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On the algebraic geometry of Kac–Moody groups

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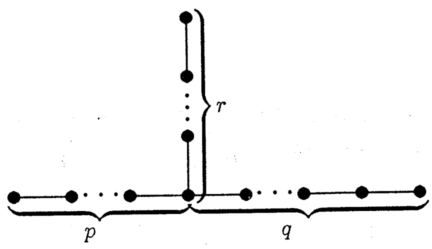
These notes are a slightly elaborated version of a talk given at the RIMS–Symposium on "Topological Field Theory and Related Topics", Kyoto, December 1996. Their aim is to give a survey of the main results obtained by Claus Mokler in his dissertation at Hamburg University ([8], October 1996) pertaining to a natural semigroup completion of Kac–Moody groups.

1. "Abstract" Kac–Moody groups

Starting point for the construction of Kac–Moody Lie algebras and associated groups is a generalized Cartan matrix, i.e. an $l \times l$–matrix $A = ((a_{ij})) \in M_l(\mathbb{Z})$ satisfying

$$\begin{array}{rcll} a_{ii} & = & 2 & \\ a_{ij} & \leq & 0 & i \neq j \\ a_{ij} & = & 0 \Rightarrow & a_{ji} = 0 \end{array}$$

We shall assume, in addition, that A is symmetrizable (cf. [2]). In fact, one might take A to be symmetric for simplicity. Also, the generalized Cartan matrices arising in singularity theory and providing the original motivation for our research in Kac–Moody groups (cf. [11], [13]) are symmetric, e.g. the matrix of type T_{pqr} encoded by the Coxeter–Dynkin diagram



Whereas the Kac–Moody algebra $\mathbf{g} = \mathbf{g}(A)$ is essentially generated by l copies of the Lie algebra $\mathbf{sl}_2(\mathbb{C})$,

$$\langle e_i, h_i, f_i\rangle \,, = i = 1, \dots, l \,,$$

subject to relations derived from A, the corresponding Kac–Moody group $G = G(A)$ is essentially generated by l copies of the Lie group $SL_2(\mathbb{C})$. Here, the relations are either imposed abstractly (Tits, cf.[15], [16]) or by the "integration" of G from the integrable representations of $\mathbf{g}$ (Moody–Teo, Marcuson, Garland, and, in the most thorough way, Kac–Peterson [10], [3], [4]).

The most important result about G as an abstract group is the existence of a "twin" BN–pair or "twin" Tits system (B^+, B^-, N, S) in G providing us, among others, with

- *positive* and *negative Borel subgroups* B^+ and B^-,
- a *maximal torus* $T = B^+ \cap B^- = N \cap B^+ = N \cap B^-$,
- a *Weyl group* $W = N/T$ with generating set S,
- *Bruhat decompositions*

$$G = \bigcup_{w \in W} B^+ w B^+ = \bigcup_{w \in W} B^- w B^- ,$$

  and a *Birkhoff–decomposition*

$$G = \bigcup_{w \in W} B^- w B^+ \,.$$

Similarly, as in the case of the Lie algebra $\mathbf{g}$ where one usually adjoins additional derivations to a "minimal" Kac–Moody algebra, the precise structure of G depends on slightly finer data than A. These data are given by an *integral realization* $(H, \Pi, \Pi^\vee)$ of A which fixes the size of the maximal torus T and its position inside G.

Here, H is the lattice of algebraic one–parameter subgroups $\mathbb{C}^* \to T$ into T with dual $P = H^* = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$, the lattice of algebraic characters $T \to \mathbb{C}^*$, and

$$\Pi = \{\alpha_1, \dots, \alpha_e\} \subset P,\ \Pi^\vee = \{h_1, \dots, h_l\} \subset H$$

are free subsets of *simple roots* in P, resp. of *simple coroots* in H, related by

$$\alpha_i(h_j) = a_{ij}.$$

More explicitly, Π and $\Pi^\vee$ are given in our context as follows:

Let $\kappa_i : SL_2(\mathbb{C}) \to G$, $i = 1, \ldots, l$ denote the basic homomorphisms of $SL_2(\mathbb{C})$ into G, and let

$$\begin{array}{lcl} h_i : \mathbb{C}^* & \to & G \\ u_i : \mathbb{C} & \to & G \end{array}$$

be given by

$$h_i(s) \ := \ \kappa_i(\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}) \quad , \ s \in \mathbb{C}^*,$$

$$u_i(c) \ := \ \kappa_i(\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}) \quad , \ c \in \mathbb{C},$$

Then $h_i(\mathbb{C}^*) \subset T$, i.e. $h_i \in H$, and there is a character $\alpha_i \in P$ such that

$$t\, u_i(c) t^{-1} = u_i(\alpha_i(t)c)$$

for all $t \in T$, $c \in \mathbb{C}$.

By its natural action on T and P, the Weyl group $W = N/T$ is identified with the subgroup of $\mathrm{Aut}_{\mathbb{Z}}(P)$ generated by the reflections $S = \{s_1, \ldots, s_l\}$

$$s_i(\omega) = \omega - \omega(h_i)\alpha_i \ , \ \omega \in P.$$

Also, s_i is given by the class of

$$\kappa_i(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \ \text{in} \ N/T.$$

We can also make the groups B^+ and B^- more explicit:

Let U_i denote the subgroup $u_i(\mathbb{C})$ and, for any *real root* $\gamma = w(\alpha_i)$ $(w \in W)$, put

$$U_\gamma := w\, U_i w^{-1}.$$

The set $\sum^{\text{real}} = W(\Pi)$ of all real roots divides naturally into positive and negative roots,

$$\sum{}^{\text{real}} = \sum{}^{\text{real},+} \cup \sum{}^{\text{real},-}$$

where $\sum{}^{\text{real},-} = -\sum{}^{\text{real},+}$, and if we put

$$U^{\pm} = \langle U_\gamma | \gamma \in \sum{}^{\text{real},\pm} \rangle$$

($\langle a, b, \ldots \rangle$ denoting the group generated by $a, b, \ldots$) we have

$$B^+ = T \ltimes U^+ \ , \ B^- = T \ltimes U^- \ .$$

Finally, the anti-involution

$$\begin{array}{ccc} SL_2(\mathbb{C}) & \longrightarrow & SL_2(\mathbb{C}) \\ g & \longmapsto & {}^t g \end{array}$$

can be lifted to all of G, i.e. there is an anti-involution $* : G \to G$ such that

- $*(t) = t$, for all $t \in T$
- $*(\kappa_i(g)) = \kappa_i({}^t g)$, for all $g \in SL_2(\mathbb{C})$.

In particular, one has $*(U^+) = U^-, *(U^-) = U^+$.

2. "Algebraic" Kac–Moody groups

If A is a Cartan matrix of "finite type" (i.e. all components are of type $A_n, B_n, \ldots$, F_n, or G_2) then G, as described in the last section, is a reductive algebraic group over $\mathbb{C}$. The algebra $\mathbb{C}[G]$ of regular functions on G is then a Hopf algebra, and the group G can be completely recovered from the Hopf algebra $\mathbb{C}[G]$, in particular

$$G = \text{Specmax}\, \mathbb{C}[G] = \text{Hom}_{\mathbb{C}-\text{alg}}(\mathbb{C}[G], \mathbb{C}).$$

If A is a proper generalized Cartan matrix, then the associated algebra $\mathbf{g}$ is of infinite dimension over $\mathbb{C}$. Thus, also G should be infinite-dimensional. A proposal for an algebra of "strongly regular" functions on G was made by Kac and Peterson in 1983 ([3]). As in the finite-dimensional case, this algebra is generated by the matrix coefficients of a suitable representation. Let us therefore recall some basic facts about the irreducible highest weight representations of G.

To simplify the presentation, we shall assume that G is of "simply–connected type", i.e. that the coroot lattice $Q^\check{} = \mathbb{Z}.\Pi^\check{}$ is a direct summand of H

$$H = Q^\check{} \oplus D.$$

Then the set $P^+ = \{\omega \in P | \omega(h_i) \geq 0, i = 1, \ldots, l\}$ of *dominant weights* can be written as a direct sum

$$P^+ = P^0 \oplus \bigoplus_{i=1}^{l} \mathbb{N}.\Lambda_i$$

where

$$P^0 = \{\omega \in P | \omega(h_i) = 0, i = 1, \ldots, l\} \cong D^*$$

and where $\Lambda_i, i = 1, \ldots, l$, are *fundamental dominant weights*

$$\Lambda_i(h_j) = \delta_{ij}\ , i, j = 1, \ldots, l,$$

uniquely determined modulo P^0.

As in the finite–dimensional case there is a bijection of P^+ onto the set of isomorphism classes of irreducible highest weight representations L of G

$$\Lambda \in P^+ \longleftrightarrow L(\Lambda)$$

determined by $L(\Lambda)$ having a unique (up to scalars) highest weight vector $v_\Lambda \in L(\Lambda)\backslash\{0\}$ of weight Λ. (If $\Lambda \in P^0$, the module $L(\Lambda)$ will be one–dimensional.)

Any such module carries a nondegenerate contravariant form (essentially unique), i.e. a symmetric bilinear form

$$\langle\ ,\ \rangle : L(\Lambda) \times L(\Lambda) \to \mathbb{C}$$

such that $\langle v, gw\rangle = \langle g^* v, w\rangle$ for all $v, w \in L(\Lambda), g \in G$, and $g^* = *(g)$ the anti–involution on G.

Let us call the function

$$c_{v,w} : G \to \mathbb{C}$$

given by $c_{v,w}(g) = \langle v, gw \rangle$ for some $v, w \in L(\Lambda)$ a *matrix coefficient* of G (in the respresentation $L(\Lambda)$). Kac and Peterson now define

$$\mathbb{C}[G] := \left(\begin{array}{c} \mathbb{C}\text{-algebra generated by} \\ \text{the matrix coefficients} \\ c_{v,w} \ \text{for all}\, v, w \in L(\Lambda) \\ \text{and all}\, \Lambda \in P^+ \end{array} \right)$$

and they prove the following

"Peter–Weyl"–Theorem: The map

$$\bigoplus_{\Lambda \in P^+} L(\Lambda) \otimes L(\Lambda) \to \mathbb{C}[G]$$

induced by $v \otimes w \longmapsto c_{v,w}$ is an isomorphism of $G \times G$–modules.

Here, the action of $G \times G$ on $\mathbb{C}[G]$ is given by $((g,h)f)(x) = f(g^* x h)$. Alternatively, one might use the usual action of $G \times G$ on functions on G and let act G on the first factor $L(\Lambda)$ by the contragredient action

$$(g, v) \longmapsto (g^*)^{-1} v.$$

It turned out that $\mathbb{C}[G]$ is not a Hopf algebra. There is neither a co–multiplication nor an antipode (basically due to the infinite–dimenionality of the $L(\Lambda)$ and the inequivalence between highest weight and lowest weight representations). Even worse, Kac and Peterson exhibited elements in Specmax $\mathbb{C}[G]$ not contained in G (which injects into Specmax $\mathbb{C}[G]$) (cf. [3] Remark 2.2). Thus they formulated the following problem (loc. cit., 4H b)):

Determine Specmax $\mathbb{C}[G]$ (possibly with respect to a topological structure on the algebra $\mathbb{C}[G]$)!

Inspired by the deformation theory of certain singularities (cf. [13]) we conjectured

$$\overline{G} := \mathrm{Specmax}\, \mathbb{C}[G] = G.\overline{T}.G$$

where $\overline{T}$ is the closure of T in $\overline{G}$ realized as the torus embedding

$$T = \mathrm{Specmax}\, \mathbb{C}[P] \subset \mathrm{Specmax}\, \mathbb{C}[P \cap I] = \overline{T}$$

for $I \subset P \otimes_{\mathbb{Z}} \mathbb{R}$, the Tits cone attached to G. This embedding, or rather a domain $\mathcal{T} \subset \overline{T}$ of discontinuity for the action of W, had been studied before by Looijenga and the quotient $\overline{T}/W$ had turned out to be the base space of a semiuniversal deformation for certain isolated singularities (cf. [6], [7]). Moreover, in [12], [13] we realized $\overline{\mathcal{T}}/W$ and $\overline{T}/W$ as target spaces for an adjoint quotient of G.

During a stay at MSRI (1984), D. Peterson announced a proof of the above conjecture including a number of structural properties of $\overline{G}$ ([9], $\overline{G}$ being considered as the continuous spectrum with respect to some topology). In connection with his infinite–dimensional algebraic–geometric approach to the flag manifolds of Kac–Moody groups, M. Kashiwara also studied the abstract maximal spectrum of $\mathbb{C}[G]$ (without topology on $\mathbb{C}[G]$), cf. [5]. Finally, C. Mokler ([8]) made a quite thorough study of $\overline{G}$ in the context of some infinite–dimensional algebraic geometry based on suitably topologized coordinate rings. In particular, he gave a detailed proof of our conjecture. This is what we want to report upon.

3. A topology on the algebra of strongly regular functions

Let V be a complex vector space. Then we may view the symmetric algebra $S(V^*)$ of its dual space V^* as the coordinate ring of the variety V. If $\dim_{\mathbb{C}} V < \infty$ we have

$$\mathrm{Hom}_{k-\mathrm{alg}}(S(V^*), \mathbb{C}) = \mathrm{Hom}\,(V^*, \mathbb{C}) = V^{**} = V.$$

However, if $\dim_{\mathbb{C}} V = \infty$ we have $V \subset V^{**}, V \neq V^{*,*}$, and $\mathrm{Specmax}\, S(V^*)$ is strictly larger than V. To remedy this defect we put the following topology on the algebra $S(V^*)$:

A basis of neighborhoods of $0 \in S^*(V^*)$ is given by the "cofinite" ideals

$$\{J(V') | V' \subset V \text{ a finite–dimensional subspace}\},$$

$$J(V') = \{f \in S(V^*) | f|_{V'} \equiv 0\}\,.$$

Now, the continuous maximal spectrum

$$\mathrm{Specm}^\circ S(V^*) = \mathrm{Hom}_{\mathrm{cont}-k-\mathrm{alg}}(S(V^*), \mathbb{C})$$

is easily identified with V (i.e. Hilbert's Nullstellensatz gives $V' = S(V^*)/J(V')$ for all finite–dimensional $V' \subset V$).

To put a topology on $\mathbb{C}[G]$ we embed G, and finally $\overline{G}$, into a larger space M constructed as follows:

We fix contravariant forms $\langle\ \ ,\ \ \rangle$ on all modules $L(\Lambda), \Lambda \in P^+$, and extend them to a form, also denoted by $\langle\ \ ,\ \ \rangle$, on the direct sum

$$L := \bigoplus_{\Lambda \in P^+} L(\Lambda)$$

by requiring $L(\Lambda)$ and $L(\Lambda')$ to be orthogonal for $\Lambda \neq \Lambda'$. Let M denote the subalgebra of End (L) satisfying

- $\varphi(L(\Lambda)) \subset L(\Lambda)$ for all $\Lambda \in P^+$,
- the adjoint φ^* of φ with respect to $\langle\ \ ,\ \ \rangle$ exists.

We let $\mathbb{C}[M]$ denote the $\mathbb{C}$–algebra generated by all matrix coefficients $c_{v,w} : M \to \mathbb{C}$, $v, w \in L$, $c_{v,w}(\varphi) = \langle v, \varphi w\rangle$, and consider the "cofinite" topology on $\mathbb{C}[M]$ given by the neighborhood basis of 0

$$\{J(M') | M' \subset M \text{ a subspace of finite dimension}\},$$

$J(M')$ being the vanishing ideal of M'.

Then we have

- $\mathrm{Specm}^\circ \mathbb{C}[M] = M$
- M is a "weak" algebraic monoid (i.e. right and left multiplication on M by given elements of M are "morphisms" of M; note that there is no comultiplication on $\mathbb{C}[M]$).

By the definition of the contravariant forms on the $L(\Lambda)$ and L we have a natural embedding $G \hookrightarrow M$. Moreover, $\mathbb{C}[G]$ is the image of $\mathbb{C}[M]$ under the restriction from M to G. We now put the quotient topology with respect to $\mathbb{C}[M] \to \mathbb{C}[G]$ on $\mathbb{C}[G]$ and we obtain

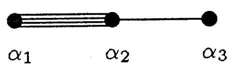
- $\mathrm{Specm}^{\circ}\mathbb{C}[G] = \overline{G} =$ Zariski–closure of G in M,
- $\overline{G}$ is a "weak" algebraic monoid (in the sense above).

4. The Tits cone and the closure of the maximal torus

Let $V = P \otimes_{\mathbb{Z}} \mathbb{R}$ be the "real" character group, $\overline{C} = \{\omega \in V | \omega(h_i) \geq 0$ for all $i = 1, \ldots, l\}$ a *fundamental Weyl chamber*, and $I = W.\overline{C}$ the union of all W–translates of $\overline{C}$. Then I is a convex solid cone, called the *Tits cone*. The interior I° of I is a domain of discontinuity of W. (For details, cf. [2]).

Example: Let A be the "hyperbolic" matrix

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{corresponding to}$$



Now, the matrix A defines a symmetric bilinear form on $V \cong \mathbb{R}^3$ of signature $(+,+,-)$, and with respect to some convention I° may be be identified with the interior of the positive light cone. The Weyl group W is isomorphic to $PGL_2(\mathbb{Z})$ acting as a group of hyperbolic motions on the unit disc $\cong \mathbb{P}(I^{\circ}) \subset \mathbb{P}(V)$.

The boundary of I is of particular interest for us. A subset $I' \subset I$ is called a (*rational*) *boundary component* of I if there is a $\gamma \in V^* = H \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. a $\gamma \in H$) such that

- $\omega(\gamma) \geq 0$ for all $\omega \in I$
- $\omega(\gamma) = 0$ for $\omega \in I$ implies $\omega \in I'$.

It is possible to classify all boundary components of I in terms of a special subset of them:
A subset $\Theta \subset \Pi$ is called *pure* if either $\Theta = \emptyset$ or if all connected components of Θ (in an obvious sense) are of infinite type.

To any pure subset $\Theta \subset \Pi$ we may associate the following subset $I(\Theta)$ of I:

$$I(\Theta) = \{\omega \in I | \omega(h_i) = 0 \text{ for all } i \text{ such that } \alpha_i \in \Theta\}.$$

We now have the following result, essentially due to Looijenga ([6]):

Theorem:

i) Let $\Theta \subset \Pi$ be pure. Then $I(\Theta)$ is a rational boundary component of I.

ii) Let $I' \subset I$ be a boundary component. Then there is a unique pure $\Theta \subset \Pi$ and a $w \in W$ such that $I' = w.I(\Theta)$. In particular, all boundary components of I are rational.

Example: We take up the previous example. There are 3 pure subsets of Π:

$$\emptyset \quad , \qquad \Theta = \{\alpha_1, \alpha_2\} \qquad , \Pi = \{\alpha_1, \alpha_2, \alpha_3\}.$$

The corresponding boundary components are

$$I \quad , \quad \text{all rational half-lines on the positive light cone} \quad , \quad \{0\}.$$

To determine the closure $\overline{T}$ of T in $\overline{G}$ we first have to describe the restriction of $\mathbb{C}[G]$ to T. Since all weights of a module $L(\Lambda)$, $\Lambda \in P^+$, are contained in $I \cap P$, and since $\overline{C} \cap P = P^+$ we obtain

$$\mathbb{C}[G]|_T = \mathbb{C}[P \cap I],$$

the semigroup algebra of $P \cap I$. It is easily seen that the induced topology on $\mathbb{C}[P \cap I]$ is discrete, thus

$$\overline{T} = \mathrm{Specm}^\circ \mathbb{C}[P \cap I] = \mathrm{Specm}\, \mathbb{C}[P \cap I].$$

Through $\mathbb{C}[P \cap I]$ is not finitely generated its maximal spectrum can be determined similarly as in the usual "finite type" theory of torus embeddings (cf. e.g. [1]), i.e. one has

$$\overline{T} = \bigcup_{I'} T/\mathrm{Ann}(I') = \bigcup_{\Theta} \bigcup_{w \in W} T/w\mathrm{Ann}(I(\Theta))w^{-1} \, ,$$

where $\mathrm{Ann}(I') = \{t \in T | \omega(t) = 1 \text{ for all } \omega \in I'\}$ and where I' (resp. Θ) runs through all rational boundary components of I (resp. all pure subsets of Π).

As a subset of M, the completion $\overline{T}$ has a quite natural representation theoretic realization:

Let $\Theta \subset \Pi$ be a pure subset. We define the pojection operator $e(\Theta) \in M$ by

$$e(\Theta)v = \begin{cases} v & \text{if} \quad v \in L(\Lambda)_\mu \quad \text{and} \quad \mu \in I(\Theta) \\ 0 & \text{if} \quad v \in L(\Lambda)_\mu \quad \text{and} \quad \mu \notin I(\Theta). \end{cases}$$

Then the boundary stratum $T/\mathrm{Ann}\,(I(\Theta))$ is realized as the T–orbit $T.e(\Theta)$ of $e(\Theta)$ under left multiplication by T. To realize $e(\Theta)$ as a boundary point of $\overline{T}$ choose a one–parameter subgroup $\gamma \in H = \mathrm{Hom}\,(\mathbb{C}^*, T)$ such that $\omega(\gamma) \geq 0$ for all $\omega \in I$ and $\omega(\gamma) = 0$ exactly when $\omega \in I(\Theta)$. Since for all $s \in \mathbb{C}^*$, $v \in L(\Lambda)_\omega$, we have

$$\gamma(s)v = s^{\omega(\gamma)}v\ ,$$

we clearly obtain (in M)

$$\lim_{s \to 0} \gamma(s) = e(\Theta)\ .$$

5. Unipotent subgroups

To study the unipotent radicals U^+, U^- of B^+, B^- as well as those of general parabolic subgroups we have to take a closer look at the action of G on $L(\Lambda), \Lambda \in P^+$. We consider $L(\Lambda)$ as a variety with the coordinate ring $\mathbb{C}[L(\Lambda)]$ generated by the functions $c_w : L(\Lambda) \to \mathbb{C}, c_w(v) = \langle v, w \rangle$, and equipped with the appropriate "cofinite" topology. Then, for any fixed $v \in L(\Lambda)$, the orbit map

$$\begin{array}{ccc} M & \longrightarrow & L(\Lambda) \\ m & \longmapsto & mv \end{array}$$

is a morphism of varieties (with continuous comorphism $\mathbb{C}[L(\Lambda)] \to \mathbb{C}[M]$). We shall make use of the following results of Kac and Peterson ([10],[3] Lemma 4.3)

- The Kostant cone $\mathcal{V}(\Lambda) = (Gv_0) \cup \{0\}$, with $v_0 \in L(\Lambda)_\Lambda \backslash \{0\}$, is Zariski closed in $L(\Lambda)$.
- If Λ is a regular dominant weight, $\Lambda \in P^{++}$ (i.e. $\Lambda(h_i) > 0$ for $i = 1, \ldots, l$), then $\mathbb{C}[G]|_{U^-}$ is generated by the matrix coefficients c_{xv_0, v_0}, x running through all elements in $\mathbf{g}$ (in fact, $x \in \mathbf{g}^- = \bigoplus_{\alpha \in \sum^-} \mathbf{g}_\alpha$, where $\sum^-$ is the system of all negative roots, is sufficient).

Theorem ([8] Satz 5.6,1)): The groups U^+ und U^- are Zariski closed in M.

Proof: Because of the existence of the anti–involution $* : G \to G$ it is sufficient to consider U^-. Assume $v_0 \in L(\Lambda)_\Lambda \backslash \{0\}$ ($\Lambda \in P^{++}$) chosen such that $\langle v_0, v_0 \rangle = 1$. This implies

$$\begin{array}{lll} c_{v_0,v_0}(u) = 1 & \text{for all} & u \in U^-, \text{ and} \\ c_{v_0,v_0}(\varphi) = 1 & \text{for all} & \varphi \in \overline{U^-}. \end{array}$$

Let $\varphi \in \overline{U^-}$. Then $\langle v_0, \varphi v_0 \rangle = 1$ implies $\varphi v_0 \neq 0$. Since $M \to L(\Lambda), m \mapsto m v_0$, is continuous and $\mathcal{V}(\Lambda)$ is closed in $L(\Lambda)$ we get $\varphi v_0 \in G v_0 \subset \mathcal{V}(\Lambda)$. Thus, using the Birkhoff decomposition of G, we find $u \in U^-, n \in N$ such that

$$\varphi v_0 = u^- n v_0 \ .$$

Because of $(u^-)^* \in U^+$ we have

$$1 = \langle v_0, \varphi v_0 \rangle = \langle (u^-)^* v_0, n v_0 \rangle = \langle v_0, n v_0 \rangle$$

and thus $n = 1$, or $\varphi v_0 = u^- v_0$. This implies $c_{x v_0, v_0}(\varphi) = c_{x v_0, v_0}(u^-)$ for all $x \in \mathbf{g}$, or $\varphi = u^- \in U^-$, q.e.d.

Recall that any subset $\Psi \in \Pi$ gives rise to a Weyl subgroup

$$W_\Psi = \langle s_{\alpha_i} | \alpha_i \in \Psi \rangle$$

and parabolic subgroups

$$P_\Psi^\pm = \langle B^\pm, W_\Psi \rangle$$

with unipotent radicals

$$U_\Psi^\pm = \bigcap_{w \in W_\Psi} w U^\pm w^{-1} \ .$$

It is obvious that $U_\Psi^\pm$ are Zariski closed in M, as well.

6. The main result

For any $i \in \{1, \ldots, l\}$ we fix a highest weight vector $v_i \in L(\Lambda_i)_{\Lambda_i} \backslash \{0\}$ and define the principal open subset $D_i \subset \overline{G}$ by

$$D_i = \{\varphi \in \overline{G} | c_{v_i,v_i}(\varphi) \neq 0\} .$$

We can almost cover $\overline{G}$ by these sets. Let $\Pi_\infty \subset \Pi$ the maximal pure subset of Π, i.e. Π is the "orthogonal" union of the set Π_∞ and a subset $\Pi \backslash \Pi_\infty$ of finite type.

Proposition A ([M], Satz 5.16): We have

$$\bigcup_{i=1}^{l} \bigcup_{g,h \in G} gD_i h = \overline{G} \backslash T.e(\Pi_\infty) .$$

Proof: To simplify our presentation, we shall assume $\Pi = \Pi_\infty$ and $P^\circ = \{0\}$. Then $e(\Pi) = e(\Pi_\infty) \in M$ is characterized by the property $e(\Pi)v = 0$, for all $v \in L(\Lambda), \Lambda \in P^+ \backslash \{0\}$. Consider $\varphi \in \overline{G}$ and assume $\varphi \notin gD_i h$ for all $i \in \{1, \ldots, l\}, g, h \in G$. Then

$$\langle gv_i, \varphi h v_i \rangle = 0, \quad \text{for all } \ i, g, h \ .$$

Since $L(\Lambda_i)$ is spanned by all $gv_i, g \in G$, we obtain $\varphi|_{L(\Lambda_i)} = 0$. Since any $L(\Lambda), \Lambda \in P^+ \backslash \{0\}$ is made up from tensor products of the $L(\Lambda_i)$ and subsequent reduction, we get

$$\varphi|_{L(\Lambda)} = 0 \quad \text{for all} \quad \Lambda \in P^+ \backslash \{0\}$$

or $\varphi = e(\Pi)$.

(This proof can be easily adopted to the general case.)

As a next step, we shall determine the structure of the open sets $D_i \subset \overline{G}$. For that recall the parabolic subgroups

$$P_i^\pm = P_{\Pi \backslash \{\alpha_i\}}^\pm$$

with unipotent radicals

$$U_i^\pm = U_{\Pi \backslash \{\alpha_i\}}^\pm ,$$

Levi subgroup $G_i = P_i^+ \cap P_i^-$ and Weyl group $W_i = W_{\Pi\setminus\{\alpha_i\}}$. Then G_i is the Kac–Moody group attached to the realization $(H, \Pi\setminus\{\alpha_i\}, \Pi^\vee\setminus\{h_i\})$. Let $\mathbb{C}[G_i]$ denote the algebra of strongly regular functions on G_i and let $\mathbb{C}[G]_i$ denote the algebra of restricted functions from $\mathbb{C}[G]$ to the subgroup G_i. Then the function c_{v_i,v_i} restricts to the character Λ_i on G_i, and representation theoretic arguments quickly show (cf. [8], section 5.1.2):

Lemma: The inclusion $\mathbb{C}[G]_i \subset \mathbb{C}[G_i]$ induces an isomorphism from the localization of $\mathbb{C}[G]_i$ with respect to Λ_i to $\mathbb{C}[G_i]$:

$$(\mathbb{C}[G]_i)_{\Lambda_i} \xrightarrow{\sim} \mathbb{C}[G_i] \; .$$

Proposition B: For any $i \in \{1, \ldots, l\}$ we have an isomorphism of infinite–dimensional varieties

$$D_i = U_i^- \times \mathrm{Specm}^\circ \mathbb{C}[G_i] \times U_i^+ \; .$$

Proof: Let us first look at $D_i \cap G$. Then the Birkhoff–decomposition

$$G = \bigcup_{w \in W} U^- wTU^+$$

gives

$$D_i \cap G = \bigcup_{w \in W_i} U^- wTU^+ = U_i^-.G_i.U_i^+ \quad \text{(direct product)} \; .$$

Recall that the $U_i^\pm$ are closed in M, therefore in $\overline{G}$ and in D_i. By the Lemma, the closure of G_i in D_i can be identified with $\mathrm{Specm}^\circ\, \mathbb{C}[G_i]$. This gives the claim.

Applying downward induction to Propositions A and B we arrive at our main result.

Theorem([8], Satz 5.18): We have

$$\overline{G} = \{ge(\Theta)h | \Theta \subset \Pi \;\text{ pure }\; g, h \in G\} = G.\overline{T}.G \; .$$

Remarks: Proposition B for the case of the minimal parabolic B^+ may already be found in [3], Lemma 4.4. Its general version for arbitrary parabolics is due to Kashiwara ([5], Proposition 5.3.5), who has also given a form of Proposition A in a somewhat different context ([5], Proposition 6.3.1).

7. An application

In [8] one finds many more results on the structure of $\overline{G}$. Here, we want to conclude with an application to the adjoint quotient of G studied in [12], [13], [14] (details are forthcoming). Recall that G admits a "parabolic" partition

$$G = \bigcup_{\substack{\Theta \subset \Pi \\ \text{pure}}} G(\Theta)$$

parallel to a stratification of $\overline{T}/W$

$$\overline{T}/W = \bigcup_{\Theta \subset \Pi} (\overline{T}/W)(\Theta)$$

$$((\overline{T}/W)(\Theta) \;\; \text{the image of} \;\; T/\mathrm{Ann}\,(I(\Theta)) \;\; \text{in}\, \overline{T}/W).$$

The adjoint quotient defined in [12], [13] is a conjugation invariant map

$$\chi : G \to \overline{T}/W$$

mapping $G(\Theta)$ to $(\overline{T}/W)(\Theta)$ for any pure $\Theta \subset \Pi$. With the help of a theory of "optimal one–parameter semisubgroups" in $\overline{G}$ the partition and the map χ can be extended to a conjugation invariant map $\overline{\chi} : \overline{G} \to \overline{T}/W$ with the following properties, basic in geometric invariant theory:

- Every fibre of $\overline{\chi}$ contains a unique closed conjugacy class,
- two elements $\varphi, \psi \in \overline{G}$ are mapped to the same point in $\overline{T}/W$ if and only if the closures of their conjugacy classes meet,

$$\overline{(Ad(G)\varphi)} \cap \overline{(Ad(G)\psi)} \neq \emptyset \, .$$

Remarks: 1) If one considers $\chi : G \to \overline{T}/W$ these statements hold only for the "classical" part $G(\emptyset)$ mapping onto T/W.
2) The closed (= minimal = semisimple) orbits in all fibres of $\overline{\chi}$ are given as the orbits of the elements $t.e(\Theta)$, $\Theta \subset \Pi$ pure, $t \in T$.

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