数理解析研究所講究録
1084 巻 1999 年 111-123

Finite Element Analysis for Parametrized Nonlinear Equations around Turning Points

愛媛大学理学部　　土屋卓也 (Takuya TSUCHIYA)

Department of Mathematical Sciences
Ehime University
tsuchiya@math.sci.ehime-u.ac.jp

Abstract. Nonlinear equations with parameters are called parametrized nonlinear equations. In this paper, a priori error estimates of finite element solutions of parametrized nonlinear elliptic equations on branches around turning points are considered. Existence of a finite element solution branch is shown under suitable conditions on an exact solution branch around a turning point. Also, some error estimates of distance between exact and finite element solution branches are given. It is shown that error of a parameter is much smaller than that of functions. Approximation of nondegenerate turning points is also considered. We show that if a turning point is nondegenerate, there exists a locally unique finite element nondegenerate turning point. At a nondegenerate turning point an elaborate error estimate of the parameter is proved.

1. Introduction.

Let A, B be Banach spaces and $\Lambda \subset \mathbb{R}^n$ a bounded interval. Let $F : \Lambda \times A \to B$ be a smooth operator. The nonlinear equations

$$F(\lambda, u) = 0,$$

with parameter $\lambda \in \Lambda$ is called **parametrized nonlinear equations**.

In [17] and [18] a thorough theory of a priori error estimates of finite element solutions of the following parametrized strongly nonlinear problems has been developed:

$$F(\lambda, u) = 0, \quad (\lambda, u) \in \Lambda \times H_0^1(\Omega),$$

$$\begin{aligned}\langle F(\lambda, u), v\rangle := \int_\Omega &[\vec{\mathbf{a}}(\lambda, x, u(x), \nabla u(x)) \cdot \nabla v(x) \\ &+ f(\lambda, x, u(x), \nabla u(x)) v(x)] dx, \quad \forall v \in H_0^1(\Omega),\end{aligned} \tag{1.1}$$

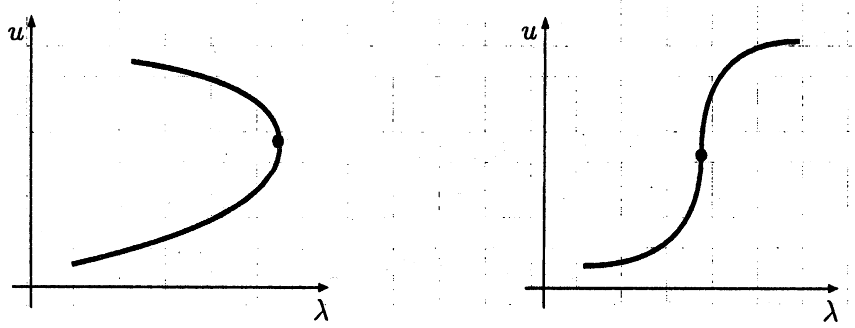

Figure 1: Nondegenerate and degenerate turning points.

where $\Omega \subset \mathbb{R}^d$ ($d = 1,2,3$) is a bounded domain with the piecewise C^2 boundary $\partial\Omega$, and $\vec{\mathbf{a}} : \Lambda \times \overline{\Omega} \times \mathbb{R}^{d+1} \to \mathbb{R}^d$, $f : \Lambda \times \overline{\Omega} \times \mathbb{R}^{d+1} \to \mathbb{R}$ are sufficiently smooth functions. Here, the equation (1.1) is called **strongly nonlinear** if $\vec{\mathbf{a}}(\lambda, x, y, z)$ ($\lambda \in \Lambda$, $x \in \Omega$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$) is nonlinear with respect to z. Otherwise, it is called **mildly nonlinear**.

Since the equation (1.1) is defined in divergence form, finite element solutions to (1.1) is defined in a natural way.

In [8], [9], and [13] Fink and Rheinboldt have shown that some subset of the solutions to (1.1) form an one-dimensional smooth manifold without boundaries, if the nonlinear operator defined by (1.1) is Fréchet differentiable and Fredholm of index 1. They have also shown that corresponding finite element solutions form an one-dimensional smooth manifold. In this paper we denote by $\mathcal{M}_0$ and $\mathcal{M}_h$ the exact solution manifold of (1.1) and the corresponding finite element solution manifold, respectively.

Here, a linear operator $P \in \mathcal{L}(A, B)$ is called **Fredholm** if (1) the dimension of $\mathrm{Ker}P$ is finite, (2) $\mathrm{Im}A \subset B$ is closed, (3) the dimension of $\mathrm{Coker}A := B/\mathrm{Im}A$ is finite. If $P \in \mathcal{L}(A, B)$ is Fredholm, its **index** $\mathrm{ind}P$ is defined by $\mathrm{ind}P := \dim\mathrm{Ker}P - \dim\mathrm{Coker}P$. Let $U \subset A$ be open and $F : U \to B$ Fréchet differentiable. F is called Fredholm in U if its Fréchet derivative $DF(u) \in \mathcal{L}(A, B)$ is Fredholm at any $u \in U$. It is shown that $\mathrm{ind}DF(u)$ is constant in each connected component of U. Hence, we define the index of F by $\mathrm{ind}F := \mathrm{ind}DF(u)$.

In [17] and [18], it is shown that, under reasonable conditions, for each compact subset $\widetilde{\mathcal{M}}_0 \subset \mathcal{M}_0$, there exists a locally unique compact subset $\widetilde{\mathcal{M}}_h \subset \mathcal{M}_h$ such that $\widetilde{\mathcal{M}}_0$ is approximated uniformly by $\widetilde{\mathcal{M}}_h$, if triangulation of Ω is sufficiently fine. Moreover, several a priori error estimates are obtained.

The aim of this paper is to refine the error analysis on branches around **turning points**.

A point $(\lambda, u) \in \mathcal{M}_0$ is called a turning point if the partial Fréchet derivative $D_uF(\lambda, u) \in \mathcal{L}(A, B)$ at (λ, u) is not an isomorphism.

To develop a refined error analysis around a turning point, we introduce a slightly different formulation of the problem from that in [17], and show a theorem which is similar to [18, Theorem 8.6] and [17, Corollary 7.8]. Next, we obtain an elaborate error estimate of parameter. In the following we explain the basic ideas of this paper.

In the error analysis of parametrized nonlinear equations, we have the following difficulty. Suppose that we are approaching a turning point during continuation process of a solution branch. Since we cannot fix the parameter λ around a turning point in (1.1), λ should be treated as an unknown parameter. Hence, correspondence of an approximated solution to an exact solution becomes ambiguous in such a situation.

Usually, this difficulty is overcome by the following manner. We introduce a (nonlinear, in general) functional $\rho : \Lambda \times A \to \mathbb{R}$, and consider the following problem:

$$H(\gamma, \lambda, u) := (\rho(\lambda, u) - \gamma, F(\lambda, u)) = (0, 0) \in \mathbb{R} \times A, \tag{1.2}$$

where $H : \mathbb{R} \times \Lambda \times A \to \mathbb{R} \times B$. We expect that the partial Fréchet derivative $D_{(\lambda,u)}H(\gamma, \lambda, u) \in \mathcal{L}(\mathbb{R} \times A, \mathbb{R} \times B)$ is an isomorphism at a turning point (λ, u) and in its neighborhood. In Section 2, it will be shown that, if $D_\lambda F(\lambda, u) \neq 0$ and $\mathrm{Ker}DF(\lambda, u) \cap \mathrm{Ker}D\rho(\lambda, u) = \{(0, 0)\}$ at $(\lambda, u) \in \mathcal{M}_0$, then the above partial Fréchet derivative is an isomorphism. If we could find a good definition of such ρ, then the solution branch would now be parametrized by γ.

Finite element solutions (λ_h, u_h) would be defined by

$$H_h(\gamma, \lambda, u) := (\rho(\lambda_h, u_h) - \gamma, F_h(\lambda_h, u_h)) = (0, 0), \tag{1.3}$$

where F_h is an approximation of F. In this setting the correspondence of an exact solution (λ, u) and an finite element solution (λ_h, u_h) is represented by $\rho(\lambda_h, u_h) = \gamma = \rho(\lambda, u)$.

In the above setting we will show that, even around a turning point, there exists a locally unique finite element solution branch near an exact solution branch under suitable conditions. Also, some error estimates of distance between the exact and finite element solution branches are given.

Next, we will consider an elaborate error estimate of parameter λ. In error analysis of the finite element method (1.3) for (1.2) around a turning point, we would have error estimates such as

$$|\lambda - \lambda_h| + \|u - u_h\|_A \le Ch^r.$$

In many practical computation, it is usually observed that the error $|\lambda - \lambda_h|$ is much smaller than $\|u - u_h\|_A$, or Ch^r.

A typical and well-known example of this phenomenon is finite element approximation of the eigenvalue problems:

$$-\Delta u = \lambda u, \qquad u \in H_0^1(\Omega). \tag{1.4}$$

Let (λ, u) be an eigen-pair of (1.4) and (λ_h, u_h) its finite element approximation. Suppose that the eigenvalue λ is simple. Then we have an error estimate such as

$$|\lambda - \lambda_h| \le C\|u - u_h\|_{H_0^1}^2,$$

where C is a positive constant independent of h (see, for example, [14, Chapter 6], [1]).

We will show that a similar estimate hold for the finite element solutions (λ_h, u_h) of (1.3) under the condition that $D_u F(\lambda, u)$ is self-adjoint. To obtain a similar estimate we introduce an auxiliary equation. Let z and z_h be the exact and finite element solutions to the auxiliary equation. We will show that the error $|\lambda - \lambda_h|$ is estimated as

$$|\lambda - \lambda_h| \le C\|u - u_h\|_A(\|u - u_h\|_A + \|z - z_h\|_A)$$

around a turning point, where C is a positive constant independent of h.

Occasionally, a turning point on the exact solution manifold $\mathcal{M}_0$ has a certain physical meaning, and, in such a case, computing its precise value will become important. If a turning point $(\lambda_0, u_0) \in \mathcal{M}_0$ is nondegenerate (see Section 3 for its definition), we can show that the associated finite element solution manifold also has a locally unique nondegenerate turning point $(\lambda_0^h, u_0^h) \in \mathcal{M}_h$. The error $|\lambda_0 - \lambda_0^h|$ is estimated accurately by a similar manner as above.

In Section 2 and 3 we develop our theory in an abstract setting. Applications of the abstract theorems obtained in Section 2 and 3 to the strongly nonlinear elliptic boundary value problem (1.1) will be found in the original version of this paper.

2. Abstract Formulation.

In this section, we formulate our problem in an abstract setting, and show a theorem which claims existence of a locally unique solution branch of a discretized problem. The setting in this section is slightly different from that of [17].

For the stage of our analysis we first introduce functional spaces.

(A1) There are Banach spaces, V, W, and X_p $(1 \le p \le \infty)$, where X_2 is a Hilbert space, such that $V \subset X_\infty \subset X_p$ $(1 \le p \le \infty)$ and $W \subset X_1{}' \subset X_q{}'$ $(1 \le q \le \infty)$. Here, X_q' is the dual space of X_q. We suppose that all inclusions are continuous. We also suppose that X_r is dense in X_p if $1 \le p \le r < \infty$.

Let $F : \Lambda \times X_p \to X_q'$ $(1/p + 1/q = 1)$ be a nonlinear map, where $\Lambda \subset \mathbb{R}$ is an interval. We consider the parametrized nonlinear equation $F(\lambda, u) = 0$. Since we will suppose that F is strongly nonlinear, the domain and the range should be taken carefully. In many cases, F is not Fréchet differentiable on $\Lambda \times X_p$, $p < \infty$, and should be restricted to a certain subspace to make it differentiable.

We also need extensions and restrictions of the Fréchet derivatives $DF(\lambda, v)$, $D_vF(\lambda, v)$ etc. at (λ, v). When we need to specify the domain of, say, $D_vF(\lambda, v)$ clearly, we will write such as $D_vF(\lambda, v) \in \mathcal{L}(P, Q)$. This means that $D_vF(\lambda, v)$ now denotes its extension (or restriction) whose domain is P and range is in Q.

Now, we take certain $p \ge 2$ and q with $1/p + 1/q = 1$, and fix them. We then assume the following:

(A2) The restriction of F to $\Lambda \times X_\infty$, denoted by F again, is a Fréchet differentiable map from $\Lambda \times X_\infty$ to X_1'. For any $\lambda \in \Lambda$ and $v \in X_\infty$ the derivative $DF(\lambda, v) \in \mathcal{L}(\mathbb{R} \times X_\infty, X_1')$ can be extended to $DF(\lambda, v) \in \mathcal{L}(\mathbb{R} \times X_p, X_q')$ and it is locally Lipschitz continuous on $\Lambda \times X_\infty$: i.e., for any bounded convex set $\mathcal{O} \subset \Lambda \times X_\infty$ there exists a positive constant $C_1(\mathcal{O})$ such that

$$\|DF(\lambda_1, v) - DF(\lambda_2, w)\|_{\mathcal{L}(\mathbb{R}\times X_p, X_q')} \le C_1(\mathcal{O})(|\lambda_1 - \lambda_2| + \|v - w\|_{X_\infty})$$

for arbitrary (λ_1, v), $(\lambda_2, w) \in \mathcal{O}$.

(A3) We suppose that there exists an open subset $\mathcal{S} \subset \Lambda \times V$ in which $F : \mathcal{S} \to W$ is a Fredholm operator of index 1. We also suppose that, for each $(\lambda, u) \in \mathcal{S}$, $DF(\lambda, u) \in \mathcal{L}(\mathbb{R} \times X_p, X_q')$ is a Fredholm operator of index 1 as well.

We define the subset $\mathcal{R}(F, \mathcal{S}) \subset \mathcal{S}$ by

$$\mathcal{R}(F, \mathcal{S}) := \{(\lambda, u) \in \mathcal{S} | DF(\lambda, u) \in \mathcal{L}(\mathbb{R} \times V, W) \text{ is onto}\}.$$

The following lemma is valid:

Lemma 2.1. (1) *For any* $(\lambda, u) \in \mathcal{R}(F, \mathcal{S})$, $\dim \mathrm{Ker} D_uF(\lambda, u)$ *is at most* 1.

(2) For $(\lambda, u) \in \mathcal{R}(F, \mathcal{S})$, we have either

Case 1: $\mathrm{Ker} D_u F(\lambda, u) = \{0\}$ *and* $D_\lambda F(\lambda, u) \in \mathrm{Im} D_u F(\lambda, u)$, *or*

Case 2: $\dim \mathrm{Ker} D_u F(\lambda, u) = 1$, *and* $D_\lambda F(\lambda, u) \notin \mathrm{Im} D_u F(\lambda, u)$. □

For the proof, see [18, Section 4].

We introduce a nonlinear functional $\rho : \Lambda \times X_p \to \mathbb{R}$ and assume that

(A4) The restriction of ρ to $\Lambda \times X_\infty$, denoted by ρ again, is Fréchet differentiable.

(A5) For $(\lambda, u) \in \Lambda \times X_\infty$, the Fréchet derivative $D\rho(\lambda, u) \in \mathcal{L}(\mathbb{R} \times X_\infty, \mathbb{R})(= \mathbb{R} \times X'_\infty)$ can be extended to $D\rho(\lambda, u) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R})(= \mathbb{R} \times X'_p)$, and it is locally Lipschitz continuous on $\Lambda \times X_\infty$, i.e., for any bounded convex set $\mathcal{O} \subset \Lambda \times X_\infty$, there exists a positive constant $C_2(\mathcal{O})$ such that

$$\|D\rho(\lambda_1, v) - D\rho(\lambda_2, w)\|_{\mathbb{R} \times X'_p} \le C_2(\mathcal{O})(|\lambda_1 - \lambda_2| + \|v - w\|_{X_\infty})$$

for any $(\lambda_1, v), (\lambda_2, w) \in \mathcal{O}$.

(A6) Let $(\lambda, u) \in \mathcal{S}$ and $D_u F(\lambda, u) \in \mathcal{L}(X_p, X'_q)$. We suppose that if $D_u F(\lambda, u)\psi = f$ for $\psi \in X_p$ and $f \in W$, then $\psi \in V$.

Lemma 2.2. *Assume that (A1)-(A6) are valid. Suppose that there is* $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})$ *such that* $D_\lambda F(\lambda_0, u_0) \neq 0 \in W$. *From (A3), there exists* $(\mu_0, \psi_0) \in \mathbb{R} \times V$ *such that* $\mathrm{Ker} DF(\lambda_0, u_0) = \mathrm{span}\{(\mu_0, \psi_0)\}$. *We assume that* $D\rho(\lambda_0, u_0)(\mu_0, \psi_0) \neq 0 \in \mathbb{R}$.

Define $G : \Lambda \times W \to \mathbb{R} \times V$ *by* $G(\lambda, u) := (\rho(\lambda, u) - \gamma, F(\lambda, u))$, *where* $\gamma \in \mathbb{R}$.

Then, $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times W, \mathbb{R} \times V)$ *is an isomorphism. Moreover,* $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R} \times X'_q)$ *is an isomorphism as well.*

Proof. From the assumptions we find that $\mathrm{Ker} DF(\lambda_0, u_0) \cap \mathrm{Ker} D\rho(\lambda_0, u_0) = \{(0, 0)\}$. This implies that $\mathrm{Ker} DG(\lambda_0, u_0)$ is trivial and $DG(\lambda_0, u_0)$ is one-to-one.

Since $DF(\lambda_0, u_0)$ is onto, for any $g \in W$, there is $(\nu, \varphi) \in \mathbb{R} \times V$ such that $DF(\lambda_0, u_0)(\nu, \varphi) = g$. Since $D\rho(\lambda_0, u_0)(\mu_0, \psi_0) \neq 0$, for any $t \in \mathbb{R}$, there is $\alpha \in \mathbb{R}$ such that $D\rho(\lambda_0, u_0)((\nu, \varphi) + \alpha(\mu_0, \psi_0)) = t$. This yields that $DG(\lambda_0, u_0)$ is onto. Therefore, $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times V, \mathbb{R} \times W)$ is an isomorphism.

To show that $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R} \times X'_q)$ is an isomorphism, we first show that $DF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, X'_q)$ is onto. Since $DF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, X'_q)$ is Fredholm with index 1 by (A3), we only have to show that the dimension of $\mathrm{Ker} DF(\lambda_0, u_0) \subset \mathbb{R} \times X_p$ is 1.

Let $(\mu, \psi) \in \mathbb{R} \times X_p$ be such that $DF(\lambda_0, u_0)(\mu, \psi) = 0 \in X'_q$. This is also written as $D_u F(\lambda_0, u_0)\psi = -\mu D_\lambda F(\lambda_0, u_0)$. Since $D_\lambda F(\lambda_0, u_0) \in W$ and (A6), we conclude that $\psi \in W$ and $\dim \mathrm{Ker}(DF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, X'_q)) = 1$.

Using this fact, we show that $DG(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times X_p, \mathbb{R} \times X'_q)$ is an isomorphism by the exactly same manner as above. □

Corollary 2.3. *Assume that (A1)-(A6) are valid. Suppose that there exists* $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})$ *such that* $F(\lambda_0, u_0) = 0$, $\rho(\lambda_0, u_0) = \gamma_0$, *and* $D_\lambda F(\lambda_0, u_0) \neq 0$. *Suppose also that* $\mathrm{Ker} DF(\lambda_0, u_0) \cap \mathrm{Ker} D\rho(\lambda_0, u_0) = \{(0, 0)\}$. *Define* $H : \mathbb{R} \times \Lambda \times V \to \mathbb{R} \times W$ *by* $H(\gamma, \lambda, u) := (\rho(\lambda, u) - \gamma, F(\lambda, u))$.

Then, we have $H(\gamma_0, \lambda_0, u_0) = (0, 0)$ *and* $D_{(\lambda,u)} H(\gamma_0, \lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times V, \mathbb{R} \times W)$ *is an isomorphism. Therefore, by the implicit function theorem, there exist a positive constant* ε *and a* C^1 *map* $(\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \ni \gamma \mapsto (\lambda(\gamma), u(\gamma)) \in \Lambda \times V$ *such that* $(\lambda(\gamma_0), u(\gamma_0)) = (\lambda_0, u_0)$ *and* $H(\gamma, \lambda(\gamma), u(\gamma)) = (0, 0)$ *for any* γ. *That is, the solution manifold of the equation* $F(\lambda, u) = 0$ *is parametrized by* $\gamma = \rho(\lambda, u)$ *around* (λ_0, u_0). □

To define discretized solutions of $F(\lambda, u) = 0$, we introduce the finite-dimensional subspaces $S_h \subset X_\infty$ which are parametrized by h, $0 < h < 1$ with the following properties:
(A7) There exists a real $r \geq 0$ and a positive constant C_3 independent of h such that

$$\|v_h\|_{X_\infty} \leq \frac{C_3}{h^r}\|v_h\|_{X_p}, \qquad \forall v_h \in S_h.$$

The relations of Banach spaces are depicted in the following:

$$\begin{array}{ccccccc}
 & & \Lambda\times S_h & & & & \\
 & & \cap & & & & \\
\Lambda\times V & \subset & \Lambda\times X_\infty & \subset & \mathbb{R}\times X_p & \subset & \mathbb{R}\times X_2 \\
\downarrow {\scriptstyle F(\lambda,u)} & & \downarrow {\scriptstyle F(\lambda,u)} & & \downarrow {\scriptstyle DF(\lambda,u)} & & \downarrow {\scriptstyle DF(\lambda,u)} \\
W & \subset & X_1' & \subset & X_q' & \subset & X_2'
\end{array}$$

The finite element solution $(\lambda_h, u_h) \in \Lambda \times S_h$ is defined naturally by

$$\langle F(\lambda_h, u_h), v_h\rangle = 0, \qquad \forall v_h \in S_h,$$

where $\langle\cdot,\cdot\rangle$ is the duality pair of X_2' and X_2. We derive an equivalent definition of the finite element solutions which is more convenient in the error analysis.

Let $Q \in \mathcal{L}(X_2, X_2')$ be a self-adjoint operator, that is, $\langle Qu, v\rangle = \langle Qv, u\rangle$ for all $u, v \in X_2$. Suppose that there exists a positive constant α such that

$$\langle Qv, v\rangle \geq \alpha\|v\|_{X_2}^2, \qquad \forall v \in X_2. \tag{2.1}$$

We define $(\cdot,\cdot)_Q$ by $(u, v)_Q := \langle Qu, v\rangle$. It is easy to show that $(\cdot,\cdot)_Q$ is an inner product and the norm $\|v\|_Q := (v, v)_Q^{1/2}$ is equivalent to the original norm $\|v\|_{X_2}$. It is also easy to show that $Q \in \mathcal{L}(X_2, X_2')$ is an isomorphism.

We define the canonical projection $\widetilde{P}_h : X_2 \to S_h$ by $(\psi - \widetilde{P}_h\psi, v_h)_Q = 0$ for all $v_h \in S_h$. Obviously, we have that $(u, \widetilde{P}_h v)_Q = (\widetilde{P}_h u, v)_Q$ for all $u, v \in X_2$. As in [18, Section 6] it follows from the definitions that (λ_h, u_h) is a finite element solution if and only if $\langle Q\widetilde{P}_h Q^{-1}F(\lambda_h, u_h), v\rangle = 0$ for all $v \in X_2$.

Following Fink and Rheinboldt ([8], [9], [13]) we define the approximation of $F(\lambda, u)$ by

$$F_h(\lambda, u) := (I - P_h)Qu + P_h F(\lambda, u), \qquad P_h := Q\widetilde{P}_h Q^{-1}, \tag{2.2}$$

where I is the identity of X_2'. It can be seen easily [13, Lemma 5.1] that $F_h(\lambda, u) = 0$ if and only if $u \in S_h$ and (λ, u) is a finite element solution.

Theorem 2.4. *Assume that (A1)-(A7) are valid. Suppose that there exists $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})$ such that $F(\lambda_0, u_0) = 0$, $\rho(\lambda_0, u_0) = \gamma_0$, and $D_\lambda F(\lambda_0, u_0) \neq 0$. Suppose also that $\mathrm{Ker}DF(\lambda_0, u_0) \cap \mathrm{Ker}D\rho(\lambda_0, u_0) = \{(0, 0)\}$. Then, by Corollary 2.3, there exist a positive constant ε_0 and a C^1 map $[\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0] \ni \gamma \mapsto (\lambda(\gamma), u(\gamma)) \in \Lambda \times V$ such that $(\lambda(\gamma_0), u(\gamma_0)) = (\lambda_0, u_0)$, $\gamma = \rho(\lambda(\gamma), u(\gamma))$, and $F(\lambda(\gamma), u(\gamma)) = 0$. We assume that $(\lambda(\gamma), u(\gamma)) \in \mathcal{R}(F, \mathcal{S})$ for all $\gamma \in [\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0]$. We also assume that there exists the projection $\Pi_h : X_p \to S_h$ for each $h > 0$ such that, for all $\gamma \in [\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0]$,*

$$\lim_{h\to 0} h^{-r}\|u(\gamma) - \Pi_h u(\gamma)\|_{X_p} = 0, \tag{2.3}$$

$$\lim_{h\to 0} \|u(\gamma) - \Pi_h u(\gamma)\|_{X_\infty} = 0, \tag{2.4}$$

and the above convergences are uniform.

We, on the other hand, suppose that $D_uF(\lambda_0,u_0)$ is decomposed into $D_uF(\lambda_0,u_0) = Q + R$, where $Q \in \mathcal{L}(X_p, X_q')$ is the principal part which is self-adjoint and satisfies (2.1), and $R \in \mathcal{L}(X_p, X_q')$ is compact. The discretized nonlinear map $F_h : X_p \to X_q'$ and the projection $P_h : X_q' \to X_q'$ is defined by (2.2). We suppose that

$$\lim_{h\to 0} \|\psi - P_h\psi\|_{X_q'} = 0, \qquad \forall\psi \in X_q'. \tag{2.5}$$

Then, for sufficiently small $h > 0$, there exist a positive constant $\varepsilon_1 \leq \varepsilon_0$ and a unique map $[\gamma_0 - \varepsilon_1, \gamma_0 + \varepsilon_1] \ni \gamma \mapsto (\lambda_h(\gamma), u_h(\gamma)) \in \Lambda \times S_h$ such that $F_h(\lambda_h(\gamma), u_h(\gamma)) = 0$ for all $\gamma \in [\gamma_0 - \varepsilon_1, \gamma_0 + \varepsilon_1]$. Moreover, we have the estimate

$$|\lambda(\gamma) - \lambda_h(\gamma)| + \|u_h(\gamma) - \Pi_h u(\gamma)\|_{X_p} \leq K_1 \|u(\gamma) - \Pi_h u(\gamma)\|_{X_p},$$

for all $\gamma \in [\gamma_0 - \varepsilon_1, \gamma_0 + \varepsilon_1]$, where K_1 is a positive constant independent of h and γ.

Proof. The proof of Theorem 2.4 is quite similar to those of [17, Theorem 7.7] and [18, Theorem 8.4]. Hence, we here skip the proof. □

3. Elaborate Error Estimates of the Parameter λ.

In this section we give elaborate error estimates of the parameter λ. To do this we need more assumptions.

(A8) The nonlinear maps $F : \Lambda \times X_\infty \to X_1'$ and $\rho : \Lambda \times X_\infty \to \mathbb{R}$ are of C^2 class.

(A9) For any $(\lambda, u) \in \mathcal{S} \subset \Lambda \times W$, $D_uF(\lambda, u) \in \mathcal{L}(X_2, X_2')$ is self-adjoint.

Now, let $(\lambda, u) \in \mathcal{R}(F, \mathcal{S})$ be a solution of $F(\lambda, u) = 0$ at which all assumptions of Theorem 2.4 and (A8), (A9) hold. Let $(\lambda_h, u_h) \in \Lambda \times S_h$ be the corresponding finite element solution with $\rho(\lambda_h, u_h) = \rho(\lambda, u)$.

We consider the following auxiliary problem: find $(\eta, z) \in \mathbb{R} \times X_p$ such that

$$\begin{aligned} &\langle (D_uF^0)z, v\rangle = \eta\langle D_u\rho^0, v\rangle, \quad \forall v \in X_p, \\ &\langle D_\lambda F^0, z\rangle - \eta D_\lambda\rho^0 = 1, \end{aligned} \tag{3.1}$$

where $D_uF^0 := D_uF(\lambda, u)$, $D_u\rho^0 := D_u\rho(\lambda, u)$, etc.

Lemma 3.1. *Suppose that all assumptions of Theorem 2.4 and (A8), (A9) hold. Then, the equation (3.1) has an unique solution $(\eta, z) \in \mathbb{R} \times X_p$.*

Proof. Recall that we have either

Case 1: $\mathrm{Ker}D_uF(\lambda, u) = \{0\}$ and $D_\lambda F(\lambda, u) \in \mathrm{Im}D_uF(\lambda, u)$, or

Case 2: $\dim \mathrm{Ker}D_uF(\lambda, u) = 1$, and $D_\lambda F(\lambda, u) \notin \mathrm{Im}D_uF(\lambda, u)$.

Suppose that we are in Case 1. Then, $\mathrm{Ker}DF(\lambda, u) = \mathrm{span}\{(1, -(D_uF^0)^{-1}(D_\lambda F^0))\}$. By the assumption we have $D\rho^0(1, -(D_uF^0)^{-1}(D_\lambda F^0)) \neq 0$, that is,

$$D_\lambda\rho^0 - \langle D_u\rho^0, (D_uF^0)^{-1}(D_\lambda F^0)\rangle \neq 0.$$

Let $\eta := (\langle D_u\rho^0, (D_uF^0)^{-1}(D_\lambda F^0)\rangle - D_\lambda\rho^0)^{-1}$ and $z := \eta(D_uF^0)^{-1}(D_u\rho^0)$. Since D_uF^0 is self-adjoint by (A9), we have

$$\eta\langle D_\lambda F^0, (D_uF^0)^{-1}(D_u\rho^0)\rangle = \eta\langle D_u\rho^0, (D_uF^0)^{-1}(D_\lambda F^0)\rangle,$$

and $\langle D_\lambda F^0, z\rangle - \eta D_\lambda \rho^0 = \eta(\langle D_u\rho^0, (D_uF^0)^{-1}(D_\lambda F^0)\rangle - D_\lambda\rho^0) = 1$. Hence (η, z) is a solution of (3.1). Uniqueness is proved by the same manner.

Now, suppose that we have Case 2. Then, there exists $\psi_0 \in V$ such that $\mathrm{Ker}DF(\lambda, u) = \mathrm{span}\{(0, \psi_0)\}$ and $\langle D_u\rho^0, \psi_0\rangle \neq 0$.

Since $DF(\lambda, u)$ is onto, there exists $(\theta, \phi) \in \mathbb{R} \times X_p$ such that

$$\theta\langle D_\lambda F^0, v\rangle + \langle (D_uF^0)\phi, v\rangle = \langle D_u\rho^0, v\rangle, \quad \forall v \in X_2, \tag{3.2}$$

and θ is determined uniquely.

We claim that $D_u\rho^0 \notin \mathrm{Im}(D_uF^0)$. If $D_u\rho^0 \in \mathrm{Im}(D_uF^0)$, then there would exist $w \in X_p$ such that $(D_uF^0)w = D_u\rho^0$. Hence, we have

$$0 \neq \langle D_u\rho^0, \psi_0\rangle = \langle (D_uF^0)w, \psi_0\rangle = \langle (D_uF^0)\psi_0, w\rangle = 0,$$

and obtain a contradiction. Therefore, we conclude that $D_u\rho^0 \notin \mathrm{Im}(D_uF^0)$ and $\theta \neq 0$.

Letting $v := \psi_0$ in (3.2), we have $\theta\langle D_\lambda F^0, \psi_0\rangle = \langle D_u\rho^0, \psi_0\rangle \neq 0$. Hence, we conclude $\langle D_\lambda F^0, \psi_0\rangle = \langle D_u\rho^0, \psi_0\rangle/\theta \neq 0$. We thus immediately notice that $(0, \alpha\psi_0)$ with $\alpha := \langle D_\lambda F^0, \psi_0\rangle^{-1}$ is a solution of (3.1). Again, the uniqueness is shown by the same manner. □

It is obvious that we may apply Theorem 2.4 to the equation (3.1) with the following setting:

$$\begin{aligned} F(\eta, z) &:= (D_uF^0)z - \eta(D_u\rho^0), \\ \rho(\eta, z) &:= \langle D_\lambda F^0, z\rangle - \eta(D_\lambda\rho^0), \end{aligned}$$

and obtain

Lemma 3.2. *For sufficiently small $h > 0$, there exists the unique finite element solution $(\eta_h, z_h) \in \mathbb{R} \times S_h$ of (3.1) such that*

$$\begin{aligned} &\langle (D_uF^0)z_h, v_h\rangle = \eta_h\langle D_u\rho^0, v_h\rangle, \quad \forall v \in S_h, \\ &\langle D_\lambda F^0, z_h\rangle - \eta_h D_\lambda\rho^0 = 1. \end{aligned}$$

Moreover, we have the estimate

$$|\eta - \eta_h| + \|z - z_h\|_{X_p} \leq C\|z - \Pi_h z\|_{X_p},$$

where C is a positive constant independent of h. □

Let $(\lambda, u) \in \mathcal{R}(F, \mathcal{S})$ is a solution of $F(\lambda, u) = 0$ which satisfies the assumptions of Theorem 2.4 and (A8), (A9), and $(\lambda_h, u_h) \in \Lambda \times S_h$ the corresponding finite element solution. By Taylor's theorem and $\langle F(\lambda_h, u_h), v_h\rangle = \langle F(\lambda, u), v_h\rangle = 0$ for any $v_h \in S_h$, we have

$$\begin{aligned} 0 =& (\lambda_h - \lambda)\langle D_\lambda F^0, v_h\rangle + \langle (D_uF^0)(u_h - u), v_h\rangle + \frac{1}{2}(\lambda_h - \lambda)^2\langle D_{\lambda\lambda}F^0, v_h\rangle \\ &+ (\lambda_h - \lambda)\langle (D_{\lambda u}F^0)(u_h - u), v_h\rangle + \frac{1}{2}\langle (D_{uu}F^0)(u_h - u)^2, v_h\rangle, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} D_{\lambda\lambda}F^0 &:= \int_0^1 (1-s)D_{\lambda\lambda}F(\lambda + s(\lambda_h - \lambda), u + s(u_h - u))ds, \\ (D_{\lambda u}F^0)(u_h - u) &:= \int_0^1 (1-s)D_{\lambda u}F(\lambda + s(\lambda_h - \lambda), u + s(u_h - u))(u_h - u)ds, \\ (D_{uu}F^0)(u_h - u)^2 &:= \int_0^1 (1-s)D_{uu}F(\lambda + s(\lambda_h - \lambda), u + s(u_h - u))(u_h - u)^2ds. \end{aligned}$$

Letting $v := u - u_h$ in (3.1), we obtain

$$\langle (D_u F^0)z, u - u_h\rangle = \langle (D_u F^0)(u - u_h), z\rangle = \eta\langle D_u\rho^0, u - u_h\rangle.$$

Since

$$\begin{aligned} 0 =&\rho(\lambda_h, u_h) - \rho(\lambda, u) \\ =&(\lambda_h - \lambda)(D_\lambda\rho^0) + \langle D_u\rho^0, u_h - u\rangle + \frac{1}{2}(\lambda_h - \lambda)^2(D_{\lambda\lambda}\rho^0) \\ &+ (\lambda_h - \lambda)(D_{\lambda u}\rho^0)(u_h - u) + \frac{1}{2}(D_{uu}\rho^0)(u_h - u)^2, \end{aligned}$$

where

$$\begin{aligned} D_{\lambda\lambda}\rho^0 &:= \int_0^1 (1-s)D_{\lambda\lambda}\rho(\lambda + s(\lambda_h - \lambda), u + s(u_h - u))ds, \\ (D_{\lambda u}\rho^0)(u_h - u) &:= \int_0^1 (1-s)\langle D_{\lambda u}\rho(\lambda + s(\lambda_h - \lambda), u + s(u_h - u)), u_h - u\rangle ds, \\ (D_{uu}\rho^0)(u_h - u)^2 &:= \int_0^1 (1-s)\langle D_{uu}\rho(\lambda + s(\lambda_h - \lambda), u + s(u_h - u))(u_h - u), u_h - u\rangle ds, \end{aligned}$$

we have

$$\begin{aligned} \langle (D_u F^0)(u - u_h), z\rangle =& -\eta(\lambda - \lambda_h)(D_\lambda\rho^0) + \frac{\eta}{2}(\lambda_h - \lambda)^2(D_{\lambda\lambda}\rho^0) \\ &+ \eta(\lambda_h - \lambda)(D_{\lambda u}\rho^0)(u_h - u) + \frac{\eta}{2}(D_{uu}\rho^0)(u_h - u)^2. \end{aligned} \tag{3.4}$$

It follows from (3.3) with $v_h := z_h$ (recall that $(\eta_h, z_h) \in \mathbb{R}\times S_h$ is the finite element solution of (3.1)) and (3.4) that

$$\begin{aligned} (\lambda - \lambda_h)(\langle D_\lambda F^0, z\rangle - \eta(D_\lambda\rho^0) + B_h) =& \langle (D_u F^0)(u - u_h), z - z_h\rangle \\ &+ \frac{1}{2}\langle (D_{uu}F^0)(u - u_h)^2, z_h\rangle - \frac{\eta}{2}(D_{uu}\rho^0)(u - u_h)^2, \end{aligned}$$

where $\lim_{h\to 0} B_h = 0$. Therefore, we have proved the following theorem:

Theorem 3.3. *Let $(\lambda, u) \in \mathcal{R}(F, \mathcal{S})$ be a solution of $F(\lambda, u) = 0$ which satisfies the assumptions of Theorem 2.4 and (A8), (A9). Let $(\lambda_h, u_h) \in \Lambda \times S_h$ be the corresponding finite element solution. Let $(\eta, z) \in \mathbb{R} \times X_p$ and $(\eta_h, z_h) \in \mathbb{R} \times S_h$ be the exact and the finite element solutions of* (3.1).

Then, for sufficiently small $h > 0$, we have the following elaborate error estimate of $|\lambda - \lambda_h|$:

$$\begin{aligned} |\lambda - \lambda_h| \le C_h\Big|&\langle (D_u F^0)(u - u_h), z - z_h\rangle + \frac{1}{2}\langle (D_{uu}F^0)(u - u_h)^2, z_h\rangle \\ &- \frac{\eta}{2}(D_{uu}\rho^0)(u - u_h)^2\Big|, \end{aligned}$$

where $D_u F^0 := D_u F(\lambda, u)$,

$$\begin{aligned} (D_{uu}F^0)(u - u_h)^2 &:= \int_0^1 (1-s)D_{uu}F(\lambda + s(\lambda_h - \lambda), u + s(u_h - u))(u - u_h)^2 ds, \\ (D_{uu}\rho^0)(u - u_h)^2 &:= \int_0^1 (1-s)\langle D_{uu}\rho(\lambda + s(\lambda_h - \lambda), u + s(u_h - u))(u - u_h), u - u_h\rangle ds, \end{aligned}$$

and C_h is a positive constant such that $\lim_{h\to 0} C_h = 1$. □

Sometimes, one may want to compute a turning point itself. For such a purpose we are able to develop a similar analysis as above. Let $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})$ be a turning point of the equation $F(\lambda, u) = 0$ at which the assumptions of Theorem 2.4 and (A8), (A9) hold. That is, $F(\lambda_0, u_0) = 0$, $DF(\lambda_0, u_0) \in \mathcal{L}(\mathbb{R} \times V, W)$ is onto, and $D_uF(\lambda_0, u_0) \in \mathcal{L}(V, W)$ is not an isomorphism. In this case we have $\dim \mathrm{Ker} D_uF(\lambda_0, u_0) = 1$ and $D_\lambda F(\lambda_0, u_0) \notin \mathrm{Im} D_uF(\lambda_0, u_0)$. It then follows from the proof of Lemma 3.1 that (3.1) has an unique solution $(0, z_0) \in \mathbb{R} \times X_p$ at (λ_0, u_0):

$$\begin{aligned} &\langle D_uF(\lambda_0, u_0)z_0, v\rangle = 0, \quad \forall v \in X_p, \\ &\langle D_\lambda F(\lambda_0, u_0), z_0\rangle = 1, \end{aligned} \tag{3.5}$$

We consider the nonlinear map $K : \Lambda \times V \times X_p \to \mathbb{R} \times W \times X_q'$ defined by

$$K(\lambda, u, z) := \begin{pmatrix} \langle D_\lambda F(\lambda, u), z\rangle - 1 \\ F(\lambda, u) \\ D_uF(\lambda, u)z \end{pmatrix}. \tag{3.6}$$

At a turning point $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})$ the equation $K(\lambda, u, z) = (0, 0, 0)$ has the solution $(\lambda_0, u_0, z_0) \in \Lambda \times V \times X_p$. A turning point $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})$ is called ***nondegenerate***, if

$$D_{uu}F(\lambda_0, u_0)\psi_0\psi_0 \notin \mathrm{Im} D_uF(\lambda_0, u_0),$$

where $\{\psi_0\} \subset X_p$ is the basis of $\mathrm{Ker} D_uF(\lambda_0, u_0)$ (see [4, Section 4]). For a nondegenerate turning point, we have the following lemma. For the proof of the lemma, see [4], [15].

Lemma 3.4. *Let $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})$ be a tuning point at which the assumptions of Theorem 2.4 and (A8), (A9) hold. Then, (λ_0, u_0) is a nondegenerate turning point if and only if the Fréchet derivative $DK(\lambda_0, u_0, z_0) \in \mathcal{L}(\mathbb{R} \times V \times X_p, \mathbb{R} \times W \times X_q')$ is an isomorphism, where $z_0 \in X_p$ is the solution of (3.5) and the nonlinear map K is defined by (3.6).* □

From Lemma 3.4, the results in [16] can be applied to the equation $K(\lambda, u, z) = (0, 0, 0)$ at a nondegenerate turning point (λ_0, u_0) and obtain the following lemma:

Lemma 3.5. *Let $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})$ is a nondegenerate tuning point. Then, for sufficiently small $h > 0$, there exist a locally unique finite element solution $(\lambda_0^h, u_0^h, z_0^h) \in \mathbb{R} \times (S_h)^2$ such that*

$$\begin{aligned} \langle D_\lambda F_h(\lambda_0^h, u_0^h), z_0^h\rangle &= 1 \\ F_h(\lambda_0^h, u_0^h) &= 0, \\ D_uF_h(\lambda_0^h, u_0^h)z_0^h &= 0, \end{aligned}$$

where F_h is the nonlinear map defined by (2.2). The finite element solution (λ_0^h, u_0^h) is a nondegenerate turning point on the finite element solution manifold $\mathcal{M}_h$.

Moreover, we have the following error estimate:

$$|\lambda_0 - \lambda_0^h| + \|u_0 - u_0^h\|_{X_p} + \|z_0 - z_0^h\|_{X_p} \le C(\|u_0 - \Pi_h u_0\|_{X_p} + \|z_0 - \Pi_h z_0\|_{X_p}),$$

where C is a positive constant independent of h, and $\Pi_h : X_p \to S_h$ is the projection which appears in Theorem 2.4. □

Now, we develop a similar elaborate error estimate for $|\lambda_0 - \lambda_0^h|$. Again, let $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})$ be a nondegenerate turning point which satisfies the assumptions of Theorem 2.4 and (A8), (A9), and $(\lambda_0^h, u_0^h) \in \Lambda \times S_h$ the corresponding finite element solution. By Taylor's theorem and $\langle F(\lambda_0^h, u_0^h), v_h\rangle = \langle F(\lambda_0, u_0), v_h\rangle = 0$ for any $v_h \in S_h$, we have

$$
\begin{aligned}
0 =& (\lambda_0^h - \lambda_0)\langle D_\lambda F(\lambda_0, u_0), v_h\rangle + \langle D_u F(\lambda_0, u_0)(u_0^h - u_0), v_h\rangle \\
&+ \frac{1}{2}(\lambda_0^h - \lambda_0)^2\langle D_{\lambda\lambda}F^0, v_h\rangle + (\lambda_0^h - \lambda_0)\langle (D_{\lambda u}F^0)(u_0^h - u_0), v_h\rangle \\
&+ \frac{1}{2}\langle (D_{uu}F^0)(u_0^h - u_0)^2, v_h\rangle,
\end{aligned} \tag{3.7}
$$

where

$$
\begin{aligned}
D_{\lambda\lambda}F^0 &:= \int_0^1 (1-s) D_{\lambda\lambda}F(\lambda_0 + s(\lambda_0^h - \lambda_0), u_0 + s(u_0^h - u_0))ds, \\
(D_{\lambda u}F^0)(u_0^h - u_0) &:= \int_0^1 (1-s) D_{\lambda u}F(\lambda_0 + s(\lambda_0^h - \lambda_0), u + s(u_0^h - u_0))(u_0^h - u_0)ds, \\
(D_{uu}F^0)(u_0^h - u_0)^2 &:= \int_0^1 (1-s) D_{uu}F(\lambda_0 + s(\lambda_0^h - \lambda_0), u + s(u_0^h - u_0))(u_0^h - u_0)^2 ds.
\end{aligned}
$$

Letting $v := u_0 - u_0^h$ in (3.5), we obtain

$$
\langle D_u F(\lambda_0, u_0) z_0, u_0 - u_0^h\rangle = \langle D_u F(\lambda_0, u_0)(u_0 - u_0^h), z_0\rangle = 0.
$$

Plugging this equation into (3.7) with $v_h := z_0^h$, we obtain

$$
\begin{aligned}
(\lambda_0 - \lambda_0^h)\big(\langle D_\lambda F(\lambda_0, u_0), z_0\rangle + B_h\big) =& \langle D_u F(\lambda_0, u_0)(u_0 - u_0^h), z_0 - z_0^h\rangle \\
&+ \frac{1}{2}\langle (D_{uu}F^0)(u_0 - u_0^h)^2, z_0^h\rangle,
\end{aligned}
$$

where $\lim_{h\to 0} B_h = 0$. Therefore, we have proved the following theorem:

Theorem 3.6. *Let $(\lambda_0, u_0) \in \mathcal{R}(F, \mathcal{S})$ be a nondegenerate turning point which satisfies the assumptions of Theorem 2.4 and (A8), (A9). Let $(\lambda_0^h, u_0^h) \in \Lambda \times S_h$ be the corresponding nondegenerate turning point on the finite element solution branch $\mathcal{M}_h$. Let $z_0 \in X_p$ and $z_0^h \in S_h$ be the exact and the finite element solutions which appear in Lemma 3.4 and 3.5.*

Then, for sufficiently small $h > 0$, we have the following elaborate error estimate of $|\lambda_0 - \lambda_0^h|$:

$$
|\lambda_0 - \lambda_0^h| \le C_h \left| \langle D_u F(\lambda_0, u_0)(u_0 - u_0^h), z_0 - z_0^h\rangle + \frac{1}{2}\langle (D_{uu}F^0)(u_0 - u_0^h)^2, z_0^h\rangle \right|
$$

where

$$
(D_{uu}F^0)(u_0 - u_0^h)^2 := \int_0^1 (1-s) D_{uu}F(\lambda_0 + s(\lambda_0^h - \lambda_0), u_0 + s(u_0^h - u_0))(u_0 - u_0^h)^2 ds,
$$

and C_h is a positive constant such that $\lim_{h\to 0} C_h = 1$. □

Remark. Apparently, Lemma 3.5 and Theorem 3.6 are very similar to [4, Theorem 7]. The main difference is the tools used in [4] and in this paper. In [4] the Liapunov-Schmidt reduction is used to parametrize solution branches around turning points. On the other hand, so-called "bordering technique" is used throughout this paper. In [15], it is pointed out that bordering technique is closely related with the Liapunov-Schmidt reduction.

Employing bordering technique, our situation becomes simpler than that of [4]. For instance, in [4] F should be C^3 map while F is C^2 map in this paper. Also, we do not need the derivatives of λ and u with respect to the newly introduced parameter, which are used frequently in [4]. The second point will be advantageous when we try to apply the results in this section to a posteriori error estimation of the parameter λ. This point will be discussed elsewhere by the author. □

References

[1] I. BABUŠKA AND J. OSBORN, Eigenvalue Problems, in Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1), ed. by P.G. Ciarlet and J.L. Lions, North-Holland, 1991.

[2] S.C. BRENNER AND L.R. SCOTT, The Mathematical Theory of Finite Element Methods, Springer, 1994.

[3] F. BREZZI, J. RAPPAZ, AND P.A. RAVIART, Finite dimensional approximation of nonlinear problems, Part I: Branches of nonsingular solutions, Numer. Math., 36 (1980) 1-25.

[4] F. BREZZI, J. RAPPAZ, AND P.A. RAVIART, Finite dimensional approximation of nonlinear problems, Part II: Limit points, Numer. Math., 37 (1981) 1-28.

[5] G. CALOZ AND J. RAPPAZ, Numerical Analysis for Nonlinear and Bifurcation Problems, in Handbook of Numerical Analysis, Vol. V, Techniques of Scientific Computing (Part 2), ed. by P.G. Ciarlet and J.L. Lions North-Holland, 1997.

[6] P.G. CIARLET, Basic Error Estimates for Elliptic Problems, in Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1), ed. by P.G. Ciarlet and J.L. Lions, North-Holland, 1991.

[7] M. CROUZEIX AND J. RAPPAZ, On Numerical Approximation in Bifurcation Theory, Springer-Verlag, 1990.

[8] J.P. FINK AND W.C. RHEINBOLDT, On the discretization error of parametrized nonlinear equations, SIAM J. Numer. Anal., 20 (1983) 732-746.

[9] J.P. FINK AND W.C. RHEINBOLDT, Solution manifolds and submanifolds of parametrized equations and their discretization errors, Numer. Math., 45 (1984) 323-343.

[10] D. GILBARG, N.S. TRUDINGER, Elliptic Partial Differential Equations of Second Order, 2nd edition, Springer-Verlag, 1983.

[11] F. KIKUCHI, An iterative finite element scheme for bifurcation analysis of semi-linear elliptic equations, Report 542, Inst. Space Aero. Sci., Univ. Tokyo, 1976.

[12] F. KIKUCHI, Finite element approximations to bifurcation problems of turning point type, Theoret. Appl. Mech., 27 (1979) 99-144.

[13] W.C. RHEINBOLDT, Numerical Analysis of Parametrized Nonlinear Equations, Wiley, 1986.

[14] G. STRANG AND G.J. FIX, An Analysis of the Finite Element Method, Prentice-Hall, 1973.

[15] T. TSUCHIYA, Enlargement procedure for resolution of singularities at simple singular solutions of nonlinear equations, Numer. Math., 52 (1988) 401-411.

[16] T. TSUCHIYA, An application of the Kantorovich theorem to nonlinear finite element analysis, *to appear in* Numer. Math.

[17] T. TSUCHIYA, Finite element approximations of parametrized strongly nonlinear boundary value problems, submitted.

[18] T. TSUCHIYA, I. BABUŠKA, A priori error estimates of finite element solutions of parametrized strongly nonlinear boundary value problems, J. Comp. Appl. Math., 79 (1997) 41-66.