数理解析研究所講究録
1086 巻 1999 年 1-16

Extension of Wess-Zumino-Witten Model to 2n Dimensions and n-Toroidal Lie Algebra*

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Abstract

A 4-dimensional (4D) extension of the 2D Wess-Zumino-Witten model has a few remarkable properties. In particular, we have shown that this model has an infinite-dimensional symmetry which generates 2-toroidal Lie algebra. Generalization of the construction of the model to higher dimensions $D = 2n$ is also given.

∗ Talk at "Topological Field Theory and Related Topics", held at RIMS, Kyoto, December 16-19, 1996.

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1. Introduction

Many Physical systems in nature have symmetries. If symmetries are continuous, they generate Lie algebras (LA). Let me give two examples, one from atomic physics and another from particle physics : Hydrogen atom has O(4) symmetric energy spectrum. The (massless) quark model has $SU(2)_L \times SU(2)_R$ chiral symmetry in addition to $SU(3)_C$ gauge symmetry.

A physical system becomes integrable, if the symmetry is large enough, namely,

$$\text{dimension of LA} \ = \ \text{number of degrees of freedom} \tag{1}$$

A classic example is the Wess-Zumino-Witten (WZW) model[1]. It is a group-valued non-linear sigma model (NLSM) on a spacetime of 2 dimensions with an addition of an anomaly term (a certain 2-cocycle term in mathematical terms). The dynamical variable is a mapping

$$g : X_2 \to \mathfrak{G}, \tag{2}$$

where X_2 is a 2-dimensional (2D) spacetime with coordinates x^μ and $\mathfrak{G}$ is a group.

As we will see in a moment, the WZW model has currents, which are composite operators composed of the field $g(x)$,

$$J(x) = g^{-1}\partial_z g, \ \ \bar{J}(x) = \partial_{\bar{z}} g g^{-1}, \tag{3}$$

where we have used complex coordinates $z = x^0 + ix^1$ and $\bar{z}$ (or $x^\pm = x^0 \pm x^1$ in light-cone coordinates). The currents obey conservation laws :

$$\partial_{\bar{z}} J = 0, \ \ \partial_z \bar{J} = 0. \tag{4}$$

The mode expansion of $J^a(z) = tr(t^a J(z))$,

$$J_n^a = \sum_{n=-\infty}^{\infty} z^{-n-1} J^a(z), \tag{5}$$

yields an infinite number of conservation laws (CL). Their commutation relations define an affine Kac-Moody algebra,

$$[J^a_m, J^b_n] = if^{abc}J^c_{m+n} + \frac{1}{2}km\delta_{m+n,0}\delta^{ab}. \tag{6}$$

The representation theory of affine Kac-Moody algebras was developed in the eighties, especially in connection with 2D integrable quantum field theories (QFT)[2]. Beyond affine Kac-Moody algebras, new classes of infinite-dimensional Lie algebras are known : hyperbolic Kac-Moody algebras and n-toroidal Lie algebras. It was conjectured that massive modes of superstring, which are infinitely many, generate a hyperbolic Kac-Moody algebras called E(10)[3]. We have recently shown[4] that the symmetry of a 4D analogue of the WZW model generates 2 - toroidal Lie algebra, a generalization of a loop algebra.

The WZW model has many remarkable physical and mathematical properties. It is a finite QFT, i.e., the β-function vanishes. It has an infinite-dimensional symmetry, as explained a moment ago, which is responsible for the model being solvable. The model can be expressed as path integral of a fermionic model[5]. One can q-deform the model to get a massive model[6].

If one could construct integrable QFT in dimensions higher than $D = 2$, it would have enormous implications from both mathematical point of view and application to particle physics. In the past there have been various attempts in this direction. When Polyakov put forward CFT in late sixties, he meant $3 + 1$ dimensional systems[7]. Cardy proposed to generalize modular invariance to $D > 2$[8]. It was conjectured that at level of classical equations of motion all integrable models are related to self-dual Yang-Mills (SDYM) equation in $D = 4$[9].

Recent studies of a 4D analogue of the 2D WZW model are motivated by this way of thought, and they have revealed a few remarkable properties of this 4D model, which we refer to 4D Kähler WZW (KWZW) model. It has an infinite-dimensional (anti)holomorphic symmetry,and it is solvable in its algebraic sector[10,11]. The model has been shown to be one-loop on-shell finite[12] in spite of apparent non-renomalizability by power counting.

In this talk we will concentrate on the infinite-dimentional symmetry of the KWZW model. We will show that this symmetry generates a Lie algebra called

2-toroidal Lie algebra, the central extension of two-loop algebra. A few mathematicians have recently begun to study this class of algebra as a possible extension of affine Kac-Moody algebras[13]. We will also mention an extension of the 2D WZW model (and of the 4D KWZW model) to general 2n dimensions.

2. Wess-Zumino-Witten model

Here we summarize the WZW model as a preparation for extension to $D = 4$. The WZW model is a $\mathfrak{g}$-valued NLSM with an addition of an anomaly term. We extends the spacetime from X_2 to X_3 so that X_2 is the boundary of $X_3, X_2 = \partial X_3$. The action S is a functional of the $\mathfrak{g}$-valued field g(x),

$$S[g] = -\frac{k}{8\pi}[\frac{1}{2}\int_{X_2} tr(g^{-1}\partial g \wedge g^{-1}\bar{\partial} g) + \frac{1}{3}\int_{X_3} tr(g^{-1}dg)], \tag{7}$$

where $\partial = \partial_- dx^-$(or $\partial_z dz$). k is a coupling constant in the usual sense in particle physics. It has to be an integer for the theory to be well-defined, and is to be identified with the centre k appearing in eq.(6) defining a Kac-Moody algebra.

We compute the variation δS of the action (7) for an infinitesimal change of the field, $\delta g = g\epsilon$. The variation of the anomaly term in (7) turns out to be reduced to an integral over X_2,

$$\delta \int_{X_3} tr(g^{-1}dg)^3 = 3\int_{X_2} d^2x \epsilon^{\mu\nu} tr(\epsilon g^{-1}\partial_\mu g g^{-1}\partial_\nu g), \tag{8}$$

where $\epsilon^{\mu\nu}$ is the so-called ϵ-tenson, $\epsilon^{01} = -\epsilon^{10} = 1$. Thanks to this expression, the equation of motion takes a very simple form,

$$\partial_+(g^{-1}\partial_- g) = 0. \tag{9}$$

This equation implies a conserved current, i.e.,

$$J_- = g^{-1}\partial_- g, \tag{10}$$

$$\partial_+ J_- = 0. \tag{11}$$

The last equation is the Cauchy-Riemann (CR) relation, whose solution is given by

$J_- = f(x^-)$. Note that the CR relation is equivalent to self-duality in 2D NLSM[14].

Eqs.(10) and (11) define a current in the right sector, a sector solely depending on x^-. Similarly one can define a current in the left sector, $\bar{J}_+ = \partial_+ g g^{-1}$ and $\partial_- \bar{J}_+ = 0$.

Eq.(11) implies an infinitely many conserved quantities by making mode expansion. Choosing x^+ to be the "time" coordinate, we have $\partial_+ J_n = 0$, where

$$J_n = \int_{x^+} dx^- (x^-)^{-n-1} J_-. \tag{12}$$

3. 4D Kähler WZW model

Aiming at constructing a 4D extension of the WZW model, we begin by writing a general NLSM in $D = 4$. The basic tool is a mapping

$$\phi^a : X_4 \to \mathcal{M}. \tag{13}$$

The X_4 is a 4D-manifold with coordinates x^μ and metric $g_{\mu\nu}(x)$. The target space (or embedding space in mathematics) $\mathcal{M}$ is an n-D manifold with coordinates $\phi^a (a = 1, \ldots, n)$ and metric $g_{ab}(\phi)$. We give an additional structure to both X_4 and $\mathcal{M}$ by assuming that there exist a 2-form on X_4,

$$\omega = \omega_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{14}$$

and another on $\mathcal{M}$,

$$B = B_{ab} d\phi^a \wedge d\phi^b. \tag{15}$$

Under the assumption that the action is bilinear in $\partial_\mu \phi^a$, it consists of two terms,

$$S[\phi^a] = a \int_{X_4} (-d\phi^a \wedge^* d\phi^b g_{ab} + \kappa \omega \wedge d\phi^a \wedge d\phi^b B_{ab}). \tag{16}$$

Note that the theory contains two coupling constants, a and κ (they have mass dimension -2 and 0, respectively).

The action consisting of the first term of (16) alone defines a usual NLSM. This theory is non-renormalizable, i.e., contains divergences which arize quantum mechanically and cannot be handled properly. What is the role of the second term, then? After a lengthy calculation of one-loop quantum effects, Ketov has shown that the NLSM defined by (16) becomes one-loop on-shell finite, provided the following three conditions are met[12] :

a) The target space $\mathcal{M}$ is a group $\mathfrak{G}$ (parallelizable more precisely). Then

$$dB = H, \tag{17}$$

is a torsion on $\mathfrak{G}$, and $H_{ijk} = f_{ijk}$ provides the structure constant of $\mathfrak{G}$.

b) The coupling constant κ is to be tuned to the value

$$\kappa = 2i. \tag{18}$$

c) The spacetime X_4 is a hyper-Kähler manifold (Ricci-flat). We can then introduce complex coordinates,

$$z^1 = x^0 + ix^1, \;\; z^2 = x^2 + ix^3 \; ; \;\; z^{\bar{1}}, z^{\bar{2}}. \tag{19}$$

We will often use the notation $z^1 = u, \; z^2 = v$. The ω is nothing but the Kähler 2-form,

$$\omega = \frac{i}{2} h_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}, \tag{20}$$

$$d\omega = 0. \tag{21}$$

After taking account of the three conditions given above, the action (16) can be reduced to the one which was constructed by Nair and Schiff[10] in extending the Chern-Simons action to $D = 5$ and by Donaldson[15] in an algebraic-geometric viewpoint (DNS action). To see this, we proceed as follows.

We now have a map

$$g\ :\ X_4 \to \mathfrak{G}. \tag{22}$$

We extend the X_4 to X_5 so that $X_4{=}\partial X_5$, to give an anomaly interpretation to the second term of (16). Let t^i be generators of $\mathfrak{G}$ and ${V_a}^i$ be vielbein on $\mathfrak{G}$, ${V_a}^i t^i = V_a \in \mathfrak{g}$. ${V_a}^i$ can be shown to obey the Maurer-Cartan equation. This means that ${V_a}^i$ is a pure-gauge,

$$V_a d\phi^a = -\frac{1}{2} i g dg. \tag{23}$$

We have replaced the scalar fields $\phi^a(x)$ by $g(x)$.

Using eqs.(17), (18), (20) and (23), the two terms of (16) can be expressed in terms of $g^{-1}dg$ and ω, and we arrive at the DNS action

$$S = -i \int_{X_4} \omega \wedge Tr(g^{-1}\partial g \wedge g^{-1}\bar{\partial} g) + \frac{i}{3}\int_{X_3} \omega \wedge tr(g^{-1}dg)^3. \tag{24}$$

Here, $\partial = \partial_\alpha dz^\alpha$. We have absorbed the coupling constant a into the redefinition of $\omega : \omega \to 2a\omega$.

We can prove the following identity, which is a 4D analogue of the Polyakov-Wiegmann formula[16] (2-cocycle condition in mathematical terms).

$$S[gh] = S[g] + S[h] - 2i \int_{X_4} \omega \wedge tr(g^{-1}\partial g \wedge \bar{\partial} h h^{-1}). \tag{25}$$

We see easily from this formula that the action is invariant under holomorphic right (HR) and antiholomorphic left (AHL) infinite symmetries,

$$g \to h_L(z^{\bar{\alpha}}) g h_R(z^\beta). \tag{26}$$

The equation of motion can be derived in the same fashion as the 2D case. It is given by

$$\bar{\partial}(\omega \wedge g^{-1}\partial g) = 0, \tag{27}$$

or equivalently,

$$\partial(\omega \wedge \bar{\partial} g g^{-1}) = 0. \tag{28}$$

These equations allow us to identify the conserved currents $J(\bar{J})$ which generate the right(left)-action symmetry in (26).

$$J = -i\omega \wedge g^{-1}\partial g, \quad \bar{J} = i\omega \wedge \bar{\partial} g g^{-1}, \tag{29}$$

$$\bar{\partial} J = 0, \quad \partial \bar{J} = 0. \tag{30}$$

You may consider the dual (one-form) of J, which we denote by $J_\alpha dz^\alpha$. Then, the conservation law (30) reads

$$\bar{\partial} J = \partial_{\bar{u}} J_u + \partial_{\bar{v}} J_v = 0. \tag{31}$$

It is curious to note that the same value of κ assures one-loop finiteness on one hand and the PW formula (25) and consequently the current conservation (30) on the other.

The equation of motion (30) (or (31)) can be shown to be equivalent to the SDYM equation. Quite some time ago, Yang cast the SDYM equation (in the flat spacetime) into the form[17]

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0, \tag{32}$$

$$\eta^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}} = 0. \tag{33}$$

The conservation low (31) is eq.(33) generalized to a Kähler spacetime.

The DNS action may be cast into a more familiar form by using Gauss decomposition. For simplicity we consider the case of $\mathfrak{g} = sl(2)$. Let $H, E^\pm$ be Chevalley generators of $\mathfrak{g}$, obeying $[E^+, E^-] = H$. The Gauss decomposition reads

$$g(x) = e^{\chi(x)E^+} e^{\varphi(x)H} e^{\psi(x)E^-}$$

$$= \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\varphi & 0 \\ 0 & e^{-\varphi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \psi & 1 \end{pmatrix}. \tag{34}$$

We have introduced three scalar fields φ, χ, ψ. Using the PW formula (25), the action is now written as an integral over X_4.

$$S[\varphi,\chi,\psi] = -i\int_{X_4} \omega \wedge (\partial\varphi \wedge \bar{\partial}\varphi + \partial\chi \wedge \bar{\partial}\psi e^{-2\varphi}). \tag{35}$$

So far X_4 is an arbitiary 4D Kähler manifold. In the following analysis we will consider the case of flat X_4.

$$h_{1\bar{1}} = h_{2\bar{2}} = \frac{1}{2}, \quad \omega_{1\bar{1}} = \omega_{2\bar{2}} = \frac{i}{2}a. \tag{36}$$

The action is then given by

$$S = -a\int_{X4} d^4z \sum_{\gamma=1,2} (\partial_\gamma\varphi\partial_{\bar{\gamma}}\varphi + \partial_\gamma\chi\partial_{\bar{\gamma}}\psi e^{-2\varphi}). \tag{37}$$

The currents J_α and $\bar{J}_{\bar{\beta}}$ are expressed in terms of the scalar fields. By writing $J_\alpha = J^0_\alpha H + J^-_\alpha E^+ + J^+_\alpha E^- \epsilon \mathfrak{G}$, we have

$$J^0_\alpha = a(\partial_\alpha\varphi + \psi\partial_\alpha\chi e^{-2\varphi}),$$

$$J^+_\alpha = -a(\psi\partial_\alpha\varphi - 2\partial_\alpha\varphi + \varphi^2\partial_\alpha\chi e^{-2\varphi}), \tag{38}$$

$$J^-_\alpha = a\partial_\alpha\chi e^{-2\varphi}.$$

4. Infinite dimensional Lie algebra

Since we are given a field theory, we should be able to compute various quantities and relations. We consider the question : What is the infinite-dimensional symmetry that the currents $J^a_\alpha(\bar{J}^a_{\bar{\beta}})$ generate?

To address to this question we use the canonical formalism and compute the Poisson brackets (commutation relations in quantum theory) of the currents. To this end we have to choose one of the four coordinates as "time" t and the remaining three are space coordinates. There are two alternatives of doing this : a) $t = x^0$, $\vec{x} = (x^1, x^2, x^3)$ being space coordinates. b) $t = \bar{u}$, $\vec{x} = (u, v, \bar{v})$ being space coordinates (light-cone scheme).

Here we present the result in the light-cone scheme, taking $\bar{u}$ as time. The coordinates are assumed to take real values by using Wick rotation. We note that the action (24) is first order in time derivatives, that is, it is already in the Hamiltonian form. Hence we can define P.B. without introducing conjugate momenta of the field $g(x)$. The symplectic two-form of this model is given by

$$\Omega_{ab}(\vec{x},\vec{x}') = a\delta_{ab}\partial_u\delta^3(\vec{x}-\vec{x}'). \tag{39}$$

The Hamiltonian structure is obtained as the inverse of Ω, $\mathcal{H}^{ab}(\vec{x},\vec{x}') = a^{-1}\delta^{ab}\partial_u^{-1}$ $\delta^3(\vec{x}-\vec{x}')$. Here, $\partial_u^{-1}\delta(u) = \frac{1}{2}\epsilon(u)$. It is then straightforward to compute P.B., e.g.

$$\{tr(Xg^{-1}\partial_A g)(\vec{x}),\ \ tr(Yg^{-1}\partial_B g)(\vec{x}')\}, \tag{40}$$

where A, B are u, v or $\bar{v}$ and X, Y are t^a.

Recall that the current is given by $J_A = \frac{1}{2}ag^{-1}\partial_A g$. In the present choice of $\bar{u}$ as time, J_u^a are the generators of the HR action symmetry : $g \to gh_R(z^\alpha)$. If we choose u as time instead, then $\bar{J}_{\bar{u}}^a$ are the generators of AHL action symmetry : $g \to h_L(z^{\bar{\beta}})g$.

The P.B. of current J_A^a can be computed from (40). We refer you to refs. [4, 19] for the explicit result and derivation. Here we only write the result for (A, B)=(u, u), which we will be concerned with hereafter.

$$\{J_u^a(\vec{x}), J_u^b(\vec{x}')\} = if^{abc}J_u^c(\vec{x})\delta^3(\vec{x}-\vec{x}') + a\delta^{ab}\partial_u\delta^3(\vec{x}-\vec{x}'), \tag{41}$$

that is, the currents J_u^a satisfy a current algebra in $D = 4$ with a c-number anomaly term.

We make mode expansion of the currents $J_u^a(\vec{x})$ to derive a more familiar-looking Lie algebra from the current algebra (41). To this end, it is convenient to compactify the three space coordinates $u, v, \bar{v}$, by letting $u, v, \bar{v}$ take values in

$I = [0, 2\pi]$. The generators of HR action symmetry are given by

$$Q^a = \int_0^{2\pi} dudv \epsilon^a(u,v) \widetilde{J}^a(\bar{u}, u, v), \tag{42}$$

where $\widetilde{J}^a$ are zero-modes with respect to $\bar{v}$,

$$\widetilde{J}^a(\bar{u}, u, v) = \int_0^{2\pi} d\bar{v} J^a_u(\bar{u}, \vec{x}). \tag{43}$$

Note that $\partial_{\bar{u}} \widetilde{J}^a = 0$, and hence that $\widetilde{J}^a$ are holomorphic functions of u and v, $\widetilde{J}^a(u,v)$. We use the notation $\vec{z} = (u, v)$. The P.B. for $\widetilde{J}^a(\vec{z})$ can be derived from eq.(41), and is given by

$$\{\widetilde{J}^a(\vec{z}),\ \widetilde{J}^b(\vec{z}\,')\} = if^{abc}\widetilde{J}^c(\vec{z})\delta^2(\vec{z} - \vec{z}\,') + 2a\delta^{ab}\partial_u \delta^2(\vec{z} - \vec{z}\,'). \tag{44}$$

The parameter $\epsilon^a(\vec{z})$ are defined on the 2-torus, and hence it can be expanded as

$$\epsilon^a(\vec{z}) = \sum_{m_0, m_1 = -\infty}^{\infty} \epsilon^a_{\vec{m}} e^{i\vec{m}\cdot\vec{z}}, \tag{45}$$

where $\vec{m} = (m_0,\ m_1)\ \epsilon Z_2$ and $\vec{m}\cdot\vec{z} = m_0 u + m_1 v$. Correspondingly, the currents $\widetilde{J}^a$ are mode-expanded as

$$\widetilde{J}^a_{\vec{m}} = \int dudv e^{i\vec{m}\cdot\vec{z}} \widetilde{J}^a(\vec{z}), \tag{46}$$

so that we have

$$Q^a = \sum_{\vec{m}} \epsilon^a_{\vec{m}} J^a_{\vec{m}}. \tag{47}$$

We obtain from eq.(44)

$$\{\widetilde{J}^a_{\vec{m}},\ \widetilde{J}^b_{\vec{\ell}}\} = if^{abc}\widetilde{J}^c_{\vec{m}+\vec{\ell}} + \lambda_0 m_0 \delta^{ab}\delta_{\vec{m},-\vec{\ell}}, \tag{48}$$

where $\lambda_0 = (area\ of\ T^2)\cdot a/8 = \pi^2 a/2$. This relation defines a 2-loop algebra with central extension, which is 2-toroidal Lie algebra.

In eq.(48) there has appeared one centre, $\lambda_0 m_0$. A general central term should consist of $\lambda_0 m_0 + \lambda_1 m_1$ (and perhaps many more terms).

In mathematical language, an affine Kac-Moody argebra is defined by using Laurent polynomial expansion. In an analogous way, a 2-toroidal Lie algebra is expressed by double Laurent polynomial expansion as

$$\mathfrak{G}_{2-tor} = \mathfrak{G} \otimes \mathbf{C}[s, t, s^{-1}, t^{-1}] \oplus \; centre \; \oplus \mathbf{C}_0 d_0 + \mathbf{C}_1 d_1. \tag{49}$$

We should mention that the representation theoretic content of 2-toroidal Lie algebras has been poorly understood. Let us compare the affine Kac-Moody algebra $\mathfrak{g}$ and the 2-toroidal Lie algebra $\mathfrak{g}_{2-tor}$ in the case of $\mathfrak{g} = sl(2)$. The root system of $\mathfrak{g}$ is two-dimensional, being infinite in one (affine) direction. That of $\mathfrak{g}_{2-tor}$ is three-dimensional, being infinite in two directions. The Cartan matrix of $\mathfrak{g}$ is

$$K = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{50}$$

The use of "Cartan matrix" in toroidal Lie algebra is yet to be understood completely. For the $\mathfrak{g}_{2-tor}$ root system we have tentatively chosen, the "Cartan matrix" is

$$K = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 2 \\ -2 & 2 & 2 \end{pmatrix}. \tag{51}$$

We note that in (54) one of the off-diagonal elements is $+2$, a feature not shared by finite-dimensional and affine Lie algebras. Extension of highest-weight representations to $\mathfrak{g}_{2-tor}$ is a diffucult problem because of the two-dimensionality of infinite directions of its root system.

5. Generalization of the KWZW model to D=2n

It is an intriguing question whether the KWZW model allows a generalization to $D = 2n$ possessing many of its remarkable properties at $D = 4$. Here we present one possible model of this type[18].

Let X_{2n} be a 2n-dimensional Kähler manifold with 2-form ω. We consider a NLSM on X_{2n} with a target space furnished with a 2-form B.

$$S[\phi^a] = a\int_{X_{2n}} (-d\phi^a \wedge^* d\phi^b g_{ab} + \kappa\omega^{n-1} \wedge d\phi^a \wedge d\phi^b b_{ab}). \tag{52}$$

By repeating an argument analogous to that we have made in the case of $D = 4$, we arrive at a a group $\mathfrak{G}$-valued NLSM with an addition of an anomaly term[19].

$$S = -i\int_{X_{2n}} \omega^{n-1} \wedge tr(g^{-1}\partial g \wedge g^{-1}\bar{\partial} g) + \frac{i}{3}\int_{X_{2n+1}} \omega^{n-1} \wedge tr(g^{-1}dg)^3. \tag{53}$$

Here $\partial = \partial_\alpha dz^\alpha$. We have tuned κ to the value

$$\kappa = 2^{n-1}i/(n-1)!, \tag{54}$$

so that the PW of the form (25) holds true.

The equation motion can be derived from (53) in the same fashion as the WZW case.

$$\bar{\partial} J = 0 \quad (equivalently\ \partial\bar{J} = 0), \tag{55}$$

where we have already introduced conserved currents

$$J = i\omega^{n-1} \wedge g^{-1}\partial g \quad (\bar{J} = -i\omega^{n-1} \wedge \bar{\partial} g g^{-1}). \tag{56}$$

For the dual of J, which we denote by $J_\alpha dz^\alpha$, eq.(55) reads

$$\partial_{\bar{\alpha}} J_\alpha = 0. \tag{57}$$

It is very curious to note that the same equation as (57) has previously been written in connection with a moduli problem of gauge fields in algebraic geometry. On a Kähler manifold X_{2n} the curvature 2-form F of a gauge field is decomposed into (2, 0), (1, 1) and (0, 2) components. Donaldson[15], Uhlenbeck and Yau[19] wrote the conditions that a holomorphic vector bundle is stable.

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0, \tag{58}$$

$$h^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}} = 0. \tag{59}$$

The connection of the DUY equation to the equation of motion of the KWZW model is as follows. We set $A_\alpha = h_R^{-1}\partial_\alpha h_R$, $A_{\bar{\beta}} = \partial_{\bar{\beta}} h_L h_L^{-1}$ and $g = h_R h_L$. Then eqs.(58) and (59) can be shown to be equivalent to eq.(57).

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