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Overconvergence Phenomena For Generalized Dirichlet Series

Daniele C. Struppa
Department of Mathematical Sciences
George Mason University
Fairfax, VA 22030
U.S.A

1 Introduction

This paper contains the text of the talk I delivered at the symposium "Resurgent Functions and Convolution Equations", and is an expanded version of a joint paper with T. Kawai, which will appear shortly and which will contain the full proofs of the results.

The goal of our work has been to provide a new approach to the classical topic of overconvergence for Dirichlet series, by employing results in the theory of infinite order differential operators with constant coefficients. The possibility of linking infinite order differential operators with gap theorems and related subjects such as overconvergence phenomena was first suggested by Ehrenpreis in [6], but in a form which could not be brought to fruition. In this paper we show how a wide class of overconvergence phenomena can be described in terms of infinite order differential operators, and that we can provide a multi-dimensional analog for such phenomena. Let us begin by stating the problem, as it was first observed by Jentzsch, and subsequently made famous by Ostrowski (see [5]). Consider a power series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \tag{1}$$

whose circle of convergence is $\Delta(0,\rho) = \{z \in \mathbb{C} : |z| < \rho\}$, with ρ given, therefore, by $\rho = \underline{\lim}|a_n|^{-1/n}$. Even though we know that the series given in (1) cannot converge outside of $\Delta = \Delta(0,\rho)$, it is nevertheless possible that some subsequence of its sequence of partial sums may converge in a region overlapping with Δ. In this case, we say that the series given in (1) is overconvergent.

A typical example can be constructed as follows [5]: let $P_n(z) = \frac{(z(1-z))^{4^n}}{p_n}$, where p_n is the highest coefficient in $(z(1-z))^{4^n}$, so that the module of every coefficient in $P_n(z)$ is at most one (and there are, in fact, infinitely many coefficients whose module equals one). Then one sees that the monomial of highest degree in P_n has degree $2 \cdot 4^n$, while the monomial of lowest degree in P_{n+1} has degree $4 \cdot 4^n$, so that there are no overlapping terms. If we now order the homogeneous terms in the sequence of polynomials $\{P_n(z)\}$, we can construct a series $\sum_{n=1}^{+\infty} a_n z^n$. It is immediate to see that $|a_n| \leq 1$ for any n, and for infinitely many values of n, one has $|a_n| = 1$. This implies that $\rho = 1$ and the series has $\Delta(0,1)$ as circle of convergence. On the other hand, if we set $y = 1 - z$ in the series $\sum_{n=1}^{+\infty} P_n(z)$ (which is a particular series of partial sums for $\sum_{n=1}^{+\infty} a_n z^n$), we see that the series, formally, does not change, and therefore we have that the series of partial sums converges in $\Delta(1,1)$, and so (according to our definition)we have the overconvergence phenomenon.

Ostrowski was probably the first to understand that this overconvergence phenomenon is strictly related to the existence, in the original series, of infinitely many gaps (i.e. intervals I_k of integers such that $a_n = 0$ for $n \in I_k,\ k = 1, \ldots$).

The first and most classical result in this direction is the following:

Theorem 1.1 *(Ostrowski) Let $\sum_{n=0}^{+\infty} a_n z^n$ be a power series with radius of convergence $\rho = 1$. Suppose there exist infinitely many gaps $m_k, \ldots, m'_k$ (i.e., $a_n = 0$ for $m_k < n < m'_k$), and suppose there exists a positive number θ such that $m'_k/m_k > 1 + \theta$. Then the sequence of partial sums*

$$S_{m_k}(z) = \sum_{n=0}^{m_k} a_n z^n$$

converges in a neighborhood of every regular point of $\Delta(0,1)$.

In this paper we want to use these ideas and results as the starting point for a generalization to the case of Dirichlet series. Indeed, every Taylor series $\sum_{n=0}^{+\infty} a_n z^n$ can be seen as a very special case of Dirichlet series, if one replaces z by e^{-w}, and obtain

$$\sum_{n=0}^{+\infty} a_n e^{-nw}$$

which is a special case of a general Dirichlet series

$$\sum_{n=0}^{+\infty} a_n e^{-\lambda_n z}.$$

So, the first idea is to consider overconvergence phenomena for Dirichlet series, rather than Taylor series, keeping in mind that now the disk of convergence $\Delta(0,\rho)$ is replaced by the half-plane $\{\text{Re } z > C\}$ where C is the unique real number such that the Dirichlet series $\sum_{n=0}^{+\infty} a_n e^{-\lambda_n z}$ converges in $\{\text{Re } z > C\}$, and diverges in $\{\text{Re } z < C\}$. Such a number C is known as the abscissa of convergence of the series.

Before we embark in a treatment of overconvergence for Dirichlet series, however, we need to point out some crucial differences between Taylor series and Dirichlet series.

To begin with, we know that every Taylor series has some singular points on the boundary of its circle of convergence. This is manifestly not true for Dirichlet series as it is demonstrated by the series

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n+1} e^{-s(\log n)}.$$

This leads us to recall the definition of "abscissa of holomorphy" for a Dirichlet series as the infimum of those numbers $\mathcal{H}$ for which the analytic continuation of the function defined by the Dirichlet series, remains holomorphic in $\{\text{Re } z > \mathcal{H}\}$. With this terminology, we have that the abscissa of convergence for the series above is zero, while its abscissa of holomorphy is minus infinity. Obviously, for Taylor series, these two abscissas coincide.

There is a second, related, difference between Taylor and Dirichlet series, as far as their convergence is concerned. By the same argument we just made, the domain of overconvergence of a Taylor series cannot contain the entire circle of convergence of the series. In fact, the boundary of such a circle always contains singular points. The situation is quite different in the case of Dirichlet series. Let us consider the following significant example. Take $a_n = (-1)^{n+1}$, $\lambda_{2k+1} = 2k$, $\lambda_{2k} = 2k + e^{-2k}$, and consider the Dirichlet series

$$f(s) := \sum_{n=1}^{+\infty} a_n e^{-\lambda_n s}.$$

In this case, since $\overline{\lim}\ \frac{\log n}{\lambda_n} = 0$, we can compute the abscissa of convergence C by the formula

$$C = \overline{\lim}\ \frac{\log |a_n|}{\lambda_n} = 0;$$

On the other hand we can group the terms of the series to ensure convergence in a larger region:

$$\sum_{k=1}^{+\infty} \left(e^{-\lambda_{2k+1} s} - e^{-\lambda_{2k} s}\right) = \sum_{k=1}^{+\infty} \left(e^{-2ks} - e^{-2ks} \cdot e^{-se^{-2k}}\right)$$

$$= \sum_{k=1}^{+\infty} \left(1 - e^{-se^{-2k}}\right) e^{-2ks}.$$

Now, since $\left|1 - e^{-se^{-2k}}\right| \leq \frac{1}{2}$ for Re $s > -1$ we have that the Dirichlet series overconverges in $\{\text{Re } s > -1\}$. Following a well known terminology, we will say that $\mathcal{O}$ is the abscissa of overconvergence for a Dirichlet series if it is the infimum of the set of real numbers σ for which the series is overconvergent in $\{\text{Re } z > \sigma\}$. As it is well known, for any given series, one has

$$H \leq \mathcal{O} \leq C$$

even though, $H = \mathcal{O} = C$ for Taylor series.

The consequence of these two simple remarks is therefore the understanding that, for Dirichlet series, the phenomenon of overconvergence is much more interesting than in the case of Taylor series.

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2 Dirichlet Series and Infinite Order Differential Equations

Dirichlet series arise as natural generalizations of Taylor series; but it is probably overly ambitious to expect that a general theory of overconvergence can be worked out for all Dirichlet series. In this paper I will restrict the attention to those Dirichlet series which arise in natural fashion as solutions of infinite order differential equations.

As it is well known, any compactly supported distribution in $\mathbb{R}$, with support in the origin, can be written as a finite sum of the Dirac delta and its derivatives. In other words if $f \in \mathcal{D}'_{\{0\}}(\mathbb{R})$, then $f = \sum_{j=0}^{N} a_j \cdot \frac{d^j\delta}{dx^j}$, for a suitable choice of N and $a_j \in \mathbb{C}$. In contrast with this situation, if f is a hyperfunction on $\mathbb{R}$, with support in the origin, then the finite sum is replaced by a series. Specifically, if $f \in B_{\{0\}}(\mathbb{R})$, then there exists a sequence $\{a_j\}$ of complex numbers such that $\lim \sqrt[j]{j!|a_j|} = 0$ and such that

$$f = \sum_{j=0}^{+\infty} a_j \frac{d^j\delta}{dx^j}.$$

If one takes the Fourier transform of such f, one obtains an entire function

$$\hat{f}(z) = \sum_{j=0}^{+\infty} a_j z^j;$$

however, the growth conditions on $\{a_j\}$ imply a global growth condition on $\hat{f}(z)$. Specifically, $\hat{f}$ is an entire function of infraexponential type, i.e. for any $\varepsilon > 0$ there exists $A_\varepsilon > 0$ such that

$$|\hat{f}(z)| \leq A_\varepsilon e^{\varepsilon|z|}.$$

The space of entire functions of infraexponential type is usually indicated by $\mathrm{Exp}_0(\mathbb{C})$, and one has the following topological holomorphism:

$$B_{\{0\}}(\mathbb{R}) \cong \mathrm{Exp}_0(\mathbb{C}) \cong [\mathcal{O}(\{0\})]',$$

where the last space is the dual of the space of germs of functions holomorphic at the origin, i.e. it is the space of analytical functionals carried (supported) by the origin. Any element in $B_{\{0\}}$ can therefore be seen as an analytic functional

$\mu : \mathcal{O}(\mathbb{C}) \to \mathbb{C}$ on the space of entire functions. As such one can define a convolution operator

$$\mu* : \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$$

by setting, for any holomorphic function g,

$$\mu * g(\zeta) = \langle \mu_z, \; z \mapsto g(z+\zeta) \rangle \,,$$

Since supp $(\mu) = \{0\}$, this convolution operator is actually a local operator and it is usually referred to as an infinite order differential operator; this terminology comes from the fact that if

$$\hat{\mu}(z) = \sum_{j=0}^{+\infty} a_j z^j,$$

then the convolution operator $\mu*$ can be seen as a differential operator $P(D)$ acting by

$$\mu * g(z) = \sum_{j=0}^{+\infty} a_j \frac{d^j g}{dz^j} \; (= P(D)g) \,.$$

The link between infinite order differential operators and Dirichlet series is made explicit by the following result (see [1] and [2]).

Theorem 2.1 *Let P be a differential operator of infinite order and let $V = \{\zeta \in \mathbb{C} : P(\zeta) = 0\} = \{\alpha_k \in \mathbb{C} : |\alpha_1| \leq |\alpha_2| \leq \ldots\}$; assume that all roots of P are simple. Then there exists a sequence of indices $0 = k_1 < k_2 < \ldots$ such that every entire solution f of*

$$P\left(\frac{d}{dz}\right) f = 0$$

can be written as

$$f(z) = \sum_{n \geq 1} \left(\sum_{k_n \leq k < k_{n+1}} a_k e^{\alpha_k z} \right) \tag{2}$$

for a suitable sequence of complex coefficients $\{a_k\}$ which satisfies the following growth condition: define by V_n the variety of points α_k such that $k_n \leq k < k_{n+1}$, by a_n the set $a_n := \{a_{k_n}, \ldots a_{k_{n+1}-1}\}$, and (for $t_n = k_{n+1} - k_n$) construct the $t_n \times t_n$ matrix

$$J^n = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 0 & \alpha_{k_n+1} - \alpha_{k_n} & \alpha_{k_n+2} - \alpha_{k_n} & \vdots \\ 0 & 0 & (\alpha_{k_n+1} - \alpha_{k_n})(\alpha_{k_n+2} - \alpha_{k_n}) & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix};$$

then for any $D > 0$ and any point $\beta_n \in X_n$, it is

$$\sum_{n\geq 1}\left(\sum_{j=1}^{t_n}|c_j^{(n)}|\right)e^{D|\beta_n|} < +\infty \tag{3}$$

where $\underline{c}^{(n)} = \left(c_1^{(n)}, \ldots, c_{t_n}^{(n)}\right)$ is defined by $\underline{c}^{(n)} := J^{(n)} \cdot a_n$. The convergence of the series is (2) uniform in every compact subset of $\mathbb{C}$. Conversely, every function $f(z)$ represented as in (2) is an entire solution of $P\left(\frac{d}{dz}\right) f = 0$, provided that the sequence $\{a_k\}$ satisfies (3).

Remark 1 This theorem shows how overconvergence phenomena for Dirichlet series arise naturally in connection with infinite order differential operators. Indeed, Theorem 2.1 can also be stated and proved for solutions which are only holomorphic in an open convex set Ω. The statement of the corresponding version of the result can be given verbatim, with the only exception that condition (3) will now be replaced by

$$\sum_{n\geq 1}\left(\sum_{j=1}^{t_n}|c_j^{(n)}|\right)e^{H_K(\beta_n)} < +\infty \tag{4}$$

where K is any compact convex set contained in Ω. and H_K is its support function.

Remark 2 We wish to point out that the groupings which appear in the statement of Theorem 1.1 cannot be eliminated. The example we provide here is taken from [10], to which we refer for details. Consider a sequence $\{\alpha_k\}$, $|\alpha_k| \nearrow +\infty$ of zero density $\left(\text{i.e. } \lim_{n\to\infty} \frac{n}{\alpha_n} = 0\right)$, so that we know, [1], that there exists an infinite order differential operator P_1 for whose symbol $\{\alpha_k\}$ is the zero-variety. Construct now $\beta_k = \alpha_k + e^{-|\alpha_k|^2}$; this gives rise to a second differential operator P_2. If we consider the differential operator

$$P\left(\frac{d}{dz}\right) := P_1\left(\frac{d}{dz}\right)P_2\left(\frac{d}{dz}\right),$$

it is not difficult [10] to show that every entire solution of

$$P\left(\frac{d}{dz}\right)f = 0$$

can be represented by the grouped series

$$f(z) = \sum_{k=1}^{+\infty}\left(A_k e^{\alpha_k z} + B_k e^{\beta_k z}\right).$$

In particular, by choosing $A_k = -B_k = 1$, we obtain the entire solution

$$f(z) = \sum_{k=1}^{+\infty} \left(e^{\alpha_k z} - e^{\beta_k z}\right);$$

this last series, however, is not entire any longer if the groupings are eliminated.

3 Overconvergence Phenomena in One Variable

We are now ready to look at some overconvergence phenomena for Dirichlet series. Before we actually state our result, we wish to give some general ideas on how the proof would proceed. We will begin with a Dirichlet series with a given abscissa of absolute convergence C. If the frequencies which appear in the exponents of the Dirichlet series satisfy suitable growth conditions, then the Dirichlet series is in fact a holomorphic solution f of an infinite order differential equation. Under suitable hypotheses, we will be able to show that such a holomorphic function actually extends holomorphically to a function $\tilde{f}$ holomorphic in $\{\text{Re } z > C - \varepsilon\}$ for some $\varepsilon > 0$; we will show that this function $\tilde{f}$ satisfies the same equation satisfied by f, and therefore, by Theorem 2.1, it has an exponential representation, as a grouped Dirichlet series: thus we have overconvergence.

In order to make this argument precise, we need to specify the acceptable growth on the frequencies of the Dirichlet series.

Definition 3.1 *We say that a sequence $\{\lambda_n\}$ of complex numbers of increasing moduli ($|\lambda_1| < |\lambda_2| < |\lambda_3| < \ldots$), $|\lambda_i| \uparrow +\infty$, is measurable of density D if*

$$\lim_{n \to +\infty} \frac{n}{|\lambda_n|} = D.$$

Let now $\mu \in (\mathcal{O}(\mathbb{C}))'$ be an analytic functional which is carried by the disk $\Delta(0, D)$. Then the zeroes of its Fourier-Borel transform $\hat{\mu}$ form a measurable sequence of density D; conversely, if $\{\lambda_n\}$ is such a sequence, one can always find an analytic functional, carried by $\Delta(0, D)$, for which $\{\lambda_n\}$ is the sequence of zeroes of $\hat{\mu}$. In terms of the growth of $\hat{\mu}$, this is equivalent to require that for every $\varepsilon > 0$ there exists $A_\varepsilon > 0$ such that

$$|\hat{\mu}(z)| \leq A_\varepsilon e^{(D+\varepsilon)|z|}.$$

Finally, we point out that if $D = 0$, $\hat{\mu}$ is of infraexponential type, μ is carried by the origin, and for every open set $\Omega \subseteq \mathbb{C}$, $\mu* : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega)$. This last fact is just another way of formulating the locality of μ; if, on the other hand, μ is carried by $\Delta(0, D)$, then, for every open set Ω, the convolution by μ causes a shift, and one has

$$\mu* : \mathcal{O}(\Omega + \Delta(0, D)) \to \mathcal{O}(\Omega).$$

Before we can prove our overconvergence result, we need to discuss the invertibility of an infinite order differential operator, as a holomorphic microlocal operator (we refer the reader to [8] for definitions and terminology form microlocal analysis). Let $P(D) = \sum_{j=0}^{+\infty} a_j \frac{d^j}{dz^j}$ be an infinite order differential operator; we can consider $P(D)$ as a holomorphic microlocal operator, and as such we may study its invertibility at a point (z_0, ζ_0) of the cotangent bundle $T^*\mathbb{C} \cong \mathbb{C} \times (\mathbb{C} \setminus \{\zeta = 0\})$. A well known result claims that $P(D)$ is in fact invertible in (z_0, ζ_0) if $P(\zeta)$ never vanishes on the set

$$\left\{\zeta \in \mathbb{C} = \left|\frac{\zeta}{|\zeta|} - \frac{\zeta_0}{|\zeta_0|}\right| < \delta,\ |\zeta| \gg 1\right\},$$

for some $\delta > 0$. This means that unless $\frac{\zeta}{|\zeta_0|}$ is a characteristic direction for P, P can be inverted.

We are now ready for our overconvergence result [9]:

Theorem 3.1 *Let $\{\lambda_n\}$ be a sequence of measurable density zero. Suppose the Dirichlet series*

$$\sum_{n=0}^{+\infty} a_n e^{\lambda_n z}$$

has abscissa of convergence C. Suppose, moreover, that the sum $f(z) = \sum_{n=0}^{+\infty} a_n e^{\lambda_n z}$ admits an analytic continuation near a point z_0 with $Re\ z_0 = C$, and that there exist finitely many unit vectors e_k ($k = 1, \ldots, K$) in $S^1 \subseteq \mathbb{R}^2$ such that:

For every $\varepsilon > 0$, and each compact set $K \subseteq S^1 - \{e_1, \ldots, e_K\}$, there exists $n_0 = n_0(\varepsilon, K)$ such that

$$\inf_{\substack{n \geq n_0 \\ e \subseteq K}} \left| \frac{(Re\ \lambda_n, Im\ \lambda_n)}{|\lambda_n|} - e \right| > \varepsilon.$$

Then, there exists $\delta > 0$ such that the abscissa of overconvergence of $\sum_{n=0}^{+\infty} a_n e^{\lambda_n z}$ is less than or equal to $C - \delta$. In other words, it is possible to group the series in such a way that f extends analytically to a function $\tilde{f}$ on $\{Re\ z > C - \delta\}$ and, there,

$$\tilde{f} = \sum_{j=0}^{+\infty} \left(\sum_{k_j \leq n < k_{j+1}} a_n e^{\lambda_n z} \right).$$

This theorem allows us to show that under our conditions on the characteristic variety of $P(D)$ $\mathcal{O} \leq C - \delta$, for some $\delta > 0$. There are, on the other hand, classical results which allow us to compute δ, in terms of the sequence $\{\lambda_n\}$. In particular, if the sequence $\{\lambda_n\}$ is given, one can construct the infraexponential entire function

$$C(z) = \prod_{n=1}^{+\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

We call "condensation index" of the sequence $\{\lambda_n\}$, the real number

$$\delta = \overline{\lim} \frac{1}{\lambda_n} \log \left| \frac{1}{C'(\lambda_n)} \right|.$$

We have the following result:

Theorem 3.2 *Under the hypothesis of Theorem 3.1, the condensation index δ of $\{\lambda_n\}$ is positive.*

Proof. A well known result of V. Bernstein shows that the condition

$$\mathcal{O} \leq C - \delta$$

is equivalent to require that the condensation index of $\{\lambda_n\}$ be at least δ. The result then follows immediately from our Theorem 3.1. □

4 Final Remarks

In Ehrenpreis' original formulation which stimulated our interest [6], Taylor series are the initial model (and indeed they inspire the kind of result we are interested in); however, as we have showed, the kind of results one may expect for Taylor series is quite different from the results one may expect for general Dirichlet series. There are several, well known, reasons for the difference in behavior from our point of view; however, the main point is that Taylor series cannot generally arise as solutions to infinite order differential equations as their frequencies (when interpreted as series of exponentials) have density one. Thus, the treatment of Taylor series suggested in [6] cannot be achieved within the framework of infinite order differential operators, but will require the use of more complex convolution operators. Unfortunately, however, we do not expect, at this point, that Kawai's results from [7] can be extended to such operators (the existing results in this area, [3], [4], do not deal with continuation across characteristic surfaces, which is really the issue at hand here).

Another interesting generalization arises when trying to extend these ideas to the case of Dirichlet series in several variables. I will not get into this at this point, but I will refer the reader to [9] where some particular cases are dealt with. Part of the difficulty lies in the lack of an accepted definition for abscissa of convergence in several variables. In [9] we provide a possible suggestion and we show that a theorem similar to Theorem 3.1 can be proved.

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