisfying

$$\Phi(C) = H = C^{-1}R_C \ , \quad C = \Phi^{-1}(H) = A(I-H)^{-1}$$
$$\Psi(H) = F_H = (I-H)^{-1} = A^{-1}C \ , \quad H = \Psi^{-1}(F_H) = I - F_H^{-1}. \qquad \square$$

### 3. General Monotonicity

The monotonicity is generalized. The ordinary definition of monotonicity is the following inverse monotonicity.

**Definition 7.** Let $A \in M_n(\mathbf{R})$. Matrix $A$ is said to be normally monotone or simply n-monotone, if $\mathbf{x} \geq \mathbf{0}$ leads to $A\mathbf{x} \geq \mathbf{0}$, and inversely monotone or i-monotone, if $A\mathbf{x} \geq \mathbf{0}$ leads to $\mathbf{x} \geq \mathbf{0}$.

**Proposition 8.**

$$A: \textit{n-monotone} \quad \Leftrightarrow \quad A \geq O .$$

**Proof.** $\Rightarrow$). Since $\mathbf{x} = \mathbf{e}_i \geq \mathbf{0}\ (i = 1,2,\cdots,n)$, $\mathbf{a}_i \overset{\text{def}}{=} A\mathbf{e}_i \geq \mathbf{0}$ and so $A = (\mathbf{a}_i) \geq O$ .

$\Leftarrow$). $A\mathbf{x} \geq \mathbf{0}$ is shown from $A \geq O$ and $\mathbf{x} \geq \mathbf{0}$. $\square$

The following corollaries are readily obtained from this proposition.

**Corollary 9.** *Let $A$ be n-monotone. If $A \leq B$ , then $B$ is n-monotone.*

**Corollary 10.** *If $I \leq A$ , then $A$ is n-monotone.*

**Proposition 11.**

$$\begin{aligned} A\textit{:i-monotone} \ &\Leftrightarrow (A\mathbf{x} \leq \mathbf{0} \Rightarrow \mathbf{x} \leq \mathbf{0}) \\ &\Leftrightarrow (A\mathbf{x} > \mathbf{0} \Rightarrow \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}) \ \textit{and} \ (A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}) \\ &\Leftrightarrow (A\mathbf{x} < \mathbf{0} \Rightarrow \mathbf{x} \leq \mathbf{0}, \mathbf{x} \neq \mathbf{0}) \ \textit{and} \ (A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}) \\ &\Leftrightarrow (A\mathbf{x} \geq \mathbf{y} \Rightarrow \mathbf{x} \geq \mathbf{y}) \end{aligned}$$

**Proof.** By taking $-\mathbf{x}$ instead of $\mathbf{x}$, $(A\mathbf{x} \geq \mathbf{0} \Rightarrow \mathbf{x} \geq \mathbf{0})$ is equivalent to $(A\mathbf{x} \leq \mathbf{0} \Rightarrow \mathbf{x} \leq \mathbf{0})$. From the i-monotonicity of $A$, $A\mathbf{x} > \mathbf{0}$ leads to $\mathbf{x} \geq \mathbf{0}$. If $\mathbf{x} = \mathbf{0}$, $A\mathbf{x} = \mathbf{0}$, so that $A\mathbf{x} > \mathbf{0} \Rightarrow \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}$. If $A\mathbf{x} = \mathbf{0}$, then $A\mathbf{x} \geq \mathbf{0}$ and $A\mathbf{x} \leq \mathbf{0}$. It follows that $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \leq \mathbf{0}$ and that $\mathbf{x} = \mathbf{0}$. Conversely, $(A\mathbf{x} > \mathbf{0} \Rightarrow \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}) \& (A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0})$ implies that $A\mathbf{x} \geq \mathbf{0} \Rightarrow \mathbf{x} \geq \mathbf{0}$. Similarly, the i-monotonicity of $A$ is equivalent to $(A\mathbf{x} < \mathbf{0} \Rightarrow \mathbf{x} \leq \mathbf{0}, \mathbf{x} \neq \mathbf{0}) \&$ $(A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0})$ and, moreover, to $(A\mathbf{x} \geq \mathbf{y} \Rightarrow \mathbf{x} \geq \mathbf{y})$ by $(A(\mathbf{x} - \mathbf{y}) \geq \mathbf{0} \Rightarrow \mathbf{x} - \mathbf{y} \geq \mathbf{0})$. $\square$

**Corollary 12.**

$$A\textit{:i-monotone} \quad \Rightarrow \quad A \in \mathrm{GL}(n;\mathbf{R})$$

**Proof.** If $A$ is i-monotone, $A\mathbf{x} = \mathbf{0}$ leads to $\mathbf{x} = \mathbf{0}$. Then, the linear transformation $T_A$ of

$\mathbf{R}^n$ to $\mathbf{R}^n$ defined as $T_A\mathbf{x} = A\mathbf{x}$ is injective. Let

$$\sum_{i=1}^{n} a_i T_A \mathbf{f}_i = \mathbf{0} \ , \quad a_i \in \mathbf{R} \ \ (1,2,\cdots,n) \ ,$$

with a basis of $\mathbf{R}^n$, $\langle \mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n \rangle$.

Since $T_A$ is an injection, $T_A\left(\sum_{i=1}^{n} a_i \mathbf{f}_i\right) = \mathbf{0}$ implies $\sum_{i=1}^{n} a_i \mathbf{f}_i = \mathbf{0}$. The vectors $\langle \mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_n \rangle$ are linearly independent. Then, $a_i = 0 \ (i = 1,2,\cdots,n)$. Thus, the set of $\langle T_A\mathbf{f}_1, T_A\mathbf{f}_2, \cdots, T_A\mathbf{f}_n \rangle$ is another basis of $\mathbf{R}^n$, so that $T_A$ is surjective. Therefore, $T_A$ is a bijection, i.e., $A \in \mathrm{GL}(n;\mathbf{R})$.

□

**Proposition 13.**

$$A\text{:}\textit{i-monotone} \ \Leftrightarrow \ A \in \mathrm{GL}(n;\mathbf{R}) \ \textit{and} \ \ A^{-1} \geq O$$
$$\Leftrightarrow \ A \in \mathrm{GL}(n;\mathbf{R}) \ \textit{and} \ \ A^{-1}\textit{; n-monotone.}$$

**Proof.** Let $\mathbf{y} = A\mathbf{x}$. Then, $\mathbf{x} = A^{-1}\mathbf{y}$. By the definition of i-monotonicity,

$$A\mathbf{x} \geq \mathbf{0} \ \Rightarrow \ \mathbf{x} \geq \mathbf{0},$$

which is equivalently expressed as

$$\mathbf{y} \geq \mathbf{0} \ \Rightarrow \ A^{-1}\mathbf{y} \geq \mathbf{0};$$

namely, $A^{-1}$ is n-monotone. By Prop. 8, $A^{-1} \geq O$. □

**Definition 14.** Let $A \in M_n(\mathbf{R})$, $B \in \mathrm{GL}(n;\mathbf{R})$. Matrix $A$ is said to be normally left or right monotone over $B$ or simply n-l or r·monotone/ $B$, if $B^{-1}A$ or $AB^{-1}$ is n-monotone, and to be inversely left or right monotone over $B$ or simply i-l or r·monotone/ $B$, if $B^{-1}A$ or $AB^{-1}$ is i-monotone, respectively. If both left and right monotonicities are valid, the monotonicity is employed without specification of left and right.

If $B = I$, the left and right monotonicities are the same. In this case $A$ is further said to be n- or i-monotone omitting "over $I$", which is identical with the previous definition of n- or i-monotonicity.

**Proposition 15.**

$$A\text{: } \textit{i-l or r·monotone/}B \ \Rightarrow \ A \in \mathrm{GL}(n;\mathbf{R}).$$

**Proof.** Since $B^{-1}A$ or $AB^{-1}$ is i-monotone, $B^{-1}A, AB^{-1} \in \mathrm{GL}(n;\mathbf{R})$ (Cor. 12). Combined with $B \in \mathrm{GL}(n;\mathbf{R})$, $A = B(B^{-1}A) = (AB^{-1})B \in \mathrm{GL}(n;\mathbf{R})$. □

**Proposition 16.**

$$A\text{: } \textit{n-l or r·monotone/}B \ \Leftrightarrow \ B^{-1}A \ \textit{ or } AB^{-1} \geq O$$
$$\Leftrightarrow \ \left(\mathbf{x}^T B \geq \mathbf{0}^T \Rightarrow \mathbf{x}^T A \geq \mathbf{0}^T\right) \textit{or} \ (B\mathbf{x} \geq \mathbf{0} \Rightarrow A\mathbf{x} \geq \mathbf{0}),$$

*respectively.*

**Proof.** By Prop. 8, the n-l or r·monotonicity of $A/B$ is equivalent to $B^{-1}A$ or $AB^{-1} \geq O$, respectively. $B^{-1}A \geq O$ is equivalently represented as $\left(\mathbf{x}^T \geq \mathbf{0}^T \Rightarrow \mathbf{x}^T B^{-1} A \geq \mathbf{0}^T\right)$. By setting $\mathbf{x}^T = \mathbf{y}^T B$, $\left(\mathbf{x}^T \geq \mathbf{0}^T \Rightarrow \mathbf{x}^T B^{-1} A \geq \mathbf{0}^T\right)$ implies $\left(\mathbf{y}^T B \geq \mathbf{0}^T \Rightarrow \mathbf{y}^T A \geq \mathbf{0}^T\right)$. Similarly, by setting $\mathbf{x} = B\mathbf{y}$, $\left(\mathbf{x} \geq \mathbf{0} \Rightarrow AB^{-1}\mathbf{x} \geq \mathbf{0}\right)$ implies $\left(B\mathbf{y} \geq \mathbf{0} \Rightarrow A\mathbf{y} \geq \mathbf{0}\right)$. □

Hereafter, proof is given only for the right monotonicity, in case the left is readily shown quite similarly to the right.

**Corollary 17.** (Transitivity of n-monotonicity) Let $A \in M_n(\mathbf{R})$, $B, C \in \mathrm{GL}(n;\mathbf{R})$.

$$A\text{: } \textit{n-monotone}/B \text{ and } B\text{: } \textit{n-monotone}/C \quad \Rightarrow \quad A\text{: } \textit{n-monotone}/C.$$

**Proof.** $AB^{-1} \geq O$, $BC^{-1} \geq O$. Then, $AC^{-1} = \left(AB^{-1}\right)\left(BC^{-1}\right) \geq O$. □

**Corollary 18.**

$$A\text{: } \textit{n-monotone} \;,\; B\text{: } \textit{i-monotone} \quad \Rightarrow \quad A\text{: } \textit{n-monotone}/B$$

**Proof.** $A \geq O$ and $B^{-1} \geq O$ gives $AB^{-1} \geq O$. □

**Proposition 19.** (Duality of Monotonicity) *Let* $A, B \in \mathrm{GL}(n;\mathbf{R})$.

$$A\text{: } \textit{n-monotone}/B \quad \Leftrightarrow \quad B\text{: } \textit{i-monotone}/A$$

**Proof.** It is readily shown from $AB^{-1} \geq O$ and $\left(BA^{-1}\right)^{-1} = AB^{-1}$. □

**Corollary 20.**

$$\begin{aligned} A\text{: } \textit{i-monotone} &\Leftrightarrow A\text{:}\textit{i-monotone}/I \\ &\Leftrightarrow I\text{: } \textit{n-monotone}/A. \end{aligned}$$

**Proof.** $A^{-1} = \left(AI^{-1}\right)^{-1} = IA^{-1}$. □

**Corollary 21.** (Transitivity of i-monotonicity) Let $A \in M_n(\mathbf{R})$, $B, C \in \mathrm{GL}(n;\mathbf{R})$.

$$A\text{: } \textit{i-monotone}/B \text{ and } B\text{: } \textit{i-monotone}/C \quad \Rightarrow \quad A\text{: } \textit{i-monotone}/C.$$

**Proof.** It follows from the transitivity of n-monotonicity and from the duality of monotonicity. □

**Proposition 22.** *Let* $A, B \in \mathrm{GL}(n;\mathbf{R})$.

*1)* $A^{-1} \leq B^{-1}$ , $A$*: i-monotone* $\Rightarrow$ $B$*: i-monotone.*

*2)* $A^{-1} \leq B^{-1}$ , $A$*: n-monotone* $\Rightarrow$ $B$*: i-monotone/*$A$.

*3)* $A \leq B$ , $A$*: i-monotone* $\Rightarrow$ $B$*: n-monotone/*$A$.

**Proof.** 1) Since $A^{-1} \geq O$ and $A^{-1} \leq B^{-1}$, $B^{-1} \geq O$.

2) $A \geq O$ and $O \leq I = AA^{-1} \leq AB^{-1} = \left(BA^{-1}\right)^{-1}$.

3) $A^{-1} \geq O$ and $O \leq I = AA^{-1} \leq BA^{-1}$. □

**Definition 23.** Let $\mathbf{a}_i$ $(i=1,2,\cdots,n)$ be linearly independent vectors.

$$\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{a}_i \overset{\mathrm{def}}{=} \left\{\mathbf{x} \,\middle|\, \mathbf{x}=\sum_{i=1}^{n} x_i\mathbf{a}_i\ ,\ x_i \geq 0\right\} = \sum_{i=1}^{n}\mathbf{R}^*\mathbf{a}_i \quad , \quad \mathbf{R}^* \overset{\mathrm{def}}{=} \mathbf{R}^+ \cup \{0\},$$

which is called a pyramid spanned by $\mathbf{a}_i$ $(i=1,2,\cdots,n)$.

The following proposition is easily obtained.

**Proposition 24.**

$$\overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{a}_i) = -\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{a}_i \ .$$

**Theorem 25.** *Let* $A \in M_n(\mathbf{R})$, $B \in \mathrm{GL}(n;\mathbf{R})$, $A=(\mathbf{a}_i)$ *and* $B=(\mathbf{b}_i)$.

*1) A:n-l or r-monotone/B* $\Leftrightarrow$ $\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{a}_i \subset \overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{b}_i$ *or* $\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{a}_i^T \subset \overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{b}_i^T$ ,

*2) A:i-l or r-monotone/B* $\Leftrightarrow$ $\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{b}_i \subset \overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{a}_i$ or $\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{b}_i^T \subset \overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{a}_i^T$ ,

*respectively.*

**Proof.** 1) Let $P=(p_{ij})=B^{-1}A$. Then, $A=BP$, or $\mathbf{a}_i=\sum_{j=1}^{n}p_{ji}\mathbf{b}_j$. Thus, it follows that $P \geq O$ implies $\mathbf{a}_i \in \overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{b}_i$. For the r-monotonicity, $P=(p_{ij})=AB^{-1}$. Then, $A^T=B^TP^T$, or $\mathbf{a}_i^T=\sum_{j=1}^{n}p_{ij}\mathbf{b}_j^T$. Thus, it follows that $P \geq O$ implies $\mathbf{a}_i^T \in \overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{b}_i^T$. Quite similarly, 2) is verified. □

**Corollary 26.**

*1) A:n-monotone* $\Leftrightarrow$ $\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{a}_i \subset \overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{e}_i$ .

*2) A:i-monotone* $\Leftrightarrow$ $\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{e}_i \subset \overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{a}_i$ .

**Proof.** 1) $IP=P=(p_{ij})=AI^{-1}=A$. $\mathbf{a}_i=\sum_{i=1}^{n}p_{ji}\mathbf{e}_j$ . 2) is shown similarly to 1). □

Let $S$ be a set. The set of all interior points of $S$ is denoted by $S^o$.

**Corollary 27.** Let $A=(\mathbf{a}_i)$.

$$A{:}i\text{-}monotone \Rightarrow \mathbf{a}_i \notin \left[\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{e}_i \cup \overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{e}_i)\right]^o = \left(\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{e}_i\right)^o \cup \left(\overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{e}_i)\right)^o$$

$$(i=1,2,\cdots,n).$$

**Proof 1.** $\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{e}_i \subset \overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{a}_i$ is equivalent to $\overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{e}_i) \subset \overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{a}_i)$ . If there exists $\mathbf{a}_i$ such that

$$\mathbf{a}_i \in \left[\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{e}_i \cup \overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{e}_i)\right]^o = \left(\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{e}_i\right)^o \cup \left(\overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{e}_i)\right)^o,$$

either $\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{e}_i \subset \overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{a}_i$ or $\overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{e}_i) \subset \overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{a}_i)$ does not hold according to $\mathbf{a}_i \in \left(\overset{n}{\underset{i=1}{\mathrm{V}}}\mathbf{e}_i\right)^o$ or $\mathbf{a}_i \in \left(\overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{e}_i)\right)^o$. □

**Proof 2.** Suppose that there exists $\mathbf{a}_{i_0}$ satisfying

$$\mathbf{a}_{i_0} \in \left(\overset{n}{\underset{i=1}{\mathrm{V}}} \mathbf{e}_i\right)^o \cup \left(\overset{n}{\underset{i=1}{\mathrm{V}}}(-\mathbf{e}_i)\right)^o \text{, i.e., } \mathbf{a}_{i_0} > \mathbf{0} \text{ or } \mathbf{a}_{i_0} < \mathbf{0}.$$

Let $A^{-1} = (\mathbf{a}_i^{'})$. Then,

$$A = |A| \begin{pmatrix} & \vdots & \\ \cdots & \tilde{a}_{ji}^{'} & \cdots \\ & \vdots & \end{pmatrix} (i \quad ,$$
$$\hat{j}$$

where $\tilde{a}_{ji}^{'}$ is the $(ji)$ cofactor of $A^{-1}$. From

$$\sum_{k=1}^{n} a_{ik}^{'} \tilde{a}_{i_0 k}^{'} = \begin{vmatrix} a_{i1}^{'} & a_{i2}^{'} & \cdots & a_{in}^{'} \\ a_{11}^{'} & a_{12}^{'} & \cdots & a_{1n}^{'} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{'} & a_{n2}^{'} & \cdots & a_{nn}^{'} \end{vmatrix}$$

with the $i_0$th row missing and the first row equal to the $i$th row, the determinant is equal to 0 in case $i \neq i_0$. Furthermore, $\mathbf{a}_{i_0} = |A|\left(\tilde{a}_{i_0 1}^{'}, \tilde{a}_{i_0 2}^{'}, \cdots, \tilde{a}_{i_0 n}^{'}\right)^T > \mathbf{0}$ or $< \mathbf{0}$, then, $a_{ik}^{'} = 0$ for $k = 1, 2, \cdots, n$. Thus, $A^{-1}$is singular. This contradicts that the i-monotone matix is nonsingular.

□

### 4. Heritage from Preconditioner

In the splitting $A = C - R_C$, $C$ is called the preconditioner. This section deals with problems on the properties of $A$ inherited from the preconditioner $C$ in the convergent splitting or convergent iterative method.

**Proposition 28.**

$$C \geq R_C \quad \Leftrightarrow \quad A\text{: } n\text{-monotone}.$$

This is obvious.

**Proposition 29.**

$$C \in \mathcal{R}_{\mathrm{CV}} \quad \Rightarrow \quad A \in \mathrm{GL}(n;\mathbf{R}),\, A \in \mathcal{R}_{\mathrm{CV}}.$$

**Proof.** $\rho(C^{-1}R_C) < 1$ asserts $I - C^{-1}R_C \in \mathrm{GL}(n;\mathbf{R})$ (Section 2). Then, $A = C(I - C^{-1}R_C)$ $\in \mathrm{GL}(n;\mathbf{R})$. Since $R_A = A - A = O$, $A^{-1}R_A = O$ and $\rho(A^{-1}R_A) = 0 < 1$. Thus, $A \in \mathcal{R}_{\mathrm{CV}}$. □

**Corollary 30.**

$$A \in \mathrm{GL}(n;\mathbf{R}) \;\Leftrightarrow\; A \in \mathcal{R}_{\mathrm{CV}} \;\Leftrightarrow\; \mathcal{R}_{\mathrm{CV}} \neq \varnothing.$$

**Proof.** From $A \in \mathrm{GL}(n;\mathbf{R})$, it follows $\rho(A^{-1}R_A) = 0 < 1$. Then, $A \in \mathcal{R}_{\mathrm{CV}}$ and so $\mathcal{R}_{\mathrm{CV}} \neq \varnothing$. Proposition 29 asserts that $\mathcal{R}_{\mathrm{CV}} \neq \varnothing$ leads to $A \in \mathrm{GL}(n;\mathbf{R})$. □

To discuss the convergent iterative method, $\mathcal{R}_{\mathrm{CV}} \neq \varnothing$ is a priori assumed. The corollary says that $\mathcal{R}_{\mathrm{CV}} \neq \varnothing$ automatically implies $A \in \mathrm{GL}(n;\mathbf{R})$. Then, hereafter $A$ is assumed to be nonsingular.

**Proposition 31.** *Let* $C \in \mathcal{Z}_{\mathrm{CV}}$.

1) $C^{-1}R_C \geq O \;\Rightarrow\; A^{-1}R_C \geq O$.

2) *C: i-monotone with* $C^{-1}R_C \geq O \;\Rightarrow\; A$*: i-monotone.*

**Proof.** Let $H = C^{-1}R_C (\geq O)$. Then, $A^{-1}C = (I-H)^{-1} = \sum_{k=0}^{\infty} H^k \geq O$.

Thus, 1) $A^{-1}R_C = A^{-1}C \cdot C^{-1}R_C \geq O$, and 2) $A^{-1} = A^{-1}C \cdot C^{-1} \geq O$. □

In this proof, the following is shown.

**Corollary 32.** *Let* $C \in \mathcal{Z}_{\mathrm{CV}}$ *and* $H = C^{-1}R_C$.

$$H\text{: n-monotone} \;\Rightarrow\; I-H\text{: i-monotone}.$$

**Definition 33.**

$$\mathcal{Z}_{\mathrm{R}} \overset{\mathrm{def}}{=} \left\{ C \in \mathrm{GL}(n;\mathbf{R}) \,\middle|\, C^{-1} \geq O, R_C \geq O \right\}$$
$$= \left\{ C \in \mathrm{GL}(n;\mathbf{R}) \,\middle|\, C: i-monotone,\ R_C : n-monotone \right\}.$$
$$\mathcal{Z}_{\mathrm{WR}} \overset{\mathrm{def}}{=} \left\{ C \in \mathrm{GL}(n;\mathbf{R}) \,\middle|\, C^{-1} \geq O, C^{-1}R_C \geq O \right\}$$
$$= \left\{ C \in \mathrm{GL}(n;\mathbf{R}) \,\middle|\, C: i-monotone,\ R_C : n-l\cdot monotone / C \right\}.$$
$$\mathcal{Z}_{\mathrm{NG}} \overset{\mathrm{def}}{=} \left\{ C \in \mathrm{GL}(n;\mathbf{R}) \,\middle|\, C^{-1}R_C \geq O \right\} = \left\{ C \in \mathrm{GL}(n;\mathbf{R}),\ R_C : n-l\cdot monotone / C \right\}.$$

$\mathcal{Z}_{\mathrm{R}}$, $\mathcal{Z}_{\mathrm{WR}}$ and $\mathcal{Z}_{\mathrm{NG}}$ are the set of all matrices associated with the regular, weak regular and nonnegative splittings, respectively. $\mathcal{Z}_{\mathrm{R}}$ is called a regular splitting set, and matrices in $\mathcal{Z}_{\mathrm{R}}$ are called regular splitting matrices. $\mathcal{Z}_{\mathrm{WR}}$, $\mathcal{Z}_{\mathrm{NG}}$ are weak regular, nonnegative splitting sets and their matrices are weak regular, nonnegative splitting matrices, respectively. The following proposition is readily derived from the definition.

**Proposition 34.**

$$\mathcal{Z}_{\mathrm{R}} \subset \mathcal{Z}_{\mathrm{WR}} \subset \mathcal{Z}_{\mathrm{NG}}.$$

Proposition 31 is alternatively represented as

**Proposition 35.**

$$C \in \mathcal{Z}_{\mathrm{CV}} \cap \mathcal{Z}_{\mathrm{WR}} \;\Rightarrow\; A\text{: i-monotone and } A^{-1}R_C \geq O.$$

**Proposition 36.** *Let* $C \in \mathcal{Z}_{\mathrm{CV}}$.

1) $C \in \mathcal{Z}_{\mathrm{R}} \;\Rightarrow\; A \in \mathcal{Z}_{\mathrm{R}}$

2) $C \in \mathcal{Z}_{\mathrm{WR}} \;\Rightarrow\; A \in \mathcal{Z}_{\mathrm{WR}}$.

Proof is obvious. Combined with $A \in \mathcal{Z}_{\mathrm{CV}}$, this is further written as

**Proposition 37.**

*1)* $\mathcal{R}_{\mathrm{R}} \cap \mathcal{R}_{\mathrm{CV}} \neq \varnothing \quad \Leftrightarrow \quad A \in \mathcal{R}_{\mathrm{R}}$

*2)* $\mathcal{R}_{\mathrm{WR}} \cap \mathcal{R}_{\mathrm{CV}} \neq \varnothing \quad \Leftrightarrow \quad A \in \mathcal{R}_{\mathrm{WR}}$ .

$A \in \mathcal{R}_{\mathrm{NG}}$ implies only $A \in \mathrm{GL}(n;\mathbf{R})$. Then, $C \in \mathcal{R}_{\mathrm{NG}} \cap \mathcal{R}_{\mathrm{CV}} \Rightarrow A \in \mathcal{R}_{\mathrm{NG}}$ and $\mathcal{R}_{\mathrm{NG}} \cap \mathcal{R}_{\mathrm{CV}} \neq \varnothing$ $\Leftrightarrow A \in \mathcal{R}_{\mathrm{NG}}$ .

Inherited properties of $A$ are necessary conditions for the convergent iteration. Now, consider sufficiency of the inherited properties for $C \in \mathcal{R}_{\mathrm{CV}}$.

**Lemma 38.**

$$\mathbf{0} \le \mathbf{x} \le \mathbf{y} \Rightarrow \|\mathbf{x}\| \le \|\mathbf{y}\| \quad \textit{(Euclidean norm)} .$$

**Proof.** If $\mathbf{x} = \mathbf{0}$, proof is trivial. By the assumption, $(\mathbf{y} - \mathbf{x}, \mathbf{x}) \ge 0$. Here, $(\mathbf{a}, \mathbf{b})$ is the inner product of two vectors $\mathbf{a}$, $\mathbf{b}$. Thus, $(\mathbf{y}, \mathbf{x}) \ge \|\mathbf{x}\|^2$. $\|\mathbf{y}\| \ge \left| \left( \mathbf{y}, \frac{\mathbf{x}}{\|\mathbf{x}\|} \right) \right| = \frac{1}{\|\mathbf{x}\|} (\mathbf{y}, \mathbf{x}) \ge \|\mathbf{x}\|$ . □

**Proposition 39.** (Weak Regular Splitting Theorem)

$$A\textit{: i-monotone} \quad \Leftrightarrow \quad \varnothing \neq \mathcal{R}_{\mathrm{WR}} \subset \mathcal{R}_{\mathrm{CV}} .$$

**Proof.** $\Leftarrow$). Obvious from Prop.35.

$\Rightarrow$). From $A$: i-monotone, $A \in \mathcal{R}_{\mathrm{WR}}$. Then, $\mathcal{R}_{\mathrm{WR}} \neq \varnothing$. Now prove $\mathcal{R}_{\mathrm{WR}} \subset \mathcal{R}_{\mathrm{CV}}$. Let $C \in \mathcal{R}_{\mathrm{WR}}$. Then, $C^{-1} \ge O$, $H = C^{-1} R_C \ge O$, and $C^{-1} = (I - H) A^{-1}$. Thus,

$$O \le \left( \sum_{j=0}^{m} H^j \right) C^{-1} = \left( I - H^{m+1} \right) A^{-1} \le A^{-1} ,$$

whence

$$O \le \left( C^{-1} \right)^T \left( \sum_{j=0}^{m} \left( H^T \right)^j \right) \le \left( A^{-1} \right)^T . \tag{3}$$

Since $H^T \ge O$, there exists an eigenvector $\mathbf{u}$ $(\neq \mathbf{0}, \ge \mathbf{0})$ of $H^T$ corresponding to the eigenvalue $\rho(H^T)$. From $\rho(H^T) = \rho(H)$, $H^T \mathbf{u} = \rho(H) \mathbf{u}$. Equation (3) and Lemma 38 asserts that

$$\left( \sum_{j=0}^{m} \rho(H)^j \right) \left\| \left( C^{-1} \right)^T \mathbf{u} \right\| \le \left\| \left( A^{-1} \right)^T \mathbf{u} \right\| < \infty .$$

From $\mathbf{u} \neq \mathbf{0}$, $\left\| \left( C^{-1} \right)^T \mathbf{u} \right\| > 0$. If $\rho(H) = 1$, $(m+1) \left\| \left( C^{-1} \right)^T \mathbf{u} \right\| \le \left\| \left( A^{-1} \right)^T \mathbf{u} \right\| < \infty$, for any $m$. This is contradiction. If $\rho(H) > 1$, $\sum_{j=0}^{m} \rho(H)^j = \frac{\rho(H)^{m+1} - 1}{\rho(H) - 1}$. From the same reason as $\rho(H) = 1$, contradiction is attained. Then, $\rho(H) < 1$. □

**Corollary 40.** (Regular Splitting Theorem)

$$A\textit{: i-monotone} \quad \Leftrightarrow \quad \varnothing \neq \mathcal{R}_{\mathrm{R}} \subset \mathcal{R}_{\mathrm{CV}} .$$

**Proof.** $\Rightarrow$). $A \in \mathcal{R}_{\mathrm{R}}$. Then, $\mathcal{R}_{\mathrm{R}} \neq \varnothing$. Proof of the rest is evident from Prop.39.

$\Leftarrow$). Trivial from Prop.35. □

**Corollary 41.**

$$A,\ C\text{: i-monotone},\ C \geq A \text{ or } R_C \geq O \quad \Rightarrow \quad C \in \mathcal{Z}_{\mathrm{CV}}\ .$$

**Proof.** $C^{-1} \geq O$, $R_C \geq O \Leftrightarrow C \in \mathcal{Z}_{\mathrm{R}}$ . □

The following is readily obtained.

**Proposition 42.**

$$\begin{aligned} A\text{: i-monotone} \quad &\Leftrightarrow \quad A \in \mathcal{Z}_{\mathrm{R}} \quad \Leftrightarrow \quad A \in \mathcal{Z}_{\mathrm{WR}} \\ &\Leftrightarrow \quad \emptyset \neq \mathcal{Z}_{\mathrm{WR}} \subset \mathcal{Z}_{\mathrm{CV}} \quad \Leftrightarrow \quad \emptyset \neq \mathcal{Z}_{\mathrm{R}} \subset \mathcal{Z}_{\mathrm{WR}} \subset \mathcal{Z}_{\mathrm{CV}} \\ &\Leftrightarrow \quad \emptyset \neq \mathcal{Z}_{\mathrm{R}} \subset \mathcal{Z}_{\mathrm{CV}} \\ &\Leftrightarrow \quad \mathcal{Z}_{\mathrm{R}} \cap \mathcal{Z}_{\mathrm{CV}} \neq \emptyset \quad \Leftrightarrow \quad \mathcal{Z}_{\mathrm{WR}} \cap \mathcal{Z}_{\mathrm{CV}} \neq \emptyset\ . \end{aligned}$$

**N.B.** Proposition 42 does not imply that $\mathcal{Z}_{\mathrm{WR}} \cap \mathcal{Z}_{\mathrm{CV}} = \mathcal{Z}_{\mathrm{R}} \cap \mathcal{Z}_{\mathrm{CV}}$.

**Theorem 43.** *Let $C \in \mathcal{Z}_{\mathrm{NG}}$, $H = C^{-1}R_C$ and $F_H = (I-H)^{-1}$. The following properties are equivalent.*

*0) $C \in \mathcal{Z}_{\mathrm{CV}}$.*

*1) $A^{-1}C \geq O$, i.e., $C$: n-l·monotone/A, $I-H$: i-monotone or $F_H$: n-monotone.*

*2) $A^{-1}R_C \geq O$, i.e., $R_C$: n-l·monotone/A.*

*3) $A^{-1}R_C \geq C^{-1}R_C$.*

*4)* $\rho(A^{-1}R_C) = \dfrac{\rho(C^{-1}R_C)}{1-\rho(C^{-1}R_C)}$ *, i.e.,* $\rho(C^{-1}R_C) = \dfrac{\rho(A^{-1}R_C)}{1+\rho(A^{-1}R_C)}$ .

**Proof.** 0)$\Rightarrow$1)$\Rightarrow$2) is shown in the proof of Prop. 31.

2)$\Leftrightarrow$3). $(A^{-1}-C^{-1})R_C = \{A^{-1}(C-A)C^{-1}\}R_C = (A^{-1}R_C C^{-1})R_C = A^{-1}R_C \cdot C^{-1}R_C \geq O$.

Conversely, $A^{-1}R_C \geq C^{-1}R_C \geq O$.

2)$\Rightarrow$1). $A^{-1}R_C \geq O \Leftrightarrow \left(\bigvee_{i=1}^{n} \mathbf{r}_i \subset \bigvee_{i=1}^{n} \mathbf{a}_i\right)$ with $R_C = (\mathbf{r}_i)$ . Let $C = (\mathbf{c}_i)$. $\mathbf{c}_i - \mathbf{a}_i = \mathbf{r}_i \in \bigvee_{i=1}^{n} \mathbf{a}_i$ .

Then, $\mathbf{c}_i = \mathbf{a}_i + \mathbf{r}_i \in \bigvee_{i=1}^{n} \mathbf{a}_i$ . Thus, $\bigvee_{i=1}^{n} \mathbf{c}_i \subset \bigvee_{i=1}^{n} \mathbf{a}_i$ , i.e., $A^{-1}C \geq O$ .

1)$\Rightarrow$0)$\Rightarrow$4). Since $C \in \mathcal{Z}_{\mathrm{NG}}$, $H = C^{-1}R_C \geq O$. Then, $\rho(H)$ is an eigenvalue of $H$. Let $\mathbf{u}$ be an eigenvector corresponding to $\rho(H)$ and $\mathbf{u} \geq \mathbf{0}$. Thus,

$$(I-H)\mathbf{u} = (1-\rho(H))\mathbf{u}\ ,$$

whence $1-\rho(H)$ is an eigenvalue of $I-H$. From the assumption of $C \in \mathcal{Z}_{\mathrm{NG}}$, $I-H$ becomes i-monotone (Cor. 32) and so nonsingular. Then, all eigenvalues of $I-H$ are nonzero. Thus, $1-\rho(H) \neq 0$ and, moreover,

$$(I-H)^{-1}\mathbf{u} = \frac{1}{1-\rho(H)}\mathbf{u}\ , \tag{4}$$

and $\qquad (I-H)^{-1}H\mathbf{u}=\dfrac{\rho(H)}{1-\rho(H)}\mathbf{u}$ .

Assuming 1), $(I-H)^{-1}\geq O$ . Combined with $\mathbf{u}\geq\mathbf{0}$, eqn. (4) leads to $\rho(H)<1$, which is the property 0). Next, assuming 0), the property 2) holds as already proved. Then, $A^{-1}R_C=A^{-1}C\cdot C^{-1}R_C=(I-H)^{-1}H\geq O$. With $\mathbf{u}\geq\mathbf{0}$, then $\dfrac{\rho(H)}{1-\rho(H)}>0$ . Since $\dfrac{\rho(H)}{1-\rho(H)}$ is an eigenvalue of $A^{-1}R_C$, $\dfrac{\rho(H)}{1-\rho(H)}\leq\rho\left(A^{-1}R_C\right)$ . All eigenvalues of $A^{-1}R_C$ are expressed in the form of $\dfrac{\lambda}{1-\lambda}$ with the eigenvalue $\lambda$ of $H$. From $\rho(H)\geq|\lambda|$ and $0<1-\rho(H)\leq 1-|\lambda|\leq|1-\lambda|$, is shown $\left|\dfrac{\lambda}{1-\lambda}\right|\leq\dfrac{\rho(H)}{1-\rho(H)}$ . Thus, $\rho\left(A^{-1}R_C\right)=\dfrac{\rho(H)}{1-\rho(H)}$ .

4)$\Rightarrow$0). It follows immediately from $\rho\left(C^{-1}R_C\right)=\dfrac{\rho\left(A^{-1}R_C\right)}{1+\rho\left(A^{-1}R_C\right)}<1$ . $\square$

**Definition 44.**

$$\mathscr{R}_1\overset{\text{def}}{=}\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,A^{-1}C\geq O\right\}.$$

$$\mathscr{R}_2\overset{\text{def}}{=}\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,A^{-1}R_C\geq O\right\}.$$

$$\mathscr{R}_3\overset{\text{def}}{=}\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,A^{-1}R_C\geq C^{-1}R_C\right\}.$$

$$\mathscr{R}_4\overset{\text{def}}{=}\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,\rho\left(A^{-1}R_C\right)=\frac{\rho(H)}{1-\rho(H)},\ H=C^{-1}R_C\right\}.$$

The following is easily obtained.

**Proposition 45.** *Let* $H=C^{-1}R_C$, $F_H=(I-H)^{-1}$, $A=(\mathbf{a}_i)$, $C=(\mathbf{c}_i)$ *and* $R_C=(\mathbf{r}_i)$.

*1)*
$$\begin{aligned}\mathscr{R}_1&=\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,C:n-l\cdot monotone/A\right\}\\&=\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,\bigvee_{i=1}^{n}\mathbf{c}_i\subset\bigvee_{i=1}^{n}\mathbf{a}_i\right\}\\&=\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,A:i-l\cdot monotone/C\right\}\\&=\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,I-H:i-monotone\right\}\\&=\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,F_H:n-monotone\right\}.\end{aligned}$$

*2)*
$$\begin{aligned}\mathscr{R}_2&=\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,R_C:n-l\cdot monotone/A\right\}\\&=\left\{C\in\mathrm{GL}(n;\mathbf{R})\,\middle|\,\bigvee_{i=1}^{n}\mathbf{r}_i\subset\bigvee_{i=1}^{n}\mathbf{a}_i\right\}.\end{aligned}$$

Theorem 43 implies

**Theorem 46.**

$$\mathscr{R}_{\mathrm{CV}}\cap\mathscr{R}_{\mathrm{NG}}=\mathscr{R}_i\cap\mathscr{R}_{\mathrm{NG}}\quad(i=1,2,3,4).$$

**Corollary 47.**

$$\mathscr{R}_i\cap\mathscr{R}_{\mathrm{NG}}\subset\mathscr{R}_{\mathrm{CV}}\quad(i=1,2,3,4).$$

If $A$ is i-monotone and $R_C \geq O$, then $C \subset \mathcal{R}_2$. Thus, $\mathcal{R}_R \subset \mathcal{R}_2 \cap \mathcal{R}_{NG}$. Accordingly, the regular splitting theorem is proved in the other way than the proof based on the weak regular splitting theorem.

## 5. Convergence Condition of the Jacobi and Gauss-Seidel Iterations

**Definition 48.**

$$\mathcal{M} \overset{\text{def}}{=} \{A \in M_n(\mathbf{R}) \mid A : \text{i} - \text{monotone}\}$$

$$\mathcal{Z} \overset{\text{def}}{=} \{A \in M_n(\mathbf{R}) \mid A \leq D_A\} \quad D_A \overset{\text{def}}{=} \operatorname{diag}(a_{11}, a_{22}, \cdots, a_{nn})$$

$$\mathcal{L} \overset{\text{def}}{=} \{A \in \mathcal{Z} \mid a_{ii} > 0 \; (i = 1, 2, \cdots, n)\}$$

$$\mathscr{M} \overset{\text{def}}{=} \mathcal{Z} \cap \mathcal{M}$$

$$\mathcal{B} \overset{\text{def}}{=} \{B \in M_n(\mathbf{R}) \mid B = I - D_A^{-1}A, A \in \mathcal{L}\} = \mathcal{Z}_p \cap \mathcal{Q}_0,$$

where $\quad \mathcal{Z}_p \overset{\text{def}}{=} -\mathcal{Z}, \quad \mathcal{Q}_0 \overset{\text{def}}{=} \{A \in M_n(\mathbf{R}) \mid D_A = 0\}$.

$\mathcal{M}$ denotes the set of all i-monotone matrices, $\mathcal{Z}$ the set of all Z-matrices, $\mathcal{L}$ the set of all L-matrices and $\mathscr{M}$ the set of all M-matrices.

Let $A = D_A - L - U$ with the strictly lower and upper triangular matrices $L, U \geq O$.

**Lemma 49.**

$$A \in \mathscr{M} \quad \Rightarrow \quad A \in \mathcal{L}.$$

**Proof.** It suffices to say that $a_{ii} > 0 \; (i = 1, 2, \cdots, n)$. $A \in \mathscr{M}$ is equivalent to $\overset{n}{\underset{i=1}{\mathrm{V}}} \mathbf{e}_i \subset \overset{n}{\underset{i=1}{\mathrm{V}}} \mathbf{a}_i$ and $A \in \mathcal{Z}$. Suppose the existence of $i_0$ such that $a_{i_0 i_0} \leq 0$. Since $\mathbf{e}_{i_0} = \sum_{j=1}^{n} p_{ji} \mathbf{a}_j$ with $p_{ji} \geq 0 \; (j = 1, 2, \cdots, n)$, the $i_0$th component of both sides gives $1 = \sum_{j=1}^{n} p_{ji} a_{ji_0}$, the righthand side of which is nonpositive. This is contradiction. □

In case $A \in \mathcal{L}$, a necessary and sufficient condition for convergence of the Jacobi iteration is given by

**Proposition 50.** (Condition for Convergent Jacobi Iteration)

*Let $A \in \mathcal{Z}$, $C = D_A$.*

$$A \in \mathscr{M} \text{ or } A\text{: i-monotone} \iff A \in \mathcal{L}, C \in \mathcal{R}_{CV}.$$

**Proof.** From Lemma 49, $C^{-1} = D_A^{-1} \geq O$. Then, $C$ is i-monotone. $R_C = L + U \geq O$.

$\Rightarrow$). Then, $C \in \mathcal{R}_R$. Thus, Cor. 40 (Regular Splitting Theorem) asserts $C \in \mathcal{R}_{CV}$. Conversely, $A \in \mathscr{M}$ follows from Prop. 35. In fact, $C^{-1} \geq O$ from $A \in \mathcal{L}$ and $R_C \geq O$. Then,

$C \in \mathscr{R}_{\mathrm{R}} \subset \mathscr{R}_{\mathrm{WR}}$. $\square$

The following supplies a necessary and sufficient condition for convergence of the Gauss-Seidel iteration for $A \in \mathscr{L}$.

**Proposition 51.** (Condition for Convergent Gauss-Seidel Iteration)

*Let* $A \in \mathscr{Z}$, $C = D_A - L$.

$A \in \mathscr{M}$ *or* $A$*: i-monotone* $\Leftrightarrow$ $A \in \mathscr{L}, C \in \mathscr{R}_{\mathrm{CV}}$ .

**Proof.** Since $\left(D_A^{-1}L\right)^n = O$,

$$\left(I - D_A^{-1}L\right)\left[I + D_A^{-1}L + \cdots + \left(D_A^{-1}L\right)^{n-1}\right] = I \ .$$

Thus,

$$\left(I - D_A^{-1}L\right)^{-1} = I + D_A^{-1}L + \cdots + \left(D_A^{-1}L\right)^{n-1}.$$

From $D_A^{-1}L \geq O$, $C = D_A\left(I - D_A^{-1}L\right)$ is i-monotone. $\Rightarrow$). Obviously, $R_C = U \geq O$. Then, $C \in \mathscr{R}_{\mathrm{R}}$. Thus, $A \in \mathscr{M} \Rightarrow C \in \mathscr{R}_{\mathrm{CV}}$. The converse is shown from Prop.35. In fact, $A \in \mathscr{L}$ gives $D_A^{-1}L \geq O$. Then, $C^{-1} \geq O$. With $R_C \geq O$, $C \in \mathscr{R}_{\mathrm{R}} \subset \mathscr{R}_{\mathrm{WR}}$. $\square$

## 6. Concluding Remarks

A general aspect of the iterative method is presented for the linear system of $A\mathbf{x} = \mathbf{b}$. The iterative method is reasonable if it converges. The convergent iteration is considered an inversion of *A*. Equivalent concepts of the convergent iterative method are described. The convergent splitting, the iterative matrix *H* with its spectral radius less than 1 and the matrix $(I - H)^{-1}$ are equivalent to the convergent iterative method.

A concept of the general monotonicity is introduced. The so-called monotonicity of *A* is identical with the inverse monotonicity of *A*. A geometrical representation of the general monotonicity facilitates to make an intuitive image of the monotonicity.

From the viewpoint of the convergent splitting, a necessary and sufficient condition for convergence of the Jacobi and Gauss-Seidel iterations is that *A* is an M-matrix in case *A* is a Z-matrix or equivalently an L-matrix. The following bibliography is the list referred to during preparation of this paper for the sake of confirming its originality.

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