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P-irreducibility of Positive Polynomials

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Abstract

The problem of deciding whether a polynomial of positive coefficients can be factored into polynomials of the same type is important for studying many physiological processes. An efficient method to decide positive irreducibility is highly valuable. The known criteria for positive irreducible polynomials need to know root location first. Here, we present a new criterion, which can be expressed only by the coefficients of the given polynomial.

1 Introduction

Protein ligand binding is a process in which the ligand can become bound and interact at a number of sites of a protein macromolecule. It can be described by the binding polynomial introduced by Wyman [7]. If the molecule has n binding sites (which generally is four in the case of hemoglobin [3]) and x represents ligand activity, then the binding polynomial can be written as $f(x) = 1 + \beta_1 x + \cdots + \beta_n x^n, \beta_i \geq 0, 1 \leq i \leq n-1$ and $\beta_n > 0$. If $f(x)$ can be factored into two polynomials with positive coefficients, then it is natural to interpret each factor as a binding polynomial for a subset of the binding sites. The binding polynomials which are positive irreducible are very important because they imply that all sites are linked. This problem has been discussed extensively in the literature [1] [2] [3] [7], some criteria were established to check the positive irreducibility for a given polynomial with degree 3 or 4. However, people need to compute all roots of the polynomial. Although, there are root finding formulas for polynomials of degree 3 or 4, these formulas consist of radical expressions of coefficients. In this paper, we give a new criterion, it only consists of a set of polynomial inequalities defined by the coefficients of $f(x)$.

In the next section, we will present some basic contents about positive polynomials and a criterion for stability. In section 3, we describe a criterion for positive irreducibility of polynomials of degree 3. Section 4 deals with positive irreducibility of polynomials of degree 4. Some examples are also included in this section.

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2 Preliminary

Definition 1 *A positive polynomial is a real polynomial whose leading and constant coefficients are positive and whose remaining coefficients are non-negative.*

Definition 2 *A positive factorization of a polynomial is a non-trivial factorization in which each factor is a positive polynomial.*

The positive factorization of a polynomial is not unique. For example:

$$\begin{aligned} f_1 &= x^4+4x^3+6x^2+19x+30 \\ &= (x+2)(x^3+2x^2+2x+15) \\ &= (x+3)(x^3+x^2+3x+10) \end{aligned} \tag{1}$$

Definition 3 *A p-irreducible polynomial is a positive polynomial which does not admit a positive factorization.*

One method determining a polynomial to be p-irreducible is to try all the possible combinations of factors over the real field. For example:

$$\begin{aligned} f_2 &= x^4+12x^3+34x^2+23x+210 \\ &= (x+7)(x+6)(x^2-x+5) \\ &= (x+7)(x^3+5x^2-x+30) \\ &= (x+6)(x^3+6x^2-2x+35). \end{aligned} \tag{2}$$

$$\begin{aligned} f_3 &= x^4+8x^3+14x^2+27x+90 \\ &= (x+6)(x+3)(x^2-x+5) \\ &= (x+3)(x^3+5x^2-x+30) \\ &= (x+6)(x^3+2x^2+2x+15). \end{aligned} \tag{3}$$

f_2 is p-irreducible and f_3 has a positive factorization.

An important class of positive polynomials consists of stable polynomials whose zeros have negative real parts. Binding polynomials which are stable can be factored uniquely into positive linear and p-irreducible quadratic factors of the forms $x+u$, x^2+vx+w $(u>0, v>0, w>0)$ so that the protein will have a number of independent sites corresponding to the linear factors and will have the remaining sites linked in pairs corresponding to the p-irreducible quadratic factors.

Routh in 1875 and Hurwitz in 1895 provided a criterion for stability. Suppose to be given a polynomial with real coefficients

$$f(x) = c_0x^n + c_1x^{n-1} + \cdots + c_n \ (c_0 = 1), \tag{4}$$

then the Routh-Hurwitz conditions can be written in the form of the inequalities

$$\Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_n > 0, \tag{5}$$

where

$$\Delta_i = \begin{vmatrix} c_1 & c_3 & c_5 & \cdots & & \\ c_0 & c_2 & c_4 & \cdots & & \\ 0 & c_1 & c_3 & \cdots & & \\ 0 & c_0 & c_2 & c_4 & & \\ & & & & \ddots & \\ & & & & & c_i \end{vmatrix} \quad (c_k = 0 \ \forall \ k > n) \tag{6}$$

However, when all the coefficients of $f(x)$ are positive, the inequalities (5) are not independent. For example: for $n = 3$, the Routh-Hurwitz conditions reduce to $\Delta_2 > 0$; for $n = 4$, reduce to: $\Delta_3 > 0$. This circumstance was investigated by the French mathematicians Liénard and Chipart in 1914 [4].

Proposition 4 *(Stability Criterion of Liénard and Chipart): Necessary and sufficient conditions for all the roots of the real polynomial $f(x) = x^n + c_1x^{n-1} + \cdots + c_n$ to have negative real parts can be given in any one of the following four forms:*

1. $c_n > 0, c_{n-2} > 0, c_{n-4} > 0, \ldots; \Delta_1 > 0, \Delta_3 > 0, \ldots,$
2. $c_n > 0, c_{n-2} > 0, c_{n-4} > 0, \ldots; \Delta_2 > 0, \Delta_4 > 0, \ldots,$
3. $c_n > 0, c_{n-1} > 0, c_{n-3} > 0, \ldots; \Delta_1 > 0, \Delta_3 > 0, \ldots,$
4. $c_n > 0, c_{n-1} > 0, c_{n-3} > 0, \ldots; \Delta_2 > 0, \Delta_4 > 0, \ldots.$

3 The case n=3

Consider the positive polynomial $f(x) = x^3 + c_1x^2 + c_2x + c_3$.

Proposition 5 *A positive polynomial of degree 3 is p-irreducible if and only if the coefficients satisfy:*

$$c_1c_2 < c_3. \tag{7}$$

Proof: According to proposition 1, $f(x)$ will be stable if and only if $c_1c_2 > c_3$. In this case, $f(x)$ can be factored into linear and quadratic p-irreducible factors. If $c_1c_2 = c_3$ then $f(x) = (x + c_1)(x^2 + c_2)$. Assume $c_1c_2 < c_3$, $f(x)$ is not stable and has no pair of conjugate pure imaginary roots. It is obvious that the roots of $f(x)$ must be in the form $-u, v \pm wI, u > 0, v > 0, w > 0$, so $f(x) = (x + u)(x^2 - 2vx + v^2 + w^2)$ is p-irreducible.

4 The case n=4

It is of particular interest to determine whether a quartic polynomial be p-irreducible because it covers a large variety of classes of proteins including hemoglobin [3]. Consider the positive polynomial $f(x) = x^4 + c_1x^3 + c_2x^2 + c_3x + c_4$, we have:

Theorem 6 *(main theorem) f is p-irreducible if and only if one of the following seven conditions is satisfied:*

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
1	$=0$										$=0$	<0
2	<0	>0										
3	<0	≤ 0		>0	>0							
4	<0	≤ 0	>0	<0		<0						
5	<0	≤ 0	>0	>0		>0	>0	≤ 0				
6	<0	≤ 0	>0	>0	≤ 0	>0	>0	>0	≤ 0			
7	<0	≤ 0	>0	>0	≤ 0	>0	>0	>0	>0	<0		

Where $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}$ are:

$$
\begin{aligned}
P_1 &= \Delta_3(f) = c_1c_2c_3 - c_1^2c_4 - c_3^2;\\
P_2 &= \mathrm{Discr}(f) = -192c_4^2c_1c_3 + 256c_4^3 - 128c_2^2c_4^2 - 4c_1^3c_3^3 + 16c_2^4c_4\\
&\quad -4c_2^3c_3^2 - 27c_1^4c_4^2 - 27c_3^4 + 144c_2c_1^2c_4^2 + 18c_1c_3^3c_2 + c_2^2c_1^2c_3^2\\
&\quad -4c_2^3c_1^2c_4 - 6c_4c_1^2c_3^2 + 144c_4c_3^2c_2 - 80c_1c_3c_2^2c_4 + 18c_1^3c_3c_2c_4;\\
P_3 &= c_1^2 - 4c_2;\\
P_4 &= f(-c_1) = c_1^2c_2 - c_1c_3 + c_4;\\
P_5 &= f'(-c_1) = -c_1^3 - 2c_1c_2 + c_3;\\
P_6 &= f(\alpha_1)f(\alpha_2) = c_2c_3^2 - c_1c_3c_4 + c_4^2;\\
P_7 &= f(\alpha_1) + f(\alpha_2) = -c_1c_3 + 2c_4;\\
P_8 &= f(-\tfrac{1}{2}c_1) = -\tfrac{1}{16}c_1^4 + \tfrac{1}{4}c_1^2c_2 - \tfrac{1}{2}c_1c_3 + c_4;\\
P_9 &= f'(\alpha_1) + f'(\alpha_2) = 2c_3 + 4c_1c_2 - c_1^3;\\
P_{10} &= f'(\alpha_1)f'(\alpha_2) = -c_1^2c_2^2 + 4c_2^3 - c_1^3c_3 + 4c_1c_2c_3 + c_3^2.\\
P_{11} &= c_1^2 + c_3^2;\\
P_{12} &= c_2^2 - 4c_4;
\end{aligned}
$$

Discr(f) *is the discriminant of f, α_1, α_2 are the roots of polynomial $x^2 + c_1x + c_2$.*

We will give several lemmas to prove the main theorem. Clearly, by proposition 1, if $\Delta_3(f) > 0$, then f is stable and has a positive factorization. Suppose $\Delta_3(f) = 0$, $f(x) = (x^2 + \frac{c_3}{c_1})(x^2 + c_1x + \frac{c_1c_4}{c_3})$ if $c_1c_3 \neq 0$; otherwise it is only possible that $c_1 = c_3 = 0$. In this case, $f(x)$ is p-irreducible if and only if $c_2^2 - 4c_4 < 0$ Now we suppose:

$$\Delta_3(f) < 0. \tag{8}$$

Proposition 7 *Let F be a squarefree real polynomial with degree n, we have:*

$$\mathrm{sign}(\mathrm{Discr}(F)) = (-1)^s, \tag{9}$$

where $2s$ is the number of nonreal roots of F.

Proof: Let $\alpha_1, \dots, \alpha_n$ be roots of F, then

$$\text{sign}(\text{Discr}(F)) = \text{sign}(\Pi_{1\le i,j\le n}(\alpha_i - \alpha_j)^2).$$

And notice that, when α is real and β is not then $\text{sign}\{(\alpha-\beta)(\alpha-\overline{\beta})\} > 0$ and $\text{sign}\{(\beta-\overline{\beta})^2\} < 0$.

Lemma 8 *If* $\Delta_3(f) < 0$ *and* $\text{Discr}(f) > 0$, *then* f *is p-irreducible.*

Proof: If $\text{Discr}(f) > 0$, according to proposition 3, we have $s = 0, 2$, i.e. f has no real roots or all the roots of f are real. Since f is a positive polynomial, it has no positive real zeros. If $s = 0$, f will have four negative real zeros, this is contradictory to the condition $\Delta_3(f) < 0$ which means f is not stable. Hence, when $\Delta_3(f) < 0$ and $\text{Discr}(f) > 0$, f must have two pairs of nonreal roots of the forms $a \pm bI, -c \pm dI, a, b, c, d > 0$ and $f(x) = (x^2 - 2ax + a^2 + b^2)(x^2 + 2cx + c^2 + d^2)$ is p-irreducible.

If $\text{Discr}(f) = 0$ then f has one double real root or one double nonreal root or two double real root. We have seen the last case is impossible when $\Delta_3(f) < 0$. If f has a double nonreal root then $f(x) = (x^2 - 2ax + a^2 + b^2)^2$. Since f is a positive polynomial, we must have $a = 0$. This implies $\Delta_3(f) = 0$. Thus, if $\Delta_3(f) < 0$ and $\text{Discr}(f) = 0$, f must have one double negative real zeros and one pair of conjugate nonreal zeros with positive real parts.

By proposition 3, $\text{Discr}(f) < 0$ implies $s = 1$, i.e., f has one pair of conjugate nonreal roots and two negative real zeros.

We suppose in the following:

$$\Delta_3(f) < 0, \text{Discr}(f) \le 0. \tag{10}$$

According to the above discussion, f has two negative real roots and a pair of conjugate nonreal roots denoted as $\alpha, \beta < 0$, $a \pm bI, a, b > 0$ respectively and

$$\begin{aligned} f(x) &= (x-\alpha)(x-\beta)(x^2 - 2ax + a^2 + b^2) \\ &= (x-\alpha)(x^3 + (c_1+\alpha)x^2 + (\alpha^2 + c_1\alpha + c_2)x - c_4/\alpha) \\ &= (x-\beta)(x^3 + (c_1+\beta)x^2 + (\beta^2 + c_1\beta + c_2)x - c_4/\beta). \end{aligned} \tag{11}$$

If $c_1 + \alpha = 0$ then $\alpha^2 + c_1\alpha + c_2 = c_2 \ge 0$, f has a positive factorization; else if $c_1 + \alpha < 0$ then $\alpha^2 + c_1\alpha + c_2 > 0$. The same discussion is suitable to β. Now it is clear that $f(x)$ is p-irreducible if and only if $(c_1 + \alpha)(\alpha^2 + c_1\alpha + c_2) < 0$ and $(c_1 + \beta)(\beta^2 + c_1\beta + c_2) < 0$.

Proposition 9 *If* $c_1^2 - 4c_2 \le 0$, $f(x)$ *is p-irreducible if and only if the real roots* α, β *of* f *satisfy* $\alpha < -c_1$ *and* $\beta < -c_1$; *otherwise,* $f(x)$ *is p-irreducible if and only if* α, β *satisfy* $\alpha, \beta < -c_1$ *or in the open interval* (α_1, α_2), *where* α_1, α_2 *are defined in theorem 1.*

Proposition 10 *Given a real polynomial*

$$f(x) = x^n + c_1x^{n-1} + \cdots + c_n = h(x^2) + xg(x^2). \tag{12}$$

If $h(x^2)$ *does not change* sign *for* $x > 0$ *and the last Hurwitz determinant* $\Delta_n \ne 0$, $n = 2m$, *then the number of roots of* $f(x)$ *in the right half plane is determined by the formula:*

$$k = 2V(1, \Delta_1, \Delta_3, \cdots, \Delta_{n-1}), \tag{13}$$

where V *is the number of sigh change in the sequence* $(1, \Delta_1, \Delta_3, \cdots, \Delta_{n-1})$.

The proof can be found in [5].

Proposition 11 *If* $f(-c_1) > 0$ *and* $f'(-c_1) > 0$ *then the two roots* α *and* β *are less than* $-c_1$*; if* $f(-c_1) > 0$ *and* $f'(-c_1) \leq 0$ *then* α *and* β *are in the open interval* $(-c_1, 0)$.

Proof: Let $y = x + c_1$, then we have:

$$\begin{aligned}
g(y) &= f(y - c_1) \\
&= y^4 + \tfrac{1}{2}f''(-c_1)y^2 + f(-c_1) + \tfrac{1}{6}f^{(3)}(-c_1)y^3 + f'(-c_1)y \\
f(-c_1) &= c_1^2c_2 - c_1c_3 + c_4, \\
f'(-c_1) &= -c_1^3 - 2c_1c_2 + c_3, \\
f''(-c_1) &= 6c_1^2 + 2c_2 \geq 0, \\
f^{(3)}(-c_1) &= -18c_1 \leq 0. \\
\Delta_1(g) &= \tfrac{1}{6}f^{(3)}(-c_1) \leq 0, \\
\Delta_3(g) &= \tfrac{1}{12}f^{(3)}(-c_1)f''(-c_1)f'(-c_1) - \tfrac{1}{36}(f^{(3)}(-c_1))^2f(-c_1) \\
&\quad -(f'(-c_1))^2
\end{aligned}$$

If $f(-c_1) > 0$ and $f'(-c_1) > 0$ then $\Delta_3(g) < 0$. By proposition 5, $g(y)$ has two zeros in the right half plane. This implies the two real zeros α, β of $f(x)$ are all less than $-c_1$. If $f(-c_1) > 0$ and $f'(-c_1) \leq 0$, it is obvious that g has no negative real zeros, i.e., α and β must be in $(-c_1, 0)$.

Lemma 12 *If* $\Delta_3(f) < 0$, $\mathrm{Discr}(f) \leq 0$, $f(-c_1) > 0$ *and* $f'(-c_1) > 0$, *then* $f(x)$ *is p-irreducible.*

Lemma 2 can be directly deduced from proposition 4 and proposition 6. In the case $\Delta_3(f) < 0$, $\mathrm{Discr}(f) \leq 0$, $c_1^2 - 4c_2 \leq 0$, if $f(-c_1) \leq 0$ then f has a real root between $[-c_1, 0)$; else $f(-c_1) > 0$ and $f'(-c_1) \leq 0$ then α, β are in $(-c_1, 0)$. So $f(x)$ is p-irreducible if and only if $f(-c_1) > 0$ and $f'(-c_1) > 0$.

Proposition 13 *Let* f *be a polynomial with real coefficients,* a *and* b *be two real numbers,* $a < b$, *such that the values* $f(a)$ *and* $f(b)$ *are nonzero, then the number of roots of* f *in the open interval (a, b), counted with their multiplicities, is even or odd depends on the product* $f(a)f(b)$ *being positive or negative.*

The proof can be found in [6].
Remark 1. We have assumed $f(x)$ have two real roots, proposition 7 tells us that there is only one root of $f(x)$ in the open interval (a, b) if $f(a)f(b) < 0$; otherwise the number of roots of $f(x)$ in (a, b) must be 0 or 2.

Lemma 14 *If* $\Delta_3(f) < 0$, $\mathrm{Discr}(f) \leq 0$, $c_1^2 - 4c_2 > 0$ *and* $f(-c_1) < 0$ *then* $f(x)$ *is p-irreducible if and only if* $f(\alpha_1)f(\alpha_2) < 0$.

Proof: If $f(-c_1) = c_1^2c_2 - c_1c_3 + c_4 < 0$, by remark 1, there is only one real root in the interval $(-c_1, 0)$ because $f(0) = c_4 > 0$, and for this root, it is in (α_1, α_2) if and only if $f(\alpha_1)f(\alpha_2) < 0$. By proposition 4 f is p-irreducible if and only if $f(\alpha_1)f(\alpha_2) < 0$.

Let us suppose $f(-c_1) > 0$. If $f(\alpha_1) \leq 0$, $f(x)$ has a root between $(-c_1, \alpha_1]$; if $f(\alpha_2) \leq 0$, $f(x)$ has a root between $[\alpha_2, 0)$. By proposition 4, f has a positive factorization in these two cases. We suppose in the following that

$$f(-c_1) > 0, f(\alpha_1) > 0, f(\alpha_2) > 0. \tag{14}$$

Lemma 15 *If $\Delta_3(f) < 0$, $\mathrm{Discr}(f) \leq 0$, $c_1^2 - 4c_2 > 0, f(-c_1) > 0, f(\alpha_1) > 0, f(\alpha_2) > 0$ and $f(-\frac{1}{2}c_1) \leq 0$, then $f(x)$ is p-irreducible.*

Proof: It is obvious that if $f(\alpha_1) > 0$, $f(\alpha_2) > 0$ and $f(-\frac{1}{2}c_1) < 0$ then α, β will belong to the intervals $(\alpha_1, -\frac{1}{2}c_1)$ and $(-\frac{1}{2}c_1, \alpha_2)$; otherwise, if $f(-\frac{1}{2}c_1) = 0$ and $f(\alpha_1)f(\alpha_2) > 0$, the other real root of $f(x)$ will also belong to the interval (α_1, α_2) according to remark 1.

Remark 2. The condition of $f(\alpha_1) > 0$ and $f(\alpha_2) > 0$ is equal to $P_4 = f(\alpha_1)f(\alpha_2) = c_4^2 + c_2c_3^2 - c_1c_3c_4 > 0$ and $P_5 = f(\alpha_1) + f(\alpha_2) = -c_1c_3 + 2c_4 > 0$.

Proposition 16 *If $f(\alpha_1) > 0$ and $f'(\alpha_1) > 0$ then α and β are less than α_1; if $f(\alpha_1) > 0$ and $f'(\alpha_1) \leq 0$ then α and β are in the open interval $(\alpha_1, 0)$.*

The proof is similar to the proof of proposition 6 and we only need to notice that

$$\begin{aligned} f''(\alpha_1) &= -6c_1\alpha_1 - 10c_2 = 3c_1^2 - 10c_2 + 3c_1\sqrt{(c_1^2 - 4c_2)} \geq 0, \\ f^{(3)}(\alpha_1) &= 24\alpha_1 + 6c_1 = -6c_1 - 12\sqrt{(c_1^2 - 4c_2)} \leq 0. \end{aligned} \tag{15}$$

Lemma 17 *If $\Delta_3(f) < 0$, $\mathrm{Discr}(f) \leq 0$, $c_1^2 - 4c_2 > 0$, $f(-c_1) > 0, f(\alpha_1) > 0, f(\alpha_2) > 0, f(-\frac{1}{2}c_1) > 0$ and $f'(-c_1) \leq 0$, then f is p-irreducible if and only if $f'(\alpha_1) \leq 0$.*

Proof: By proposition 6, if $f(-c_1) > 0$ and $f'(-c_1) \leq 0$ then α, β all belong to $(-c_1, 0)$. Otherwise if $f(-\frac{1}{2}c_1) > 0$ then $f(-\frac{1}{2}c_1)f(0) > 0$, the number of roots in $[-\frac{1}{2}c_1, 0]$ must be 0 or 2. If $\alpha, \beta > -\frac{1}{2}c_1$, then $-c_1 = \alpha + \beta + 2a > -c_1 + 2a$. So α, β must be in the interval $(-c_1, -\frac{1}{2}c_1)$ when $f(-c_1) > 0$, $f'(-c_1) \leq 0$ and $f(-\frac{1}{2}c_1) > 0$. Furthermore, if $f(\alpha_1) > 0$ and $f'(\alpha_1) \leq 0$, then α, β must be in $(\alpha_1, -\frac{1}{2}c_1)$, i.e., f is p-irreducible; if $f'(\alpha_1) > 0$ then α, β are in $(-c_1, \alpha_1)$ which implies $f(x)$ has positive factorizations by proposition 4.

The condition $f'(\alpha_1) \leq 0$ in lemma 5 can be replaced by $P_7 = f'(\alpha_1) + f'(\alpha_2) = f'(\alpha_1) + f'(\alpha_2) = 2c_3 + 4c_1c_2 - c_1^3 \leq 0$ or $P_7 > 0$ and $P_8 = f'(\alpha_1)f'(\alpha_2) = -c_1^2c_2^2 + 4c_2^3 - c_1^3c_3 + 4c_1c_2c_3 + c_3^2 \leq 0$.

Remark 3.

1. The conditions in theorem 1 can be checked one by one from left to right and top to down.
2. If $c_1^2 - 4c_2 \leq 0$ then the last two inequalities in condition 3 are sufficient and necessary.
3. The last inequality in condition 4 is sufficient and necessary.
4. The inequalities $P_6 > 0, P_7 > 0$ in condition 5,6,7 are necessary.

Let us check several examples. The first three are given in Section 1.
Example 1. $f_1 = x^4 + 4x^3 + 6x^2 + 19x + 30$.

$$\begin{array}{lcrcl} P_1 & = & -385 & < & 0 \\ P_2 & = & -6644411 & < & 0 \\ P_3 & = & -8 & < & 0 \\ P_4 & = & 50 & > & 0 \\ P_5 & = & -93 & < & 0 \end{array}$$

So f_1 is not p-irreducible according to condition 3 and remark 3.
Example 2. $f_2 = x^4 + 12x^3 + 34x^2 + 23x + 210$.

$$\begin{array}{lcrcl} P_1 & = & -21385 & < & 0 \\ P_2 & = & -156174091 & < & 0 \\ P_3 & = & 8 & > & 0 \\ P_4 & = & 483 & > & 0 \\ P_5 & = & -2521 & < & 0 \\ P_6 & = & 4126 & > & 0 \\ P_7 & = & 144 & > & 0 \\ P_8 & = & 0 & & \end{array}$$

f_2 is p-irreducible because it satisfies inequality condition 5.
Example 3. $f_3 = x^4 + 8x^3 + 14x^2 + 27x + 90$.

$$\begin{array}{lcrcl} P_1 & = & -3465 & < & 0 \\ P_2 & = & -109166571 & < & 0 \\ P_3 & = & 8 & > & 0 \\ P_4 & = & 770 & > & 0 \\ P_5 & = & -709 & < & 0 \\ P_6 & = & -1134 & < & 0 \end{array}$$

By condition 5 and remark 3, f_3 has a positive factorization.
Example 4. $f_4 = x^4 + 4x^3 + 3x^2 + 2x + 9$.

$$\begin{array}{lcrcl} P_1 & = & -117 & < & 0 \\ P_2 & = & -13136 & < & 0 \\ P_3 & = & 4 & > & 0 \\ P_4 & = & 49 & > & 0 \\ P_5 & = & -86 & < & 0 \\ P_6 & = & 21 & > & 0 \\ P_7 & = & 10 & > & 0 \\ P_8 & = & 1 & > & 0 \\ P_9 & = & -12 & < & 0 \end{array}$$

f_4 is p-irreducible since it satisfies inequality condition 6.
Example 5. $f_5 = x^4 + x^3 + 2x^2 + 3x + 4$.

$$\begin{array}{rcrcl} P_1 & = & -7 & < & 0 \\ P_2 & = & 9217 & > & 0 \end{array}$$

f_5 is p-irreducible according to inequality condition 2.
Example 6. $f_6 = x^4 + 5x^3 + x^2 + 6x + 2$.

$$\begin{array}{rcccl} P_1 & = & -56 & < & 0 \\ P_2 & = & -175800 & < & 0 \\ P_3 & = & 21 & > & 0 \\ P_4 & = & -3 & < & 0 \\ P_6 & = & -20 & < & 0 \end{array}$$

f_6 is p-irreducible according to inequality condition 4.
Example 7. $f_7 = x^4 + 5x^3 + 6x^2 + 3x + 91/10$.

$$\begin{array}{rcccl} P_1 & = & -293/2 & < & 0 \\ P_2 & = & -5314327/500 & < & 0 \\ P_3 & = & 1 & > & 0 \\ P_4 & = & 1441/10 & > & 0 \\ P_5 & = & -182 & < & 0 \\ P_6 & = & 31/100 & > & 0 \\ P_7 & = & 16/5 & > & 0 \\ P_8 & = & 3/80 & > & 0 \\ P_9 & = & 1 & > & 0 \\ P_{10} & = & -42 & < & 0 \end{array}$$

f_7 is p-irreducible according to inequality condition 7.
Example 8. $f_8 = x^4 + x^3 + 1/5x^2 + 20x + 21$.

$$\begin{array}{rcccl} P_1 & = & -417 & < & 0 \\ P_2 & = & -2159748159/625 & < & 0 \\ P_3 & = & 1/5 & > & 0 \\ P_4 & = & 6/5 & < & 0 \\ P_5 & = & 93/5 & > & 0 \end{array}$$

f_8 is p-irreducible according to inequality condition 3.
Example 9. $f_9 = x^4 + 0.1134x^2 + 0.00642978$.

$$\begin{array}{rccl} P_1 & = & 0 & \\ P_{11} & = & 0 & \\ P_{12} & = & -0.01285956 & < 0 \end{array}$$

f_9 is p-irreducible according to inequality condition 1.
Example 10.

$$\begin{aligned} f_{10} &= x^4 + 10000800000888888800x^3 \\ &\quad +6888889999999999999999999999999999999x^2 \\ &\quad +39999999999x + 914444448888888. \end{aligned}$$

$$P_1 = 0.275577644465 \cdot 10^{55} > 0$$

f_{10} is not p-irreducible. Using Maple's function *fsolve*, it will return: Error, (in fsolve/polynom) evalhf/polynom malfunction. If we only cut down the last digit of the coefficient of x^3 in f_{10}, by *fsolve* we find four complex roots:

$$\begin{array}{rcl} -0.5000400000 \cdot 10^{16} & - & 0.8299783106 \cdot 10^{19} I, \\ -0.5000400000 \cdot 10^{16} & + & 0.8299783106 \cdot 10^{19} I, \\ -0.2903225338 \cdot 10^{-27} & - & 0.3643376384 \cdot 10^{-11} I, \\ -0.2903225338 \cdot 10^{-27} & + & 0.3643376384 \cdot 10^{-11} I. \end{array}$$

This example shows that even we have the radical expressions of roots of polynomials with degree four but in the ill-condition cases, it is still not trivial to find all roots.
Example 11. $f_{11} = x^4 + 1.96x^3 + 10^{36}x^2 + 2 \cdot 10^{36}x + 10^{36}$.

$$\begin{array}{rcrcl} P_1 & = & -0.8000000 \cdot 10^{71} & < & 0 \\ P_2 & = & 0.4262400000 \cdot 10^{147} & > & 0 \end{array}$$

So it is $p-$irreducible according to condition 2. Maple's *fsolve* returns a double real root -1 and two non-real roots. If we check by the values of roots [3], f_{11} is not p-irreducible.

$$\begin{aligned} f_{11} &\approx (x+1)(x+1)(x - 0.02 + 10^{18} I)(x - 0.02 - 10^{18} I) \\ &\approx (x+1)(x^3 + 0.96x^2 + 10^{36}x + 10^{36}). \end{aligned}$$

5 Remarks

We have given an approach to determine whether positive polynomials of degree 3 or 4 are p-irreducible. The inequality conditions are consisted of polynomials with the coefficients of the given polynomial. Previous criteria [3] [7] need to find all zeros of a polynomial, then check if zeros satisfy a set of inequalities combined by rational functions. It is well known that roots of a polynomial are highly sensitive to even slight variation of coefficients. So it is not stable to check by roots. Especially for ill-condition polynomials. Our criterion can be computed without error. It is exact and easy to be checked.

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